13. Homomorphic mappings (deformations) of groupoids

13.1. Definition

Let $\mathfrak{G}$, $\mathfrak{G}^*$ be arbitrary groupoids. As we have already said (in 12.2), a mapping of $\mathfrak{G}$ into $\mathfrak{G}^*$ is a mapping of the field G of $\mathfrak{G}$ into the field G^* of $\mathfrak{G}^*$. In a similar way we apply to groupoids all the other concepts and symbols we have described (in Chapter 6) while studying the mappings of sets. By the above definition, the concept of a mapping of $\mathfrak{G}$ into $\mathfrak{G}^*$ concerns only the fields and does in no way depend on the multiplications in the groupoids. Some mappings may, however, be in certain relations with the multiplications in $\mathfrak{G}$ and $\mathfrak{G}^*$. Of great importance to the theory of groupoids are the so-called homomorphic mappings characterized by preserving the multiplications of both groupoids. A detailed definition:

A mapping $\boldsymbol{d}$ of the groupoid $\mathfrak{G}$ into $\mathfrak{G}^*$ is called *homomorphic* if the product ab of an arbitrary element $a \in \mathfrak{G}$ and an element $b \in \mathfrak{G}$ is mapped onto the product of the $\boldsymbol{d}$-image of a and the $\boldsymbol{d}$-image of b, i.e., if, for $a, b \in \mathfrak{G}$, there holds $\boldsymbol{d}ab = \boldsymbol{d}a \, . \, \boldsymbol{d}b$.

For convenience, a homomorphic mapping of the groupoid $\mathfrak{G}$ into $\mathfrak{G}^*$ is called a *deformation of the groupoid* $\mathfrak{G}$ *into* $\mathfrak{G}^*$. A deformation of $\mathfrak{G}$ *onto* $\mathfrak{G}^*$ is sometimes called a *homomorphism*.

While studying the mapping of sets, we have realized that there need not always exist a mapping of a given set onto another set; consequently, a mapping of $\mathfrak{G}$ onto $\mathfrak{G}^*$ and, of course, a deformation of $\mathfrak{G}$ onto $\mathfrak{G}^*$ need not exist at all. If it exists, then *the groupoid* $\mathfrak{G}^*$ *is* said to be *homomorphic with* $\mathfrak{G}$.

13.2. Example of a deformation

Let n denote a positive integer and $\boldsymbol{d}$ the mapping of the groupoid $\mathfrak{Z}$ onto $\mathfrak{Z}_n$, defined as follows: $\boldsymbol{d}a \in \mathfrak{Z}_n$ is, for $a \in \mathfrak{Z}$, the remainder of the division of a by n. It is easy to verify that $\boldsymbol{d}$ is a deformation and therefore a homomorphism of $\mathfrak{Z}$ onto $\mathfrak{Z}_n$. Indeed, let a, b stand for arbitrary elements of $\mathfrak{Z}$. The product ab of a and b is, by the definition of the multiplication in $\mathfrak{Z}$, the sum $a + b$ and $\boldsymbol{d}a, \boldsymbol{d}b, \boldsymbol{d}ab$ are, by the definition of the mapping $\boldsymbol{d}$, the remainders of the division of $a, b, a + b$ by n, respectively. The product $\boldsymbol{d}a\boldsymbol{d}b$ of $\boldsymbol{d}a$ and $\boldsymbol{d}b$ is, by the definition, the remainder of the division of $\boldsymbol{d}a + \boldsymbol{d}b$ by n and, since the numbers $\boldsymbol{d}a + \boldsymbol{d}b$ and $a + b$ differ only by an integral multiple of n, the product $\boldsymbol{d}a\boldsymbol{d}b$ is the remainder of the division $a + b$ by n. Hence we have $\boldsymbol{d}a\boldsymbol{d}b = \boldsymbol{d}ab$ and see that $\boldsymbol{d}$ is a deformation. In the following study of groupoids we shall often meet with cases of deformation, so we shall, meanwhile, be satisfied with this single example.

13.3. Properties of deformations

Let $\boldsymbol{d}$ be an arbitrary deformation of $\mathfrak{G}$ into $\mathfrak{G}^*$.

Suppose A, B, C are nonempty subsets of $\mathfrak{G}$.

