10. Series of decompositions of sets

In: Otakar Borůvka (author): Foundations of the Theory of Groupoids and Groups. (English). Berlin: VEB Deutscher Verlag der Wissenschaften, 1974. pp. 74--89.

Persistent URL: http://dml.cz/dmlcz/401549

Since there hold a) and a′) we generally speak about the join and the meet of two elements without drawing any distinction as to their arrangement.

To give an example of a join and a meet, let us note the antisymmetric congruence on the system of all decompositions of G under which there correspond, to each decomposition of G, all its coverings or refinements. Every two decompositions $\bar{A}$, $\bar{B}$ of G have the join $[\bar{A}, \bar{B}]$ or $(\bar{A}, \bar{B})$ and the meet $(\bar{A}, \bar{B})$ or $[\bar{A}, \bar{B}]$.

9.5. Exercises

1. Let the set G be mapped, under the single-valued functions $\boldsymbol{a}, \boldsymbol{b}$, onto the set A or B, respectively, and let its decompositions corresponding to these mappings be equal. Show that, in that case, $\boldsymbol{f} = \boldsymbol{b}\boldsymbol{a}^{-1}$ is a single-valued and simple mapping of A onto B and $\boldsymbol{f}^{-1} = \boldsymbol{a}\boldsymbol{b}^{-1}$ the inverse mapping of B onto A. Hence, in this case the sets A, B are equivalent.
2. Let n denote an arbitrary positive integer. Associating, with every integer a, each number $a + \nu n$ where $\nu = \ldots, -2, -1, 0, 1, 2, \ldots$, we obtain a symmetric congruence on the set of all integers. The corresponding decomposition consists of n classes; the numbers $0, 1, \ldots, n - 1$ form a system of representatives of the congruence.
3. Associating, with every positive integer, each of its positive multiples (each of its positive divisors), we obtain an antisymmetric congruence on the set of all positive integers. Every two positive integers have, with regard to this congruence, a join formed by their least common multiple (greatest common divisor) and a meet formed by their greatest common divisor (least common multiple). Either of the congruences is the inverse of the other.
4. Associating, with every part of G, each of its supersets (subsets), we obtain an antisymmetric congruence on the set of all parts of G. Every two parts of G have, with regard to this congruence, a join formed by their sum (intersection) and a meet formed by their intersection (sum). Either of the congruences is the inverse of the other.
5. If $\boldsymbol{g}$ is an antisymmetric congruence on G and some elements $a, b \in G$ have the join $a \cup b$, then:

   a) $\boldsymbol{g}(a \cup b) = \boldsymbol{g}a \cap \boldsymbol{g}b$ (the right-hand side denotes, of course, the intersection of $\boldsymbol{g}a$, $\boldsymbol{g}b$),
   b) $\boldsymbol{g}^{-1}(a \cup b) \supset \boldsymbol{g}^{-1}a \cup \boldsymbol{g}^{-1}b$.

10. Series of decompositions of sets

In this chapter we shall develop a theory of the so-called series of decompositions of sets. We shall make use of many results arrived at in the previous considerations and concerning decompositions and mappings of sets. The mentioned theory describes the set-structure of the appropriate sections of the theory of groupoids and groups and admits of a better understanding of the results of the theory of groups arrived at by classical methods. A study of the series of decompositions of sets has, moreover, proved most useful in connection with mappings onto sets of sequences and the domain of scientific classifications.

10.1. Basic concepts

Let $\bar{A} \geqq \bar{B}$ stand for arbitrary decompositions of the set G.

A *series of decompositions of the set G from $\bar{A}$ to $\bar{B}$* (briefly, a *series of decompositions from $\bar{A}$ to $\bar{B}$*) is a finite sequence of the decompositions $\bar{A}_1, \ldots, \bar{A}_\alpha$ on G, of length $\alpha(\geqq 1)$, with the following properties: 1. The first member of the sequence is the decomposition $\bar{A}$, the last member is $\bar{B}$, hence $\bar{A}_1 = \bar{A}$, $\bar{A}_\alpha = \bar{B}$. 2. Every decomposition is a refinement of the one directly preceding it, so that

$$(\bar{A} =)\ \bar{A}_1 \geqq \cdots \geqq \bar{A}_\alpha\ (= \bar{B}).$$

Such a series is briefly denoted $(\bar{A})$. The decompositions $\bar{A}_1, \ldots, \bar{A}_\alpha$ are called *members of* $(\bar{A})$; $\bar{A}_1$ is the *initial* and $\bar{A}_\alpha$ the *final* member of $(\bar{A})$. By the *length of* $(\bar{A})$ we mean the number α of the members of $(\bar{A})$.

For example, a decomposition $\bar{A}$ on G forms a series of length 1; its initial as well as final member coincides with $\bar{A}$.

Suppose $\big((\bar{A}) = \big)\ \bar{A}_1 \geqq \cdots \geqq \bar{A}_\alpha$ is a series of decompositions from $\bar{A}$ to $\bar{B}$.

A member of $(\bar{A})$ is called *essential* if it is either the initial member $\bar{A}_1$ or a proper refinement of the member directly preceding it. In the opposite case it is *inessential*. If $(\bar{A})$ contains at least one inessential member $\bar{A}_{\gamma+1}$, then it is called (because $\bar{A}_{\gamma+1} = \bar{A}_\gamma$) a *series with iteration*. If all the members of $(\bar{A})$ are essential, then $\bar{A}$ is said to be *without iteration*. The number α' of essential members of $(\bar{A})$ is the *reduced length of* $(\bar{A})$. There evidently holds $1 \leqq \alpha' \leqq \alpha$ and the equality $\alpha' = \alpha$ is characteristic of series without iteration. If any iterations in $(\bar{A})$ occur, then $(\bar{A})$ may be reduced by omitting all the inessential members, that is to say, shortened to a series $(\bar{A}')$ without iteration. The length of the reduced series $(\bar{A}')$ equals the reduced length α' of the series $(\bar{A})$. Conversely, $(\bar{A})$ may be lengthened by inserting a finite number of inessential members between any two neighbouring members $\bar{A}_\gamma$, $\bar{A}_{\gamma+1}$ or, if convenient, before (after) the first (last) member $\bar{A}_1$ $(\bar{A}_\alpha)$ of $(\bar{A})$. Every series of decompositions, generated by reducing or extending (lengthening) $(\bar{A})$, naturally, has the same reduced length as $(\bar{A})$.

If $\alpha_1 < \cdots < \alpha_\beta$ are arbitrary numbers of the set $\{1, \ldots, \alpha\}$, then even

$$\bar{A}_{\alpha_1} \geqq \cdots \geqq \bar{A}_{\alpha_\beta}$$

is a series of decompositions on G, called a *partial series* or a *part of* $(\bar{A})$.

If, moreover, A is a nonempty subset of G, then the sequence

$$\bar{A}_{\alpha_1} \sqcap A \geqq \cdots \geqq \bar{A}_{\alpha_\beta} \sqcap A$$

is a series of decompositions on A.

10.2. Local chains

Suppose $\big((\bar{A}) =\big)\ \bar{A}_1 \geqq \cdots \geqq \bar{A}_\alpha$ is a series of decompositions on G of length $\alpha \geqq 1$.

Let $\bar{a} \in \bar{A}_\alpha$ be an arbitrary element and $\bar{a}_\gamma \in \bar{A}_\gamma$ the element of $\bar{A}_\gamma$ containing $\bar{a}$ $(\gamma = 1, \ldots, \alpha)$. There evidently holds:

$$\bar{a}_1 \supset \cdots \supset \bar{a}_\alpha \ (\bar{a}_\alpha = \bar{a}).$$

Furthermore,

$$\bar{K}_\gamma = \bar{a}_\gamma \sqcap \bar{A}_{\gamma+1} \qquad (\bar{A}_{\alpha+1} = \bar{A}_\alpha)$$

is a decomposition on $\bar{a}_\gamma$, forming a part of $\bar{A}_{\gamma+1}$ and, simultaneously, $\bar{a}_{\gamma+1} \in \bar{K}_\gamma$ $(\bar{a}_{\alpha+1} = \bar{a}_\alpha)$. We observe that

$$([\bar{K}] =)\ \bar{K}_1 \to \ldots \to \bar{K}_\alpha$$

is a chain of decompositions of sets from $\bar{a}_1$ to $\bar{a}_{\alpha+1}$ $(= \bar{a})$ (2.5). It is called the *local chain of the series* $(\bar{A})$, *corresponding to the element* $\bar{a} \in \bar{A}_\alpha$, briefly: the local chain with the base $\bar{a}$. Notation as above or, more accurately: $([\bar{K}\bar{a}] =)\ \bar{K}_1\bar{a} \to \cdots \to \bar{K}_\alpha\bar{a}$. The element $\bar{a} \in \bar{A}_\alpha$ is called the *base of the chain* $[\bar{K}]$. By its base $\bar{a}$ the chain $[\bar{K}]$ is uniquely determined.

