4.4. Exercises

1. Two finite coupled decompositions have the same number of elements.
2. On taking account of the last theorem of 4.3, show that there holds:

$$((\bar{x} \cap \bar{y}) \sqsubset \mathring{A}) \sqcap \mathbf{s}((\bar{x} \cap \bar{y}) \sqsubset \mathring{B}) = ((\bar{x} \cap \bar{y}) \sqsubset \mathring{B}) \sqcap \mathbf{s}((\bar{x} \cap \bar{y}) \sqsubset \mathring{A})$$
$$= (\bar{x} \cap \bar{y}) \sqcap [\bar{A}, \bar{B}].$$

5. Complementary (commuting) decompositions

Further particular situations generated by decompositions on the set G arise from the so-called complementary or commuting decompositions. As the latter play an important part in the following deliberations, we shall discuss them in a special chapter.

5.1. The notion of complementary (commuting) decompositions

Let $\bar{A}$, $\bar{B}$, $\bar{C}$ stand for arbitrary decompositions on G.

By the definition of the least common covering $[\bar{A}, \bar{B}]$, every element $\bar{u} \in [\bar{A}, \bar{B}]$ is the sum of certain elements $\bar{a} \in \bar{A}$ and, at the same time, the sum of certain elements $\bar{b} \in \bar{B}$. The *decomposition* $\bar{A}$ *is* called *complementary to* or *commuting with the decomposition* $\bar{B}$ if every element $\bar{a} \in \bar{A}$ is incident with each element $\bar{b} \in \bar{B}$ that lies in the same element $\bar{u} \in [\bar{A}, \bar{B}]$ as $\bar{a}$.

If. for example, $\bar{A}$ is a covering of $\bar{B}$, then $\bar{A}$ is complementary to $\bar{B}$. The new notion generalizes the concept of a covering.

There holds:

a) $\bar{A}$ *is complementary to* $\bar{A}$.
b) *If* $\bar{A}$ *is complementary to* $\bar{B}$, *then* $\bar{B}$ *is complementary to* $\bar{A}$.

Indeed, a) is obviously true. To prove b), let us accept the assumption but reject the assertion. Then there exists an element $\bar{b} \in \bar{B}$, lying in a certain element $\bar{u} \in [\bar{B}, \bar{A}]$, which is not incident with every element of $\bar{A}$ that lies in $\bar{u}$. Consequently, $\bar{b}$ is not incident with an element $\bar{a} \in \bar{A}$ lying in $\bar{u}$. Hence, $\bar{a}$ is not incident with all the elements of $\bar{B}$ lying in $\bar{u}$, which contradicts our assumption that $\bar{A}$ is complementary to $\bar{B}$ and the proof is accomplished.

With regard to b), we generally speak about complementary (commuting) decompositions without stressing which is complementary to (commuting with) which.

The following example proves the fact that *if* $\bar{A}$, $\bar{B}$ *and, at the same time,* $\bar{B}$, $\bar{C}$ *are complementary, then* $\bar{A}$, $\bar{C}$ *need not be complementary.*

Suppose $G = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ is a set consisting of six elements. Denote, furthermore,

$$\bar{a}_1 = \{a_1, a_2\}, \quad \bar{a}_2 = \{a_3, a_4\}, \quad \bar{a}_3 = \{a_5, a_6\};$$

$$\bar{b}_1 = \{a_1, a_3, a_5\}, \quad \bar{b}_2 = \{a_2, a_4, a_6\};$$

$$\bar{c}_1 = \{a_1, a_2, a_3\}, \quad \bar{c}_2 = \{a_4, a_5, a_6\},$$

so that we have following decompositions on G:

$$\bar{A} = \{\bar{a}_1, \bar{a}_2, \bar{a}_3\}, \quad \bar{B} = \{\bar{b}_1, \bar{b}_2\}, \quad \bar{C} = \{\bar{c}_1, \bar{c}_2\}.$$

Every element $\bar{a}_\alpha$ is incident with every element $\bar{b}_\beta$ and every element $\bar{b}_\beta$ is incident with every element $\bar{c}_\gamma$ ($\alpha = 1, 2, 3$; $\beta, \gamma = 1, 2$). So we have $[\bar{A}, \bar{B}] = \bar{G}_{\max}$, $[\bar{B}, \bar{C}] = \bar{G}_{\max}$ and it is clear that $\bar{A}$, $\bar{B}$ and, at the same time, $\bar{B}$, $\bar{C}$ are complementary. Moreover, both elements $\bar{c}_1$, $\bar{c}_2$ are incident with $\bar{a}_2$ so that $[\bar{A}, \bar{C}] = \bar{G}_{\max}$ but the elements $\bar{a}_1$, $\bar{c}_2$, for example, are not incident. Hence $\bar{A}$, $\bar{C}$ are not complementary.

.2. Characteristic properties

Suppose, again, that $\bar{A}$, $\bar{B}$, $\bar{C}$ are decompositions on G.