1. The symbol $\boldsymbol{d}A$ denotes, as we know, the image of the set A under the extended mapping $\boldsymbol{d}$, i.e., the subset of $\mathfrak{G}^*$ consisting of the $\boldsymbol{d}$-images of the individual elements of A.

It is easy to show that *there holds*

$$\boldsymbol{d}(AB) = \boldsymbol{d}A \,.\, \boldsymbol{d}B.$$

Every element $c^* \in \boldsymbol{d}(AB)$ is, on the one hand, the $\boldsymbol{d}$-image of the product ab of an element $a \in A$ and an element $b \in B$ so that $c^* = \boldsymbol{d}ab = \boldsymbol{d}a \,.\, \boldsymbol{d}b \in \boldsymbol{d}A \,.\, \boldsymbol{d}B$; consequently, there holds $\boldsymbol{d}(AB) \subset \boldsymbol{d}A \,.\, \boldsymbol{d}B$. On the other hand, every element $c^* \in \boldsymbol{d}A \,.\, \boldsymbol{d}B$ is the product of an element $a^* \in \boldsymbol{d}A$ and an element $b^* \in \boldsymbol{d}B$ so that there exist elements $a \in A$, $b \in B$ such that $a^* = \boldsymbol{d}a$, $b^* = \boldsymbol{d}b$ and we have: $c^* = a^*b^* = \boldsymbol{d}a \,.\, \boldsymbol{d}b = \boldsymbol{d}ab \in \boldsymbol{d}(AB)$; consequently: $\boldsymbol{d}A \,.\, \boldsymbol{d}B \subset \boldsymbol{d}(AB)$ and the proof is complete.

2. With respect to this result we conclude that *if the set AB is a part of C, then the set $\boldsymbol{d}A \,.\, \boldsymbol{d}B$ is a part of $\boldsymbol{d}C$*; that is to say, $AB \subset C$ yields $\boldsymbol{d}A \,.\, \boldsymbol{d}B \subset \boldsymbol{d}C$.

3. If A is the field of a subgroupoid $\mathfrak{A} \subset \mathfrak{G}$ so that it is groupoidal, then we have $AA \subset A$ whence $\boldsymbol{d}A \,.\, \boldsymbol{d}A \subset \boldsymbol{d}A$ and we see that *the $\boldsymbol{d}$-image of the field of the subgroupoid $\mathfrak{A}$ is a groupoidal subset of $\mathfrak{G}^*$*. The subgroupoid of $\mathfrak{G}^*$ whose field is $\boldsymbol{d}A$ is called the *image of the subgroupoid $\mathfrak{A}$ under the deformation $\boldsymbol{d}$* and is denoted $\boldsymbol{d}\mathfrak{A}$; the subgroupoid $\mathfrak{A}$ is called an *inverse image of $\boldsymbol{d}\mathfrak{A}$ under the deformation $\boldsymbol{d}$*. It is obvious that $\boldsymbol{d}$ is a deformation of $\mathfrak{A}$ onto $\boldsymbol{d}\mathfrak{A}$ so that $\boldsymbol{d}\mathfrak{A}$ is homomorphic with $\mathfrak{A}$.

The above notions and results apply, in particular, in case of the field G of $\mathfrak{G}$. We observe that *the $\boldsymbol{d}$-image $\boldsymbol{d}\mathfrak{G}$ of $\mathfrak{G}$ is a subgroupoid of $\mathfrak{G}^*$, homomorphic with $\mathfrak{G}$*. If $\boldsymbol{d}$ is a deformation of $\mathfrak{G}$ onto $\mathfrak{G}^*$, then we, naturally, have $\mathfrak{G}^* = \boldsymbol{d}\mathfrak{G}$.

4. If $\boldsymbol{d}$ is a deformation of $\mathfrak{G}$ into $\mathfrak{G}^*$ and $\boldsymbol{f}$ a deformation of $\mathfrak{G}^*$ into a groupoid $\mathfrak{F}$, then $\boldsymbol{fd}$ is a deformation of $\mathfrak{G}$ into $\mathfrak{F}$. Indeed, in accordance with the definition of the composite mapping $\boldsymbol{fd}$, and $\boldsymbol{d}, \boldsymbol{f}$ being deformations, there holds, for $a, b \in \mathfrak{G}$:

$$\boldsymbol{fd}(ab) = \boldsymbol{f}(\boldsymbol{d}ab) = \boldsymbol{f}(\boldsymbol{d}a \,.\, \boldsymbol{d}b) = \boldsymbol{f}(\boldsymbol{d}a) \,.\, \boldsymbol{f}(\boldsymbol{d}b) = \boldsymbol{fd}a \,.\, \boldsymbol{fd}b,$$

and therefore, in fact, $\boldsymbol{fd}(ab) = \boldsymbol{fd}a \,.\, \boldsymbol{fd}b$.