Let us remark that the final member $\bar{K}_\alpha$ of $[\bar{K}]$ is the greatest decomposition of $\bar{a}$, hence inessential. $\bar{K}_\gamma$ may, with respect to $\bar{A}_\gamma \geqq \bar{A}_{\gamma+1}$, also be defined by the formula $\bar{K}_\gamma = \bar{a}_\gamma \sqsubset \bar{A}_{\gamma+1}$.

The local chain $[\bar{K}]$ *is an elementary chain from* $\bar{a}_1$ *to* $\bar{a}_{\alpha+1}$ $(= \bar{a})$ *over* $\bar{A}_{\alpha+1}$.

Indeed, since $\bar{A}_{\gamma+1}$ is a covering of $\bar{A}_{\alpha+1}$ $(\gamma = 1, \ldots, \alpha)$, $\bar{a}_\gamma \sqcap \bar{A}_{\gamma+1}$ is a covering of $\bar{a}_\gamma \sqcap \bar{A}_{\alpha+1}$.

The length of $[\bar{A}]$ is, obviously, α and therefore equal to the length of $(\bar{A})$. If a member $\bar{A}_{\gamma+1}$ of $(\bar{A})$ is inessential and so $\bar{A}_{\gamma+1} = \bar{A}_\gamma$, then there holds $\bar{a}_{\gamma+1} = \bar{a}_\gamma$; hence $\bar{K}_\gamma$ is an inessential member of $[\bar{K}]$. Consequently, for the reduced lengths α' and $\varkappa'$ of $(\bar{A})$ and the local chain $[\bar{K}]$, there holds: $\varkappa' \leqq \alpha'$. Thus, if a local chain of (A) has no iteration, except the final member which is always inessential, then $(\bar{A})$ is a series without iteration.

10.3. Refinements of series of decompositions

Suppose, again, that $\big((\bar{A}) =\big)\ \bar{A}_1 \geqq \cdots \geqq \bar{A}_\alpha$ is a series of decompositions of length $\alpha \geqq 1$ on the set G.

By a *refinement of* $(\bar{A})$ we mean a series of decompositions on G such that $(\bar{A})$ is a part of that series. Thus every refinement of $(\bar{A})$ is of the form:

$$\bar{A}_{1,1} \geqq \cdots \geqq \bar{A}_{1,\beta_1-1} \geqq \bar{A}_{1,\beta_1} \geqq \bar{A}_{2,1} \geqq \cdots \geqq \bar{A}_{2,\beta_2-1} \geqq \bar{A}_{2,\beta_2} \geqq \cdots \geqq$$
$$\geqq \bar{A}_{\alpha,\beta_\alpha} \geqq \bar{A}_{\alpha+1,1} \geqq \cdots \geqq \bar{A}_{\alpha+1,\beta_{\alpha+1}-1}.$$

In the above formulae, $\bar{A}_{\gamma,\beta_\gamma} = \bar{A}_\gamma$ holds for $\gamma = 1, \ldots, \alpha$, whereas $\beta_1, \ldots, \beta_{\alpha+1}$ are natural numbers. If $\beta_\delta = 1$, then the members $\bar{A}_{\delta,1} \geqq \ldots \geqq \bar{A}_{\delta,\beta_\delta - 1}$ are not read. From the definition it is clear that any refinement of $(\bar{A})$ is obtained by way of inserting between two neighbouring members $\bar{A}_\gamma, \bar{A}_{\gamma+1}$ and, maybe, also before $\bar{A}_1$ and after $\bar{A}_\alpha$, a suitable series of decompositions. Note that every lengthening of $(\bar{A})$ is its own refinement.

Let us consider a refinement $(\mathring{A})$ of $(\bar{A})$ and use the same notation as above. In particular, $\bar{A}_{\gamma,\beta_\gamma} = \bar{A}_\gamma$ for $\gamma = 1, \ldots, \alpha$. The indices μ, ν will, in what follows, denote: for $\beta_{\alpha+1} = 1$, the numbers $\mu = 1, \ldots, \alpha$; $\nu = 1, \ldots, \beta_\mu$ and for $\beta_{\alpha+1} > 1$, even the numbers $\mu = \alpha + 1$, $\nu = 1, \ldots, \beta_{\alpha+1} - 1$.

Let $\bar{a} \in \bar{A}_\alpha$ or $\bar{a} \in \bar{A}_{\alpha+1,\beta_{\alpha+1}-1}$ stand for an element of $\bar{A}_\alpha$ or of $\bar{A}_{\alpha+1,\beta_{\alpha+1}-1}$ according as $\beta_{\alpha+1} = 1$ or $\beta_{\alpha+1} > 1$. Let, moreover, $\bar{a}_{\mu,\nu}$ and $\bar{a}_\gamma$ denote the elements of $\bar{A}_{\mu,\nu}$, $\bar{A}_\gamma$ for which $\bar{a} \subset \bar{a}_{\mu,\nu} \in \bar{A}_{\mu,\nu}$ and $\bar{a} \subset \bar{a}_\gamma \in \bar{A}_\gamma$, respectively; so we have, in particular, $\bar{a}_{\gamma,\beta_\gamma} = \bar{a}_\gamma$.

The local chain $[\mathring{K}]$ of $(\mathring{A})$, with the base $\bar{a}$, is

$$([\mathring{K}] =)\, \bar{K}_{1,1} \to \cdots \to \bar{K}_{1,\beta_1} \to \bar{K}_{2,1} \to \cdots \to \bar{K}_{2,\beta_2} \to \cdots \to \bar{K}_{\alpha,\beta_\alpha}$$
$$\to \bar{K}_{\alpha+1,1} \to \cdots \to \bar{K}_{\alpha+1,\beta_{\alpha+1}-1},$$

where $\bar{K}_{\mu,\nu} = \bar{a}_{\mu,\nu} \sqcap \bar{A}_{\mu,\nu+1}$, $\bar{A}_{\mu,\beta_\mu+1} = \bar{A}_{\mu+1,1}$ and, moreover, $\bar{A}_{\alpha+1,1} = \bar{A}_{\alpha,\beta_\alpha}$ in case of $\beta_{\alpha+1} = 1$ and $\bar{A}_{\alpha+1,\beta_{\alpha+1}} = \bar{A}_{\alpha+1,\beta_{\alpha+1}-1}$ in case of $\beta_{\alpha+1} > 1$.

We observe that the local chain $[\mathring{K}]$ is obtained by replacing each member $\bar{K}_\gamma = \bar{a}_\gamma \sqcap \bar{A}_{\gamma+1}$ of the local chain $[\bar{K}]$ of $(\bar{A})$, with the base $\bar{a}_\alpha \in \bar{A}_\alpha$, by a chain from the set $\bar{a}_\gamma$ to $\bar{a}_{\gamma+1}$:

$$\bar{K}_{\gamma,\beta_\gamma} \to \bar{K}_{\gamma+1,1} \to \cdots \to \bar{K}_{\gamma+1,\beta_{\gamma+1}-1}.$$

(if $\beta_{\gamma+1} = 1$, then we read only the initial member $\bar{K}_{\gamma,\beta_\gamma}$) and, moreover, if $\beta_1 > 1$, we add, at the beginning of $[\bar{K}]$, a chain from the set $\bar{a}_{1,1}$ to $\bar{a}_1$: $\bar{K}_{1,1} \to \cdots \to \bar{K}_{1,\beta_1-1}$. The above chains are, evidently, elementary chains from $\bar{a}_\gamma$ to $\bar{a}_{\gamma+1}$ or from $\bar{a}_{1,1}$ to $\bar{a}_1$ over the decompositions $\bar{a}_\gamma \sqcap \bar{A}_{\gamma+1}$ or $\bar{a}_{1,1} \sqcap \bar{A}_1$, respectively. Thus *the local chain of every refinement of $(\bar{A})$, with the base $\bar{a} \subset \bar{a}_\alpha$ is a refinement of the local chain of $(\bar{A})$, with the base $\bar{a}_\alpha$.*

10.4. Manifolds of local chains

Let us consider a series of decompositions on the set G:

$$\big((\bar{A}) =\big)\, \bar{A}_1 \geqq \cdots \geqq \bar{A}_\alpha \ (\alpha \geqq 1).$$

To every element $\bar{a} \in \bar{A}_\alpha$ there corresponds a local chain of $(\bar{A})$, with the base $\bar{a}$:

$$([\bar{K}\bar{a}] =)\, \bar{K}_1\bar{a} \to \cdots \to \bar{K}_\alpha\bar{a}.$$

The set consisting of local chains whose bases are the individual elements of $\bar{A}_\alpha$ is called the *manifold of local chains, corresponding to* $(\bar{A})$; notation: $\tilde{A}$. It is obviously an α-grade structure with regard to the sequence of decompositions $\bar{A}_2, \ldots, \bar{A}_{\alpha+1}$ $(\bar{A}_{\alpha+1} = \bar{A}_\alpha)$ in the sense of the definition introduced in 1.9.