If every two elements $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$ *lying in the same element of a common covering* $\bar{C}$ *of the decompositions* $\bar{A}$, $\bar{B}$ *are incident, then* $\bar{C} = [\bar{A}, \bar{B}]$ *and therefore the decompositions* $\bar{A}$, $\bar{B}$ *are complementary.*

Indeed, let $\bar{C}$ stand for a common covering of $\bar{A}$, $\bar{B}$ and let $\bar{c} \in \bar{C}$. Then $\bar{c}$ is the sum of certain elements of the decomposition $[\bar{A}, \bar{B}]$. Let $\bar{u}, \bar{v}$ be elements of $[\bar{A}, \bar{B}]$, lying in $\bar{c}$. Every element $\bar{a}_1 \in \bar{A}$ lying in $\bar{u}$ is incident with some element $\bar{b} \in \bar{B}$ which must, therefore, lie in $\bar{u}$ and, consequently, in $\bar{c}$. If $\bar{A}$, $\bar{B}$ have the above property, then $\bar{b}$ is incident with every element $\bar{a}_2 \in \bar{A}$ lying in $\bar{v}$ so that the two-membered sequence $\bar{a}_1, \bar{a}_2$ forms a binding $\{\bar{A}, \bar{B}\}$ from $\bar{a}_1$ to $\bar{a}_2$. Hence $\bar{v} = \bar{u}$ as well as $\bar{c} = \bar{u}$ and, furthermore, $\bar{C} \subset [\bar{A}, \bar{B}]$. Since every element of $[\bar{A}, \bar{B}]$ lies in an element of $\bar{C}$, there also holds the relation $\supset$, hence even the equality and the proof is complete.

The decompositions $\bar{A}$, $\bar{B}$ are complementary if and only if for every two elements $\bar{a}_1, \bar{a}_2 \in \bar{A}$ lying in the same element $\bar{u} \in [\bar{A}, \bar{B}]$ there holds $\bar{a}_1 \sqsubset \bar{B} = \bar{a}_2 \sqsubset \bar{B}$.

Proof. a) Suppose the decompositions $\bar{A}, \bar{B}$ are complementary. If an element $\bar{b} \in \bar{B}$ is incident with $\bar{a}_1$, then it lies in $\bar{u}$ and is, therefore, incident with $\bar{a}_2$. Hence $\bar{a}_1 \sqsubset \bar{B} \subset \bar{a}_2 \sqsubset \bar{B}$ and, analogously, $\bar{a}_2 \sqsubset \bar{B} \subset \bar{a}_1 \sqsubset \bar{B}$.

b) Suppose $\bar{a}_1 \sqsubset \bar{B} = \bar{a}_2 \sqsubset \bar{B}$. Let the elements $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$ lie in the same element $\bar{u} \in [\bar{A}, \bar{B}]$. The element $\bar{b}$ is incident with an element $\bar{x} \in \bar{A}$ and the latter lies in $\bar{u}$. So we have $\bar{b} \in \bar{x} \sqsubset \bar{B} = \bar{a} \sqsubset \bar{B}$ and, consequently, $\bar{a}$ and $\bar{b}$ are incident.

5.3. Further properties

Suppose $\bar{A}$, $\bar{B}$ are complementary decompositions on G.

For every two elements $\bar{a} \in \bar{A}$, $\bar{u} \in [\bar{A}, \bar{B}]$ where $\bar{a} \subset \bar{u}$ there holds $\bar{u} = \mathbf{s}(\bar{a} \sqsubset \bar{B})$.

In fact, let $\bar{a} \in \bar{A}$, $\bar{u} \in [\bar{A}, \bar{B}]$ be arbitrary elements such that $\bar{a} \subset \bar{u}$. Every point $u \in \bar{u}$ lies in a certain element $\bar{b} \in \bar{B}$ which is, of course, a part of $\bar{u}$. Since the decompositions $\bar{A}, \bar{B}$ are complementary, the elements $\bar{a}$, $\bar{b}$ are incident and, therefore, $\bar{b}$ is an element of the closure $\bar{a} \sqsubset \bar{B}$, namely $\bar{b} \in \bar{a} \sqsubset \bar{B}$. There follows $u \in \bar{b} \subset \mathbf{s}(\bar{a} \sqsubset \bar{B})$ and $\bar{u} \subset \mathbf{s}(\bar{a} \sqsubset \bar{B})$. Furthermore, every point $a \in \mathbf{s}(\bar{a} \sqsubset \bar{B})$ lies in a certain element $\bar{b} \in \bar{B}$ incident with $\bar{a}$ and $\bar{b}$ is a part of $\bar{u}$. Consequently, $a \in \bar{u}$ as well as $\mathbf{s}(\bar{a} \sqsubset \bar{B}) \subset \bar{u}$ and the above statement is correct.

Every decomposition $\bar{C}$ on G that satisfies $[\bar{A}, \bar{B}] \geqq \bar{C} \geqq \bar{A}$ is complementary to $\bar{B}$.