13.4. Isomorphic mappings

1. The concept of a deformation includes other important notions, first of all, the notion of a simple deformation of the groupoid $\mathfrak{G}$ into $\mathfrak{G}^*$, i.e., a deformation in which each element of $\mathfrak{G}^*$ has, at most, one inverse image. A simple deformation of $\mathfrak{G}$ into (onto) $\mathfrak{G}^*$ is called *isomorphic mapping* of $\mathfrak{G}$ into (onto) $\mathfrak{G}^*$.

From the results in 6.7 and 13.3.4 there follows that *if $\boldsymbol{d}$ is an isomorphic mapping of $\mathfrak{G}$ into $\mathfrak{G}^*$ and $\boldsymbol{f}$ an isomorphic mapping of $\mathfrak{G}^*$ into $\mathfrak{F}$, then the composite mapping $\boldsymbol{fd}$ of $\mathfrak{G}$ into $\mathfrak{F}$ is also isomorphic.*

2. An isomorphic mapping of $\mathfrak{G}$ *onto* $\mathfrak{G}^*$ is called *isomorphism.* To every simple deformation $\boldsymbol{d}$ of $\mathfrak{G}$ onto $\mathfrak{G}^*$ there, naturally, exists an inverse mapping $\boldsymbol{d}^{-1}$ of $\mathfrak{G}^*$ onto $\mathfrak{G}$ which is simple and, as we shall easily verify, a deformation. Assuming a^*, b^* to be arbitrary elements of $\mathfrak{G}^*$, let $a, b \in \mathfrak{G}$ be their inverse images under $\boldsymbol{d}$ so that $\boldsymbol{d}a = a^*$, $\boldsymbol{d}b = b^*$, $\boldsymbol{d}ab = a^*b^*$. Hence we have, by the definition of the inverse mapping $\boldsymbol{d}^{-1}$, the equalities: $a = \boldsymbol{d}^{-1}a^*$, $b = \boldsymbol{d}^{-1}b^*$, $ab = \boldsymbol{d}^{-1}a^*b^*$ which, in fact, yield $\boldsymbol{d}^{-1}a^*b^* = \boldsymbol{d}^{-1}a^* \,.\, \boldsymbol{d}^{-1}b^*$. Thus, if there exists an isomorphism $\boldsymbol{d}$ of $\mathfrak{G}$ onto $\mathfrak{G}^*$, then there exists an isomorphism $\boldsymbol{d}^{-1}$ of $\mathfrak{G}^*$ onto $\mathfrak{G}$; in that case we say that $\mathfrak{G}$ ($\mathfrak{G}^*$) is *isomorphic* with $\mathfrak{G}^*$ ($\mathfrak{G}$) or that $\mathfrak{G}$, $\mathfrak{G}^*$ are isomorphic and write $\mathfrak{G} \simeq \mathfrak{G}^*$ or $\mathfrak{G}^* \simeq \mathfrak{G}$. It is obvious that the fields of any two isomorphic groupoids are equivalent sets.

A mapping composite of two isomorphisms is again an isomorphism.

3. *Examples.* The abstract groupoid with the field $\{\boldsymbol{e}\}$ and the multiplication described in the first multiplication table in 11.4.2 is isomorphic with the groupoid $\mathfrak{S}_1$. The abstract groupoid with the field $\{\boldsymbol{e}, \boldsymbol{a}\}$ and the multiplication described in the second multiplication table in 11.4.2 is isomorphic with the groupoid $\mathfrak{S}_2$; the abstract groupoid with the field $\{\boldsymbol{e}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}\}$ and the multiplication described in the third multiplication table in 11.4.2 is isomorphic with the groupoid $\mathfrak{S}_3$.