Associating, with every point $a \in G$, the local chain $[\bar{K}\bar{a}] \in \tilde{A}$ with the base $\bar{a} = \bar{a}_\alpha \in \bar{A}_\alpha$ for which $a \in \bar{a}$, we obtain a mapping called the *natural mapping of the set G onto the manifold of the local chains* $\tilde{A}$. The decomposition of G corresponding to this mapping, naturally, coincides with $\bar{A}_\alpha$. By a *local chain of* $(\bar{A})$, *corresponding to a*, we mean the local chain $[\bar{K}\bar{a}]$.

Now let

$$\big((\bar{A}) =\big)\, \bar{A}_1 \geqq \cdots \geqq \bar{A}_\alpha,$$
$$\big((\bar{B}) =\big)\, \bar{B}_1 \geqq \cdots \geqq \bar{B}_\beta \; (\alpha, \beta \geqq 1)$$

be series of decompositions on G such that their end-members $\bar{A}_\alpha$, $\bar{B}_\beta$ coincide: $\bar{A}_\alpha = \bar{B}_\beta$.

Consider the manifolds of local chains, $\tilde{A}$ and $\tilde{B}$, corresponding to the series $(\bar{A})$ and $(\bar{B})$, respectively.

Associating, with every element $[\bar{K}\bar{a}] \in \tilde{A}$, the local chain $[\bar{L}\bar{a}] \in \tilde{B}$ with the same base $a \in \bar{A}_\alpha = \bar{B}_\beta$, we obtain a simple mapping of $\tilde{A}$ onto $\tilde{B}$, called *co-basal.*

We see that the manifolds of local chains, corresponding to two series of decompositions with coinciding end-members, are equivalent sets and that the co-basal mapping is a one-to-one mapping of one onto the other.

10.5. Chain-equivalent series of decompositions

Suppose

$$\big((\bar{A}) =\big)\, \bar{A}_1 \geqq \cdots \geqq \bar{A}_\alpha,$$
$$\big((\bar{B}) =\big)\, \bar{B}_1 \geqq \cdots \geqq \bar{B}_\alpha$$

are arbitrary chains of decompositions on G of the same length α $(\geqq 1)$.

Let again $\tilde{A}$, $\tilde{B}$ denote the manifolds of local chains corresponding to $(\bar{A}), (\bar{B})$.

$(\bar{B})$ is said to be *chain-equivalent to* $(\bar{A})$ if the manifold of the local chains, $\tilde{B}$, is strongly equivalent to the manifold $\tilde{A}$.

If $(\bar{B})$ is chain-equivalent to $(\bar{A})$, then $(\bar{A})$ is chain-equivalent to $(\bar{B})$, (6.9.1). With respect to this symmetry, we speak about chain-equivalent series $(\bar{A})$, $(\bar{B})$.

By the above definition, $(\bar{B})$ is chain-equivalent to $(\bar{A})$ if there exists a strong equivalence-mapping of the manifold of the local chains, $\tilde{A}$, onto the manifold $\tilde{B}$ (6.9.1). If, in particular, the end-members $\bar{A}_\alpha, \bar{B}_\alpha$ of $(\bar{A}), (\bar{B})$, respectively, coincide and, simultaneously, the co-basal mapping of $\tilde{A}$ onto $\tilde{B}$ is a strong equivalence, then $(\bar{B})$ is said to be *co-basally chain-equivalent to* $(\bar{A})$ and we speak about *co-basally chain-equivalent series* $(\bar{A})$, $(\bar{B})$.

Let us now assume that $(\bar{A})$, $(\bar{B})$ are chain-equivalent.

Let $\boldsymbol{f}$ be a strong equivalence-mapping of the manifold $\tilde{A}$ onto $\tilde{B}$. By 6.9.1, $\boldsymbol{f}$ is a one-to-one mapping of $\tilde{A}$ onto $\tilde{B}$, where every two associated elements of $\tilde{A}$, $\tilde{B}$ are in certain mutual relations. This situation can more accurately be described as follows:

There exists a permutation $\boldsymbol{p}$ of the set $\{1, \ldots, \alpha\}$ with the following effect:

Let $[\bar{K}] \in \tilde{A}$, $\boldsymbol{f}[\bar{K}] = [\bar{L}] \in \tilde{B}$ be two arbitrary local chains of the series $(\bar{A})$, $(\bar{B})$, respectively:

$$([\bar{K}] =)\, \bar{K}_1 \to \cdots \to \bar{K}_\alpha,$$
$$([\bar{L}] =)\, \bar{L}_1 \to \cdots \to \bar{L}_\alpha,$$

where $[\bar{L}]$ is the image of $[\bar{K}]$ under the mapping $\boldsymbol{f}$. We know that every member $\bar{K}_\gamma$ $(\bar{L}_\gamma)$ $(\gamma = 1, \ldots, \alpha)$ is a decomposition in G which is a part of $\bar{A}_{\gamma+1}$ $(\bar{B}_{\gamma+1})$ while $\bar{A}_{\alpha+1} = \bar{A}_\alpha$, $\bar{B}_{\alpha+1} = \bar{B}_\alpha$. The effect of $\boldsymbol{p}$ consists in that to every member $\bar{K}_\gamma$ of $[K]$ there exists a one-to-one function $\boldsymbol{a}_\gamma$ mapping the member $\bar{K}_\gamma$ onto the member $\bar{L}_\delta$ of $[\bar{L}]$ while $\delta = \boldsymbol{p}\gamma$.

We observe that any two members $\bar{K}_\gamma$, $\bar{L}_\delta$ of the local chains $[\bar{K}]$, $[\bar{L}]$ with the indices $\gamma, \delta = \boldsymbol{p}\gamma$ are equivalent sets. Consequently, such members $\bar{K}_\gamma$, $\bar{L}_\gamma$ are, in the local chains $[\bar{K}]$, $[\bar{L}]$, simultaneously either essential or inessential. Hence *any two local chains corresponding to each other under f are of the same reduced length.*

Our object now is to show that even $(\bar{A})$, $(\bar{B})$ *are of the same reduced length.*

That is, first of all, evident if $\alpha = 1$, as the initial members $\bar{A}_1$, $\bar{B}_1$ of $(\bar{A})$, $(\bar{B})$ are aways essential.

Let $\alpha > 1$. Consider an arbitrary essential member $\bar{A}_{\gamma+1}$ $(1 \leqq \gamma < \alpha)$ of $(\bar{A})$. Then there exists an element $\bar{a}_\gamma \in \bar{A}_\gamma$ such that $\bar{a}_\gamma \sqcap \bar{A}_{\gamma+1}$ comprises more than one element. Let $\bar{a} = \bar{a}_\alpha \in \bar{A}_\alpha$ be an arbitrary element of $\bar{A}_\alpha$ such that: $\bar{a} \sqsubset \bar{a}_\gamma$. Furthermore, let $[\bar{K}]$ be the local chain $(\bar{A})$ with the base $\bar{a}$ and $[\bar{L}] = \boldsymbol{f}[\bar{K}]$ denote the local chain of $(\bar{B})$ associated with $[\bar{K}]$ under the function $\boldsymbol{f}$. The members of $[\bar{K}]$, $[\bar{L}]$ are denoted as above. Then we have, in particular, $\bar{K}_\gamma = \bar{a}_\gamma \sqcap \bar{A}_{\gamma+1}$, $\bar{L}_\delta = \bar{b}_\delta \sqcap \bar{B}_{\delta+1}$ where $\delta = \boldsymbol{p}\gamma$ and $\bar{b}_\delta \in \bar{B}_\delta$. According to the above considerations, $\bar{L}_\delta$ is a set equivalent to $\bar{K}_\gamma$ and therefore contains more than one element. Consequently, the member $\bar{B}_{\delta+1}$ of $(\bar{B})$ is essential; in particular, we have $1 \leqq \delta < \alpha$. It is obvious that $(\bar{B})$ contains at least as many essential members as $(\bar{A})$ so that, for the reduced lengths α', β' of $(\bar{A})$, $(\bar{B})$, there holds $\alpha' \leqq \beta'$. For analogous reasons there also holds $\beta' \leqq \alpha'$ and the proof is accomplished.

10.6. Semi-joint (loosely joint) and joint series of decompositions

Let us again consider two series of decompositions $(\bar{A})$, $(\bar{B})$ on the set G, of length α $(\geqq 1)$, and use the above notation. The symbols $\tilde{A}$, $\tilde{B}$ then denote the manifolds of the local chains, corresponding to the series (A), $(\bar{B})$.

$(\bar{B})$ *is* said to be *semi-joint* or *loosely joint* (*joint*) *with* $(\bar{A})$ if the manifold of the local chains, $\tilde{B}$, is equivalent to and loosely coupled with (equivalent to and coupled with) the manifold $\tilde{A}$.

If $(\bar{B})$ is semi-joint (joint) with $(\bar{A})$, then $(\bar{A})$ is also semi-joint (joint) with $(\bar{B})$ (6.9.2). Taking account of this symmetry, we speak about the semi-joint or loosely joint (joint) series $(\bar{A})$, $(\bar{B})$.

By the above definition, $(\bar{B})$ is semi-joint (joint) with $(\bar{A})$ if there exists an equivalence connected with loose coupling (equivalence connected with coupling) of the manifold of the local chains, $\tilde{A}$, onto the manifold $\tilde{B}$ (6.9.2). If, in particular, the final members $\bar{A}_\alpha$, $\bar{B}_\beta$ of $(\bar{A})$, (B) coincide and the co-basal mapping of the manifold of the local chains, $\tilde{A}$, onto the manifold $\tilde{B}$ is an equivalence connected with loose coupling (an equivalence connected with coupling) then $(\bar{B})$ *is* said to be *co-basally semi-joint* or *co-basally loosely joint* (*co-basally joint*) *with* $(\bar{A})$; in that case we also speak of *co-basally semi-joint* or *co-basally loosely joint* (*co-basally joint*) series $(\bar{A})$, $(\bar{B})$.

Let us now assume $(\bar{A})$, $(\bar{B})$ to be loosely joint (joint).

Let $\boldsymbol{f}$ stand for an equivalence-mapping connected with loose coupling (equivalence-mapping connected with coupling) of the manifold of the local chains, $\tilde{A}$, onto the manifold $\tilde{B}$. The mapping $\boldsymbol{f}$ is therefore simple (one-to-one) (6.9) and the situation may be described as follows (6.9.2):

There exists a permutation $\boldsymbol{p}$ of the set $\{1, \ldots, \alpha\}$ with the following effect:

Let $[\bar{K}] \in \tilde{A}$, $\boldsymbol{f}[\bar{K}] = [\bar{L}] \in \tilde{B}$ be arbitrary local chains of $(\bar{A})$, $(\bar{B})$ associated with each other under the function $\boldsymbol{f}$. Then every two members $\bar{K}_\gamma$, $\bar{L}_\delta$ of $[\bar{K}]$, $[\bar{L}]$ are loosely coupled (coupled) decompositions in G; at the same time, $\delta = \boldsymbol{p}\gamma$. More accurately: each member of either of the mentioned decompositions is incident with at most one (exactly one) element of the other while there always occurs at least one incidence. The closures $\mathrm{H}\bar{K}_\gamma = \bar{L}_\delta \sqsubset \bar{K}_\gamma$, $\mathrm{H}\bar{L}_\delta = \bar{K}_\gamma \sqsubset \bar{L}_\delta$ $(= \emptyset)$ are coupled.

If $(\bar{A})$, $(\bar{B})$ are joint, then the mapping $\boldsymbol{a}_\gamma$ of $\bar{K}_\gamma$ onto $\bar{L}_\delta$, given by the incidence of the elements $\bar{K}_\gamma$, $\bar{L}_\delta$, is simple.

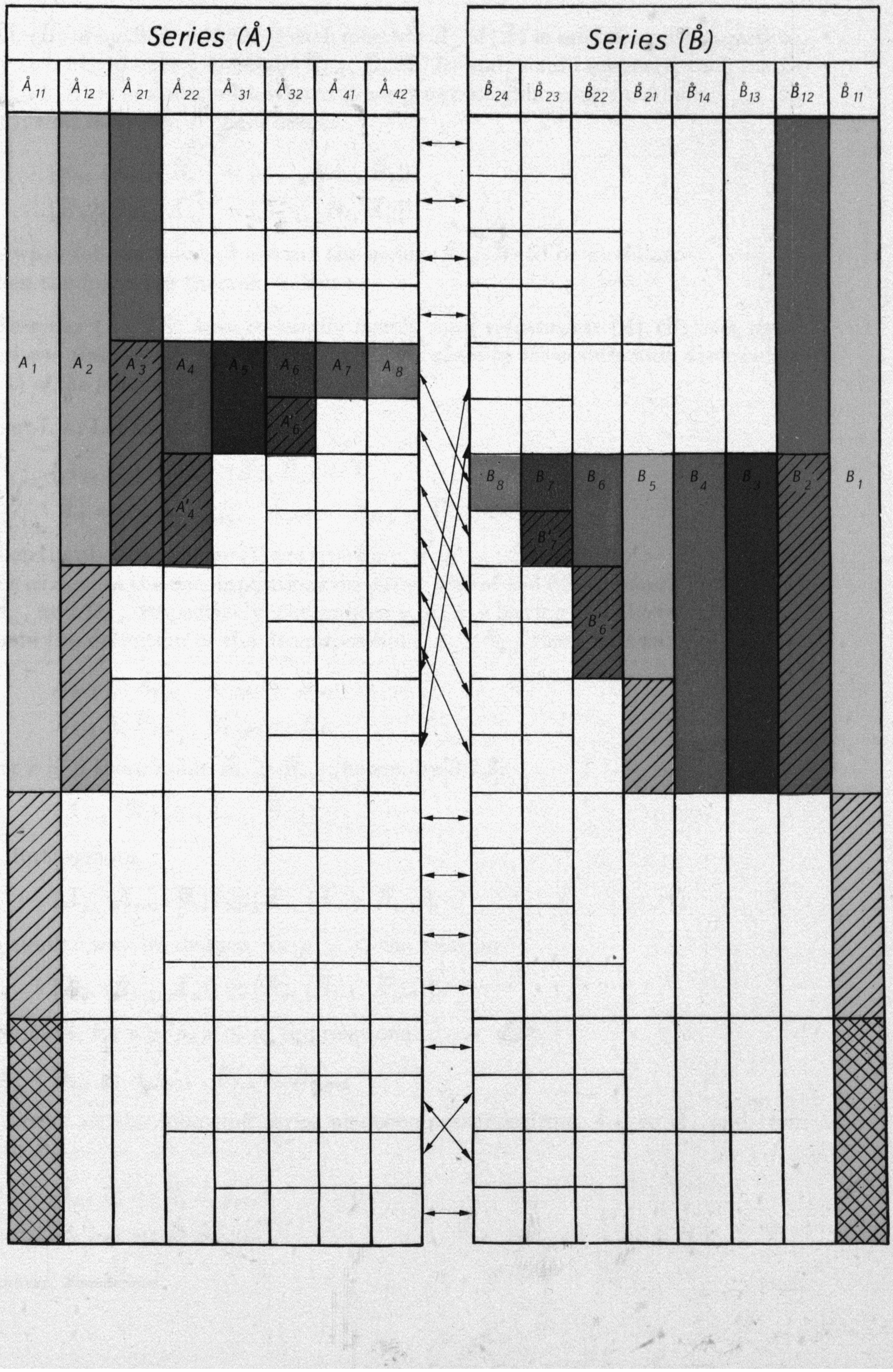
We see that *two joint series of decompositions are chain-equivalent*. In particular, hey are of the same reduced length.

10.7. Modular series of decompositions

Suppose that

$$\big((\bar{A}) =\big)\ \bar{A}_1 \geqq \cdots \geqq \bar{A}_\alpha,$$
$$\big((\bar{B}) =\big)\ \bar{B}_1 \geqq \cdots \geqq \bar{B}_\beta$$

are series of decompositions on G, of lengths α, $\beta \geqq 1$, respectively.


Series (Å)
Series (B̊)
$\mathring{A}_{11}$ $\mathring{A}_{12}$ $\mathring{A}_{21}$ $\mathring{A}_{22}$ $\mathring{A}_{31}$ $\mathring{A}_{32}$ $\mathring{A}_{41}$ $\mathring{A}_{42}$
$\mathring{B}_{24}$ $\mathring{B}_{23}$ $\mathring{B}_{22}$ $\mathring{B}_{21}$ $\mathring{B}_{14}$ $\mathring{B}_{13}$ $\mathring{B}_{12}$ $\mathring{B}_{11}$
A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8
A'_6
A'_4
B_8 B_7 B_6 B_5 B_4 B_3 B_2 B_1
B'_7
B'_6

$(\bar{A})$, $(\bar{B})$ are called *modular* if each member $\bar{A}_\mu$ of $(\bar{A})$ is modular with regard to every two neighbouring members $\bar{B}_{\delta-1}$, $\bar{B}_\delta$ of (B) and, simultaneously, each member $\bar{B}_\nu$ of $(\bar{B})$ is modular with regard to every two neighbouring members $\bar{A}_{\gamma-1}$, $\bar{A}_\gamma$ of $(\bar{A})$; that is to say, if there holds:

$$[\bar{A}_\gamma, (\bar{A}_{\gamma-1}, \bar{B}_\nu)] = (\bar{A}_{\gamma-1}, [\bar{A}_\gamma, \bar{B}_\nu]),$$
$$[\bar{B}_\delta, (\bar{B}_{\delta-1}, \bar{A}_\mu)] = (\bar{B}_{\delta-1}, [\bar{B}_\delta, \bar{A}_\mu]). \tag{1}$$

In what follows we shall assume the series $(\bar{A})$, $(\bar{B})$ to be modular.