13.5. Operators, meromorphic and automorphic mappings

1. Further notions included in the concept of a deformation concern the case of a deformation of $\mathfrak{G}$ into or onto itself.

A deformation of $\mathfrak{G}$ into itself is also called an *operator on* (or *of*) *the groupoid* $\mathfrak{G}$ or an *endomorphic mapping of* $\mathfrak{G}$.

A simple operator on $\mathfrak{G}$, i.e., an isomorphic mapping of $\mathfrak{G}$ into itself is sometimes called a *meromorphic mapping of* $\mathfrak{G}$. If the image of $\mathfrak{G}$ is a proper subgroupoid of $\mathfrak{G}$, then the meromorphic mapping of $\mathfrak{G}$ is said to be *proper*.

2. An isomorphic mapping of $\mathfrak{G}$ onto itself is also called an *automorphic mapping of* $\mathfrak{G}$, briefly, an *automorphism of* $\mathfrak{G}$.

3. *Examples.* The mapping of the groupoid $\mathfrak{Z}$ into itself where each element $a \in \mathfrak{Z}$ is mapped onto the product (in arithmetic sense) $ka \in \mathfrak{Z}$, k denoting a non-negative integer, is an operator on $\mathfrak{Z}$. For $k \geqq 1$ it is a meromorphic mapping of $\mathfrak{Z}$, for

$k = 1$ it is an automorphism of $\mathfrak{Z}$ and for $k = 0$ an operator but not a meromorphic mapping of $\mathfrak{Z}$.

The simplest example of an automorphism of any groupoid $\mathfrak{G}$ is the identical mapping of $\mathfrak{G}$, the so-called *identical automorphism of* $\mathfrak{G}$.

13.6. Exercises

1. If any two elements of $\mathfrak{G}$ are interchangeable, then their images under every deformation of $\mathfrak{G}$ into $\mathfrak{G}^*$ are also interchangeable. The image of every Abelian groupoid is also Abelian.
2. If the product of a three-membered sequence of elements $a, b, c \in \mathfrak{G}$ consists of a single element, then the same holds for the sequence of images $\boldsymbol{d}a, \boldsymbol{d}b, \boldsymbol{d}c \in \mathfrak{G}^*$ under any deformation $\boldsymbol{d}$ of $\mathfrak{G}$ into $\mathfrak{G}^*$. The image of every associative groupoid under any deformation is also associative.
3. If $\mathfrak{G}$ is associative and has a center, then the image of the center under any deformation of $\mathfrak{G}$ onto $\mathfrak{G}^*$ lies in the center of $\mathfrak{G}^*$.
4. The inverse image of a groupoidal subset of $\mathfrak{G}^*$ under a deformation of $\mathfrak{G}$ onto $\mathfrak{G}^*$ need not be groupoidal.
5. Every meromorphic mapping of a finite groupoid $\mathfrak{G}$ is an automorphism of $\mathfrak{G}$.
6. For isomorphisms of the groupoids $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ the following statements are true: a) $\mathfrak{A} \simeq \mathfrak{A}$ (reflexivity); b) $\mathfrak{A} \simeq \mathfrak{B}$ yields $\mathfrak{B} \simeq \mathfrak{A}$ (symmetry); c) from $\mathfrak{A} \simeq \mathfrak{B}$, $\mathfrak{B} \simeq \mathfrak{C}$ there follows $\mathfrak{A} \simeq \mathfrak{C}$ (transitivity).
7. It is left to the reader to give some examples of deformation himself.

14. Generating decompositions

14.1. Basic concepts

Suppose $\mathfrak{G}$ is an arbitrary groupoid.

Definition. Any decomposition $\bar{A}$ in $\mathfrak{G}$ is called *generating* if there exists, to any two-membered sequence of the elements $\bar{a}, \bar{b} \in \bar{A}$, an element $\bar{c} \in \bar{A}$ such that $\bar{a}\bar{b} \in \bar{c}$.

As to the generating decompositions on the groupoid $\mathfrak{G}$, note that the greatest decomposition $\bar{G}_{\max}$ and the least decomposition $\bar{G}_{\min}$ are generating. On every groupoid there exist at least these two extreme generating decompositions.

The equivalence belonging to a generating decomposition (9.3) is usually called a *congruence*.