Then the following theorem is true:

The series $(\bar{A})$, $(\bar{B})$ have co-basally loosely joint refinements $(\mathring{A})$, $(\mathring{B})$ with equal initial and final members. The refinements are given by the construction described in part a) *of the following proof.*

Proof. a) Let us denote:

$$[\bar{A}_1, \bar{B}_1] = \bar{U}, \quad (\bar{A}_\alpha, \bar{B}_\beta) = \bar{V},$$
$$\bar{A}_0 = \bar{B}_0 = \bar{G}_{\max}, \quad \bar{A}_{\alpha+1} = \bar{B}_{\beta+1} = \bar{V}.$$

Then the above formulae (1) are true for $\gamma, \mu = 1, \ldots, \alpha + 1; \delta, \nu = 1, \ldots, \beta + 1$.

Let us denote the decompositions on either side of the first (second) formula (1) by $\mathring{A}_{\gamma,\nu}$ and $\mathring{B}_{\delta,\mu}$, respectively, the indices γ, μ; δ, ν having the above values.

From the definition of the decompositions $\mathring{A}_{\gamma,\nu}$, $\mathring{B}_{\delta,\mu}$ there follows:

$$\bar{A}_{\gamma-1} \geqq \mathring{A}_{\gamma,\nu}, \quad \mathring{A}_{\gamma,\beta+1} = \bar{A}_\gamma,$$
$$\bar{B}_{\delta-1} \geqq \mathring{B}_{\delta,\mu}, \quad \mathring{B}_{\delta,\alpha+1} = \bar{B}_\delta.$$

For $\nu \leqq \beta$ there holds $\bar{B}_\nu \geqq \bar{B}_{\nu+1}$; hence, by 3.7.2,

$$(\bar{A}_{\gamma-1}, \bar{B}_\nu) \geqq (\bar{A}_{\gamma-1}, \bar{B}_{\nu+1})$$

and, furthermore,

$$[\bar{A}_\gamma, (\bar{A}_{\gamma-1}, \bar{B}_\nu)] \geqq [\bar{A}_\gamma, (\bar{A}_{\gamma-1}, \bar{B}_{\nu+1})].$$

In a similar way we deduce, for $\mu \leqq \alpha$, the relation

$$[\bar{B}_\delta, (\bar{B}_{\delta-1}, \bar{A}_\mu)] \geqq [\bar{B}_\delta, (\bar{B}_{\delta-1}, \bar{A}_{\mu+1})].$$

So we have, for $\nu \leqq \beta$, $\mu \leqq \alpha$, the relations:

$$\mathring{A}_{\gamma,\nu} \geqq \mathring{A}_{\gamma,\nu+1}, \quad \mathring{B}_{\delta,\mu} \geqq \mathring{B}_{\delta,\mu+1}$$

and arrive at the following series of decompositions from $\mathring{A}_{\gamma,1}$ to $\bar{A}_\gamma$ and from $\mathring{B}_{\delta,1}$ to $\bar{B}_\delta$:

$$\mathring{A}_{\gamma,1} \geqq \cdots \geqq \mathring{A}_{\gamma,\beta+1},$$
$$\mathring{B}_{\delta,1} \geqq \cdots \geqq \mathring{B}_{\delta,\alpha+1}.$$

We observe that the following series of the decompositions $(\mathring{A})$, $(\mathring{B})$ on G are refinements of $(\bar{A})$, $(\bar{B})$:

$$\begin{aligned}\big((\mathring{A}) =\big)\ \bar{U} = \mathring{A}_{1,1} \geqq \cdots \geqq \mathring{A}_{1,\beta+1} \geqq \mathring{A}_{2,1} \geqq \cdots \geqq \mathring{A}_{2,\beta+1} \\ \geqq \cdots \geqq \mathring{A}_{\alpha+1,1} \geqq \cdots \geqq \mathring{A}_{\alpha+1,\beta+1} = \bar{V},\\ \big((\mathring{B}) =\big)\ \bar{U} = \mathring{B}_{1,1} \geqq \cdots \geqq \mathring{B}_{1,\alpha+1} \geqq \mathring{B}_{2,1} \geqq \cdots \geqq \mathring{B}_{2,\alpha+1} \\ \geqq \cdots \geqq \mathring{B}_{\beta+1,1} \geqq \cdots \geqq \mathring{B}_{\beta+1,\alpha+1} = \bar{V}.\end{aligned}$$

The series $(\mathring{A})$, $(\mathring{B})$ obviously have the same length $(\alpha + 1)(\beta + 1)$ and their initial and final members coincide: $(\bar{U} =)\ \mathring{A}_{1,1} = \mathring{B}_{1,1}$, $\mathring{A}_{\alpha+1,\beta+1} = \mathring{B}_{\beta+1,\alpha+1} (= \bar{V})$. *The series $(\mathring{A})$, $(\mathring{B})$ are the mentioned co-basally loosely joint refinements of the series $(\bar{A})$, $(\bar{B})$, respectively.*

b) Now let us show that $(\mathring{A})$, $(\mathring{B})$ are co-basally loosely joint.

We shall, first, define the permutation $\boldsymbol{p}$ of the set

$$\{1, \ldots, (\alpha + 1)(\beta + 1)\}$$

as follows:

$$\begin{aligned}&\boldsymbol{p}[(\mu - 1)(\beta + 1) + \nu - 1] = (\nu - 1)(\alpha + 1) + \mu - 1\\ &(\mu = 1, \ldots, \alpha + 1;\quad \nu = 1, \ldots, \beta + 1;\quad \mu + \nu > 2),\\ &\boldsymbol{p}(\alpha + 1)(\beta + 1) = (\beta + 1)(\alpha + 1).\end{aligned}$$

Let $\bar{a} \in \bar{V}$ be an arbitrary element and

$$\begin{aligned}&([\mathring{K}\bar{a}] =)\quad \mathring{K}_1 \to \cdots \to \mathring{K}_{(\alpha+1)(\beta+1)},\\ &([\mathring{L}\bar{a}] =)\quad \mathring{L}_1 \to \cdots \to \mathring{L}_{(\beta+1)(\alpha+1)}\end{aligned}$$

the local chains of $(\mathring{A})$, $(\mathring{B})$ corresponding to the base $\bar{a}$.

Let, moreover, $\bar{a}_{\mu-1}, \bar{b}_{\nu-1}; \mathring{a}_{\mu,\nu}, \mathring{b}_{\nu,\mu}$ be elements given by the relations:

$$\begin{aligned}&\bar{a} \subset \bar{a}_{\mu-1} \in \bar{A}_{\mu-1},\quad \bar{a} \subset \bar{b}_{\nu-1} \in \bar{B}_{\nu-1},\\ &\bar{a} \subset \mathring{a}_{\mu,\nu} \in \mathring{A}_{\mu,\nu},\quad \bar{a} \subset \mathring{b}_{\nu,\mu} \in \mathring{B}_{\nu,\mu}\\ &(\mu = 1, \ldots, \alpha + 1;\quad \nu = 1, \ldots, \beta + 1;\quad \bar{a}_0 = \bar{b}_0 = G).\end{aligned}$$

Then we have:

$$\begin{aligned}&\mathring{K}_{(\mu-1)(\beta+1)+\nu-1} = \mathring{a}_{\mu,\nu-1} \sqcap \mathring{A}_{\mu,\nu},\\ &\mathring{L}_{(\nu-1)(\alpha+1)+\mu-1} = \mathring{b}_{\nu,\mu-1} \sqcap \mathring{B}_{\nu,\mu}\\ &(\mu + \nu > 2;\quad \mathring{a}_{\mu,0} = \bar{a}_{\mu-1},\quad \mathring{b}_{\nu,0} = \bar{b}_{\nu-1}),\\ &\mathring{K}_{(\alpha+1)(\beta+1)} = \bar{a} \sqcap \bar{V} = \mathring{L}_{(\beta+1)(\alpha+1)}.\end{aligned}\tag{1}$$

We shall show that the decompositions $\mathring{K}_{(\mu-1)(\beta+1)+\nu-1}$ and $\mathring{L}_{(\nu-1)(\alpha+1)+\mu-1}$ ($\mu + \nu > 2$) as well as $\mathring{K}_{(\alpha+1)(\beta+1)}$ and $\mathring{L}_{(\beta+1)(\alpha+1)}$ associated under the permutation $\boldsymbol{p}$ are loosely coupled.

From $\bar{a}_{\mu,\nu-1} \in (\bar{A}_{\mu-1}, [\bar{A}_\mu, \bar{B}_{\nu-1}])$ we have $\mathring{a}_{\mu,\nu-1} = \bar{a}_{\mu-1} \cap \bar{v}$ where $\bar{v} \in [\bar{A}_\mu, \bar{B}_{\nu-1}]$) is the sum of all the elements of the decomposition $\bar{B}_{\nu-1}$ that can be connected with the element $\bar{b}_{\nu-1}$ in $\bar{A}_\mu$. In particular, there holds $\bar{b}_{\nu-1} \subset \bar{v}$ and, therefore, even $\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1} \subset \mathring{a}_{\mu,\nu-1}$.

Analogously, $\mathring{b}_{\nu,\mu-1} = \bar{b}_{\nu-1} \cap \bar{u}$ where $\bar{u} \in [\bar{B}_\nu, \bar{A}_{\mu-1}]$ is the sum of all the elements that can be connected with the element $\bar{a}_{\mu-1}$ in $\bar{B}_\nu$. In particular, we have $\bar{a}_{\mu-1} \subset \subset \bar{u}$ and, therefore, even $\bar{b}_{\nu-1} \cap \bar{a}_{\mu-1} \subset \mathring{b}_{\nu,\mu-1}$.

Consequently:

$$(\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1}) \subset (\mathring{a}_{\mu,\nu-1} \cap \mathring{b}_{\nu,\mu-1}) = (\bar{a}_{\mu-1} \cap \bar{v}) \cap (\bar{b}_{\nu-1} \cap \bar{u}) \subset (\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1})$$

so that we have

$$\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1} = \mathring{a}_{\mu,\nu-1} \cap \mathring{b}_{\nu,\mu-1}.$$

By (1), $\mathring{K}_{(\mu-1)(\beta+1)+\nu-1}$ is a decomposition on $\mathring{a}_{\mu,\nu-1}$ and $\mathring{L}_{(\nu-1)(\alpha+1)+\mu-1}$ a decomposition on $\mathring{b}_{\nu,\mu-1}$. To simplify the notation, let us put

$$\mathring{K}_{\mu,\nu} = \mathring{K}_{(\mu-1)(\beta+1)+\nu-1}, \quad \mathring{L}_{\nu,\mu} = \mathring{L}_{(\nu-1)(\alpha+1)+\mu-1}.$$

Then the above equality may be written in the form:

$$\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1} = \boldsymbol{s}\mathring{K}_{\mu,\nu} \cap \boldsymbol{s}\mathring{L}_{\nu,\mu}.$$

Any element $\mathring{x} \in \mathring{K}_{\mu,\nu}$ is incident with an element of $\mathring{L}_{\nu,\mu}$ if and only if there holds $\mathring{x} \in (\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1}) \sqsubset \mathring{K}_{\mu,\nu}$. In fact, if $\mathring{x}$ is incident with some element of $\mathring{L}_{\nu,\mu}$, then it is incident with the set $\boldsymbol{s}\mathring{K}_{\mu,\nu} \cap \boldsymbol{s}L_{\nu,\mu}$ and therefore also with $\bar{a}_{\mu-1} \cap b_{\nu-1}$ so that we have: $\mathring{x} \in (\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1}) \sqsubset \mathring{K}_{\mu,\nu}$; if, conversely, the latter relation applies, then $\mathring{x}$ is incident with the set $\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1}$, hence even with $\boldsymbol{s}\mathring{K}_{\mu,\nu} \cap \boldsymbol{s}\mathring{L}_{\nu,\mu}$ and, consequently, with at least one element of $\mathring{L}_{\nu,\mu}$.

In a similar way we can verify that any element $\mathring{y} \in \mathring{L}_{\nu,\mu}$ is incident with some element of $\mathring{K}_{\mu,\nu}$ if and only if there holds $\mathring{y} \in (\bar{b}_{\nu-1} \cap \bar{a}_{\mu-1}) \sqsubset \mathring{L}_{\nu,\mu}$.

It is easy to show that $\mathring{K}_{\mu,\nu}$ and $\mathring{L}_{\nu,\mu}$ are loosely coupled.

Let us, first, note that the intersection $\mathring{K}_{\mu,\nu} \sqcap \mathring{L}_{\nu,\mu}$ is not empty, for $\bar{a} \subset \subset \mathring{a}_{\mu,\nu} \cap \mathring{b}_{\nu,\mu}$. Moreover, we shall find that each element of $\mathring{K}_{\mu,\nu}$ is incident with, at most, one element of $\mathring{L}_{\nu,\mu}$. Indeed, if an element $\mathring{x} \in \mathring{K}_{\mu,\nu}$ does not lie in the closure $(\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1}) \sqsubset \mathring{K}_{\mu,\nu}$, then it is not incident with any element of $\mathring{L}_{\nu,\mu}$. In the opposide case, $\mathring{x} \in \mathring{K}_{\mu,\nu}$ is incident with at least one element of $\mathring{L}_{\nu,\mu}$, and all the elements of $\mathring{L}_{\mu,\nu}$ incident with $\mathring{x}$ belong to the closure $(\bar{b}_{\nu-1} \cap \bar{a}_{\mu-1}) \sqsubset \mathring{L}_{\nu,\mu}$; by 4.3, the closures $(\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1}) \sqsubset \mathring{K}_{\mu,\nu}$ and $(\bar{b}_{\nu-1} \cap \bar{a}_{\mu-1}) \sqsubset \mathring{L}_{\nu,\mu}$ are coupled and so, in $L_{\nu,\mu}$, there is exactly one element incident with $\mathring{x}$. Thus we have shown that each element of $\mathring{K}_{\mu,\nu}$ is incident with, at most, one element of $\mathring{L}_{\nu,\mu}$. In a similar way we can verify that each element of $\mathring{L}_{\nu,\mu}$, is incident with, at most, one element of $\mathring{K}_{\mu,\nu}$. It follows that the decompositions $\mathring{K}_{\mu,\nu}$, $\mathring{L}_{\nu,\mu}$ are loosely coupled.

To accomplish the proof it remains to verify that even $\mathring{K}_{(\alpha+1)(\beta+1)}$ and $\mathring{L}_{(\beta+1)(\alpha+1)}$ are loosely coupled. But that is obvious, since these decompositions consist of the single element $\bar{a}$.

10.8. Complementary series of decompositions

Let again $(\bar{A})$, $(\bar{B})$ stand for arbitrary series of decompositions on G, of lengths $\alpha, \beta \geqq 1$; notation as above.

$(\bar{A})$, $(\bar{B})$ *are* called *complementary* if any member of $(\bar{A})$ is complementary to any member of $(\bar{B})$.

Let us assume that $(\bar{A})$, $(\bar{B})$ are complementary. Then, on taking account of 5.5 and 5.4, the following theorems apply:

Every two local chains with the same ends, corresponding to the series $(\bar{A})$, $(\bar{B})$, respectively, are adjoint.

The series $(\bar{A})$, $(\bar{B})$ are modular.

Furthermore, we shall prove that

$(\bar{A})$, $(\bar{B})$ have co-basally joint refinements $(\mathring{A})$, $(\mathring{B})$ with the same initial and final members. $(\mathring{A})$, $(\mathring{B})$ are given by the above construction of co-basally loosely joint refinements of modular series (part a) of the above proof).

Proof. Since $(\bar{A})$, $(\bar{B})$ are not only modular but even complementary, we have to modify the part b) of the above proof so as to show that the decompositions

$$(\mathring{K}_{\mu,\nu} =)\ \mathring{K}_{(\mu-1)(\beta+1)+\nu-1} = \mathring{a}_{\mu,\nu-1} \sqcap \mathring{A}_{\mu,\nu},$$
$$(\mathring{L}_{\mu,\nu} =)\ \mathring{L}_{(\nu-1)(\alpha+1)+\mu-1} = \mathring{b}_{\nu,\mu-1} \sqcap \mathring{B}_{\nu,\mu},$$
$$(\mathring{a}_{\mu,0} = \bar{a}_{\mu-1},\quad \mathring{b}_{\nu,0} = \bar{b}_{\nu-1};\quad \mu + \nu > 2)$$

are coupled.

As we know from 5.3, the decompositions $\bar{A}_{\mu}$, $(\bar{A}_{\mu-1}, \bar{B}_{\nu-1})$ are complementary; hence, on taking account of the first theorem in 5.3, we observe that the element $\mathring{a}_{\mu,\nu-1} \in [\bar{A}_{\mu}, (\bar{A}_{\mu-1}, \bar{B}_{\nu-1})]$ is the sum of all the elements of the decomposition $\bar{A}_{\mu}$ that are incident with the element $\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1} \in (\bar{A}_{\mu-1}, \bar{B}_{\nu-1})$. Even an arbitrary element $\mathring{x} \in \mathring{A}_{\mu,\nu}$ is the sum of certain elements of $\bar{A}_{\mu}$; we observe that $\mathring{x} \in \mathring{A}_{\mu,\nu}$ is incident with $\mathring{a}_{\mu,\nu-1}$ if and only if it is incident with the set $\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1}$. It follows:

$$\mathring{K}_{\mu,\nu} = (\bar{a}_{\mu-1} \cap \bar{b}_{\nu-1}) \sqsubset \mathring{A}_{\mu,\nu}.$$

In a similar way we obtain:

$$\mathring{L}_{\nu,\mu} = (\bar{b}_{\nu-1} \cap \bar{a}_{\mu-1}) \sqsubset \mathring{B}_{\nu,\mu}.$$

As the decompositions on both sides of the above equalities are coupled (5.5), the proof is complete.

10.9. Example of co-basally joint series of decompositions

In the figure behind p. 80 we find an example of co-basally joint series of decompositions $(\mathring{A})$, $(\mathring{B})$ on the set G consisting of 20 elements (cf. p. 205, N°39). The elements of G, or the one-point sets formed by these elements, are in the inner columns, denoted by A_8, B_8, ...; the arrows show which of the elements are the same. The individual members of the co-basally joint series

$$\big((\mathring{A}) =\big)\ \mathring{A}_{11} \geqq \mathring{A}_{12} \geqq \mathring{A}_{21} \geqq \mathring{A}_{22} \geqq \mathring{A}_{31} \geqq \mathring{A}_{32} \geqq \mathring{A}_{41} \geqq \mathring{A}_{42},$$
$$\big((\mathring{B}) =\big)\ \mathring{B}_{11} \geqq \mathring{B}_{12} \geqq \mathring{B}_{13} \geqq \mathring{B}_{14} \geqq \mathring{B}_{21} \geqq \mathring{B}_{22} \geqq \mathring{B}_{23} \geqq \mathring{B}_{24}$$

are in the appropriate columns.

The starting-point for the construction of the series $(\mathring{A})$, $(\mathring{B})$ are the complementary series of the decompositions of G:

$$\big((\bar{A}) =\big)\ \bar{A}_1 \geqq \bar{A}_2 \geqq \bar{A}_3,$$
$$\big((\bar{B}) =\big)\ \bar{B}_1 \geqq \bar{B}_2\ (=\bar{A}_3),$$

the individual members of which are, in $(\mathring{A})$, $(\mathring{B})$, denoted by $\mathring{A}_{12}$, $\mathring{A}_{22}$, $\mathring{A}_{32}$ and $\mathring{B}_{14}$, $\mathring{B}_{24}$, respectively. From the figure it is clear that each member of $(\bar{B})$ is complementary to each member of $(\bar{A})$.

The coupled members contained in two local chains of $(\mathring{A})$, $(\mathring{B})$, with the same base, are found in the columns marked by $\mathring{A}_{\gamma\delta}$, $\mathring{B}_{\delta\gamma}$.

The local chains of $(\mathring{A})$, $(\mathring{B})$, with the base $A_8 = B_8$, are marked in colours. We observe that the members of these local chains, introduced in the columns $\mathring{A}_{\gamma\delta}$, $\mathring{B}_{\delta\gamma}$, are coupled decompositions. Incident elements of two coupled decompositions are marked in the same colour. For example, to the decomposition consisting of the elements A_4, A_4' there corresponds the decomposition formed by the elements B_6, B_6'; A_4 (B_6) is incident with the single element B_6 (A_4) and A_4' (B_6') with the single element B_6' (A_4').

10.10. Connection with the theory of mappings of sets onto sets of finite sequences

The above theory of the series of decompositions of sets is closely connected with a study of mappings of sets onto sets formed by finite sequences of the same length.

Consider a nonempty set $\mathcal{A}$ consisting of finite α-membered sequences ($\alpha \geqq 1$) and a mapping $\boldsymbol{a}$ of the set G onto $\mathcal{A}$.

To the set $\mathcal{A}$ there belongs, as we know from 1.7, a number α of sets of the main parts, $\mathcal{A}_1, \ldots, \mathcal{A}_\alpha\ (= \mathcal{A})$.

Choosing an arbitrary $\gamma\ (= 1, \ldots, \alpha)$, we first define the mapping $\boldsymbol{a}_\gamma$ of G onto $\mathcal{A}_\gamma$ by associating, with each point $a \in G$, the γ-th main part $a^{(\gamma)} \in \mathcal{A}_\gamma$ of the sequence $\boldsymbol{a}a$. The mapping $\boldsymbol{a}_\alpha$ is, of course, the same as $\boldsymbol{a}$.

To the mapping $\boldsymbol{a}_\gamma$ there belongs a certain decomposition of G, denoted by $\bar{A}_\gamma$. $\bar{A}_\alpha$, naturally, coincides with the decomposition belonging to $\boldsymbol{a}$.

Let $a \in G$ be an arbitrary point of G.

To the element $\boldsymbol{a}_\gamma a = a^{(\gamma)} \in \mathcal{A}_\gamma$ there corresponds the set of its successors (1.7) $M(a^{(\gamma)}) \subset \mathcal{A}_{\gamma+1}$ $(1 \leqq \gamma < \alpha)$. It is useful to employ the notation $M(a^{(\alpha)}) = \{a^{(\alpha)}\}$. The sequence of the sets

$$([Ma] =)\ M(a^{(1)}) \to \cdots \to M(a^{(\alpha)})$$

is called the *chain of successor-sets* that belongs to a.

Consider the element $a^{(\gamma)} \in \mathcal{A}_\gamma$ and the element $\bar{a}_\gamma \in \bar{A}_\gamma$ consisting of $\boldsymbol{a}_\gamma$-inverse images of $a^{(\gamma)}$. Under the mapping $\boldsymbol{a}_{\gamma+1}$ $(1 \leqq \gamma < \alpha)$ every point lying in $\bar{a}_\gamma$ is mapped onto a certain successor of $a^{(\gamma)}$; at the same time, each successor of $a^{(\gamma)}$ has, under the mapping $\boldsymbol{a}_{\gamma+1}$, one or more inverse images lying in G; all of them are contained in $\bar{a}_\gamma$. It is obvious that the sets of the $\boldsymbol{a}_{\gamma+1}$-inverse images of the individual successors of $a^{(\gamma)}$, i.e., the sets of the $\boldsymbol{a}_{\gamma+1}$-inverse images of the individual elements of the set $M(a^{(\gamma)})$, form a decomposition of the element $\bar{a}_\gamma \in \bar{A}_\gamma$; it is the decomposition $(\bar{K}_\gamma a =)\ \bar{a}_\gamma \sqcap \bar{A}_{\gamma+1}$ belonging to the partial mapping $\boldsymbol{a}_{\gamma+1}$ of $\bar{a}_\gamma$ onto the successor-set $M(a^{(\gamma)})$. The latter is, by the first equivalence theorem (6.8), equivalent to the decomposition $\bar{K}_\gamma a$. The set $M(a^{(\alpha)})$ is, of course, equivalent to $\bar{K}_\alpha a$.

Thus we arrive at the following description of the situation:

The set of sequences, $\mathcal{A}$, and the mapping $\boldsymbol{a}$ of G onto $\mathcal{A}$ determine, on G, a series of decompositions, of length α, the so-called *model series*

$$\big((\bar{A}) =\big)\ \bar{A}_1 \geqq \cdots \geqq \bar{A}_\alpha$$

whose members are the decompositions belonging to the individual mappings $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_\alpha$.

To each point $a \in G$ there corresponds a chain of successor-sets

$$([Ma] =)\ M(a^{(1)}) \to \cdots \to M(a^{(\alpha)})$$

and a local chain of the series $(\bar{A})$

$$([\bar{K}a] =)\ \bar{K}_1 a \to \cdots \to \bar{K}_\alpha a.$$

Every two members $M(a^{(\gamma)})$, $\bar{K}_\gamma a$ of these chains, with the same index γ, are equivalent sets.

Let us now consider two nonempty sets $\mathcal{A}$, $\mathcal{B}$ consisting of finite $\alpha(\geqq 1)$-membered sequences and arbitrary mappings $\boldsymbol{a}, \boldsymbol{b}$ of G onto $\mathcal{A}, \mathcal{B}$, respectively. Then we have the corresponding sets of the main parts, $\mathcal{A}_1, \ldots, \mathcal{A}_\alpha$ $(= \mathcal{A})$; $\mathcal{B}_1, \ldots, \mathcal{B}_\alpha$ $(= \mathcal{B})$, furthermore, the mappings $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_\alpha$ $(= \boldsymbol{a})$; $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_\alpha$ $(= \boldsymbol{b})$ of G onto the corresponding sets of the main parts and, finally, the model-series

$$\big((\bar{A}) =\big)\ \bar{A}_1 \geqq \cdots \geqq \bar{A}_\alpha,$$

$$\big((\bar{B}) =\big)\ \bar{B}_1 \geqq \cdots \geqq \bar{B}_\alpha.$$

To each point $a \in G$ there correspond two chains of successor-sets:

$$([Ma] =)\ M(a^{(1)}) \rightarrow \cdots \rightarrow M(a^{(\alpha)}),$$
$$([Na] =)\ N(b^{(1)}) \rightarrow \cdots \rightarrow N(b^{(\alpha)})$$

and, furthermore, the local chains of the series $(\bar{A})$, $(\bar{B})$:

$$([\bar{K}a] =)\ \bar{K}_1 a \rightarrow \cdots \rightarrow \bar{K}_\alpha a,$$
$$([\bar{L}a] =)\ \bar{L}_1 a \rightarrow \cdots \rightarrow \bar{L}_\alpha a.$$

Every two members $M(a^{(\gamma)})$, $\bar{K}_\gamma a$ or $N(b^{(\gamma)})$, $\bar{L}_\gamma a$ of these chains, respectively, with the same index γ, are equivalent sets.

Let us now assume that the model-series $(\bar{A})$, $(\bar{B})$ are co-basally chain-equivalent.

In that case, first, the final members $\bar{A}_\alpha$, $\bar{B}_\alpha$ of the model series $(\bar{A})$, $(\bar{B})$ coincide, hence $\bar{A}_\alpha = \bar{B}_\alpha$. Moreover, we can show that:

There exists a permutation $\boldsymbol{p}$ of the set $\{1, \ldots, \alpha\}$, such that the member $M(a^{(\gamma)})$, with an arbitrary index γ, of the chain of successor-sets, $[Ma]$, corresponding to an arbitrary point $a \in G$ and the member $N(b^{(\delta)})$, with the index $\delta = \boldsymbol{p}\gamma$, of the chain of successor-sets, $[Na]$, corresponding to the same point a, are equivalent sets.

Proof. The co-basal mapping of the manifold of the local chains, $\tilde{\boldsymbol{A}}$, of $(\bar{A})$ onto the manifold of the local chains, $\tilde{B}$, of $(\bar{B})$ is, on our assumption, a strong equivalence. That means that there exists a permutation $\boldsymbol{p}$ of the set $\{1, \ldots, \alpha\}$ with the following effect:

Let $a \in G$ stand for an arbitrary point and $\bar{a}$ for that element of the decomposition $\bar{A}_\alpha = \bar{B}_\alpha$ which comprises it.

Let, moreover, $[\bar{K}\bar{a}]$, $[\bar{L}\bar{a}]$ be the local chains of $(\bar{A})$, $(\bar{B})$ with the base $\bar{a}$. Then every two members $\bar{K}_\gamma\bar{a}$, $\bar{L}_\delta\bar{a}$ of $[\bar{K}\bar{a}]$, $[\bar{L}\bar{a}]$ for which $\delta = \boldsymbol{p}\gamma$ are equivalent sets.

Consider the member $M(a^{(\gamma)})$ of $[Ma]$, with an arbitrary index γ, corresponding to the point a and the member $N(b^{(\delta)})$ of $[Na]$, with index $\delta = \boldsymbol{p}\gamma$, corresponding to the same point a. Then we have $\bar{K}_\gamma a = \bar{K}_\gamma \bar{a}$, $\bar{L}_\delta a = \bar{L}_\delta \bar{a}$. Since $M(a^{(\gamma)})$, $\bar{K}_\gamma a$ and $\bar{L}_\delta a$ are equivalent to $\bar{K}_\gamma a\ (= \bar{K}_\gamma \bar{a})$, $\bar{L}_\delta a\ (= \bar{L}_\delta \bar{a})$ and $N(b^{(\delta)})$, respectively, it is obvious (6.10.7) that $M(a^{(\gamma)})$ is equivalent to $N(b^{(\delta)})$ and the proof is accomplished.

The above theorem leads to the following observation: If an arbitrary point $a \in G$ is mapped, under the functions $\boldsymbol{a}_\gamma$, $\boldsymbol{b}_\delta$, into the sets of the main parts, $\mathcal{A}_\gamma$, $\mathcal{B}_\delta$, γ and δ being in the above relation, then the successor-sets of both images are equivalent.

10.11. Some remarks on the use of the preceding theory in scientific classifications

The theory of the series of decompositions of sets is of interesting use in scientific classifications. In this respect, however, we shall content ourselves with a few remarks, for a more detailed study would exceed the limits of this book.

A *scientific classification* ($\mathcal{A}$) *of the set* G, briefly, *a classification of* G is a non-empty set $\mathcal{A}$ formed by finite α-membered sequences ($\alpha \geqq 1$) and a mapping $\boldsymbol{a}$ of G onto $\mathcal{A}$. The γ-th member of the sequence $\boldsymbol{a}a$ is called the *γ-th characteristic* or the *characteristic of order* γ of the element a. The elements of $\mathcal{A}$ are therefore sometimes called *sequences of the characteristics*. The above notions concerning mappings onto sets of sequences may, of course, be directly applied to classifications.

In case of scientific classifications, the elements of G are called *individuals*, the sets of the main parts are the *characteristic-sets* and the model-series is the so-called *classification-series*.

In an actual construction of a classification, the choice of the characteristics is restricted by special conditions which influence, in particular, the properties of the classification-series. In natural sciences, for example, the chosen characteristics of the individuals are particular properties of the latter, given by nature herself.

Any individual a in the classification ($\mathcal{A}$) is determined by finding the corresponding sequence of the characteristics, $\boldsymbol{a}a$. In actual cases, however, it sometimes happens that some of the characteristics cannot be ascertained, e.g., for deficiency of adequate means to do so or if the individual is damaged or pathological. In such cases the given individual cannot be determined by means of ($\mathcal{A}$).

Hence there arises the following problem:

We are to describe the principle of constructing two so-called *harmonious classifications* of the set G in convenient mutual relations. It is required that:
1) both classifications lead to the same result, i.e., that the individuals which are not considered to be different from one another be in both classifications the same;
2) that the characteristics missing in one classification may, for each individual, be replaced by adequate characteristics in the other.

Our results concerning the functions whose values are sequences point out the way of solving this, rather difficult, problem. Let us start with two suitably chosen complementary series of decompositions of the classified set G and choose, according to the construction introduced in 10.7, the characteristics in both classifications in a way that the corresponding classification-series be co-basally joint (10.8). If we have succeeded, then we are able to determine, for each individual, the $(\gamma + 1)$-th characteristic in one of these classifications from the knowledge of its first γ characteristics in the latter and of its $\delta + 1$ characteristics in the other classification; we can do so by means of the simple mappings existing between the corresponding successor-sets. But the possibility of constructing such harmonious classifications is, in actual cases, rarely available, as the choice of the characteristics depends on the postulates imposed on them. In this respect, however, the latter grants a certain freedom because the complementary series of decompositions from which it starts may be arbitrarily chosen.

10.12. Exercises

1. The manifold of local chains corresponding to a series of decompositions $((\bar{A}) =) \bar{A}_1 \geqq \cdots \geqq \bar{A}_\alpha$ on G is a set of sequences, $\mathscr{A}$. Associating, with every point $a \in G$, the corresponding local chain $[\bar{K}a]$, we obtain a mapping $\boldsymbol{a}$ of G onto $\mathscr{A}$. The corresponding model-series is $(\bar{A})$. The γ-th main part ($\gamma = 1, \ldots, a$) of the sequence $\boldsymbol{a}a$ associated with an arbitrary point $a \in G$ is the chain $\bar{K}_1 a \to \cdots \to \bar{K}_\gamma a$. For $1 \leqq \gamma < \alpha$, all the successors of the latter are obtained by adding, at its end, always one decomposition $\bar{x}_{\gamma+1} \sqcap \bar{A}_{\gamma+2}$ while $\bar{x}_{\gamma+1}$ runs over all the elements of $\bar{a}_\gamma \sqcap \bar{A}_{\gamma+1}$ ($a \in \bar{a}_\gamma \in \bar{A}_\gamma$; $\bar{A}_{\alpha+1} = \bar{A}_\alpha$). There exist mappings of G onto sets of sequences with arbitrarily given model-series.
2. The figure behind p. 80 may be regarded as a scheme of two harmonious classifications (with co-basally joint classification-series). The sequences of characteristics corresponding to the single individuals or classes of individuals that are not distinguished from one another are introduced in the single rows; the arrows point to both sequences of characteristics belonging to the same individual. The corresponding equivalent sets of successors are introduced in two columns denoted by $\mathring{A}_{\gamma\delta}$ and $\mathring{B}_{\delta\gamma}$. If, for example, a certain individual has, in the classification $(\mathring{A})$, the characteristics A_1, A_2, A_3, A_4, A_5 and, in the classification $(\mathring{B})$, the characteristics $B_1, B_2, B_3, B_4, B_5, B_6, B_7$ (or B_7'), then it has, in $(\mathring{A})$, also the characteristic A_6 (or A_6'). A detailed study of this problem may be left to the reader.