In fact, suppose $\bar{C}$ is a decomposition on G, satisfying the above relations. Then (3.7.2a; 3.4): $[\bar{A}, \bar{B}] \geqq [\bar{C}, \bar{B}] \geqq [\bar{A}, \bar{B}]$, so that (3.2): $[\bar{C}, \bar{B}] = [\bar{A}, \bar{B}]$. Consider arbitrary elements $\bar{c} \in \bar{C}$, $\bar{b} \in \bar{B}$ lying in the same element $\bar{u} \in [\bar{C}, \bar{B}]$. Since $[\bar{C}, \bar{B}] = [\bar{A}, \bar{B}]$, the elements $\bar{c}$, $\bar{b}$ are subsets of the same element $\bar{u} \in [\bar{A}, \bar{B}]$. From $\bar{C} \geqq \bar{A}$ there follows that $\bar{c}$ is the sum of some elements $\bar{a} \in \bar{A}$. As $\bar{A}$, $\bar{B}$ are complementary, $\bar{c}$, $\bar{b}$ are incident. Therefore $\bar{C}$, $\bar{B}$ are complementary.

Furthermore, there holds:

If the decomposition $\bar{X}$ on G is a covering of $\bar{A}$, i.e., $\bar{X} \geqq \bar{A}$, then $\bar{A}$ is complementary to $(\bar{X}, \bar{B})$.

If the decomposition $\bar{Z}$ on G is a refinement of $\bar{A}$, i.e., $\bar{Z} \leqq \bar{A}$, then $\bar{A}$ is complementary to $[\bar{Z}, \bar{B}]$.

Proof. Suppose $\bar{X} \geqq \bar{A}$. Consider an element $\bar{u} \in [\bar{A}, (\bar{X}, \bar{B})]$. We are to show that every two elements $\bar{a} \in \bar{A}$, $\bar{b}' \in (\bar{X}, \bar{B})$ contained in $\bar{u}$ are incident, so that $\bar{a} \cap \bar{b}' \neq \emptyset$. Indeed, by 3.7.2a and for convenient elements $\bar{x} \in \bar{X}$, $\bar{w} \in [\bar{A}, \bar{B}]$ we have $\bar{u} \subset \bar{x} \cap \bar{w}$; moreover, with regard to $\bar{b}'$ and for convenient $\bar{x}' \in \bar{X}$, $\bar{b} \in \bar{B}$, there holds $\bar{b}' = \bar{x}' \cap \bar{b}$. From $\bar{x}' \cap \bar{b} \subset \bar{u} \subset \bar{x} \cap \bar{w}$ there follows $\bar{x}' = \bar{x}$ and $\bar{b} \subset \bar{w}$.

Furthermore: $\bar{a} \subset \bar{u} \subset \bar{x} \cap \bar{w}$. Since $\bar{a}, \bar{b} \subset \bar{w}$ and the decompositions $\bar{A}, \bar{B}$ are complementary, there holds $\bar{a} \cap \bar{b} \neq \emptyset$ and since $\bar{a} \subset \bar{x}$, we have:

$$\bar{a} \cap \bar{b} = (\bar{a} \cap \bar{x}) \cap \bar{b} = \bar{a} \cap (\bar{x} \cap \bar{b}) = \bar{a} \cap \bar{b}'.$$

Consequently: $\bar{a} \cap \bar{b}' \neq \emptyset$.

b) Suppose $\bar{Z} \leqq \bar{A}$. Then we have (3.7.2a): $[\bar{B}, \bar{A}] \geqq [\bar{Z}, \bar{B}] \geqq \bar{B}$ and, by the above (second) statement, the assertion is correct.

5.4. Modularity

Let again $\bar{A}, \bar{B}$ stand for complementary decompositions on G.

If $\bar{X} \geqq \bar{A}$, then $\bar{B}$ is modular with respect to $\bar{X}, \bar{A}$.

Proof. Suppose $\bar{X}$ is a covering of $\bar{A}$, i.e., $\bar{X} \geqq \bar{A}$. Taking account of 3.7.2, our object is to show that $(\bar{X}, [\bar{A}, \bar{B}]) \leqq [\bar{A}, (\bar{X}, \bar{B})]$. Consider an element $\bar{u}' \in (\bar{X}, [\bar{A}, \bar{B}])$ so that $\bar{u}' = \bar{x} \cap \bar{u}$ for convenient elements $\bar{x} \in \bar{X}$, $\bar{u} \in [\bar{A}, \bar{B}]$. The element $\bar{u}$ is the sum of certain elements of $\bar{A}$ some of which, let us denote them $\bar{a}$, are incident with $\bar{x}$ whereas others, if there are any, are disjoint with $\bar{x}$. Since $\bar{X} \geqq \bar{A}$, there applies to every $\bar{a}$ the relation $\bar{x} \supset \bar{a}$. Hence $\bar{u}'$ is the sum of all the elements $\bar{a}$ and we have $\bar{u}' = \bigcup \bar{a}$. It remains to be shown that any two elements $\bar{a}$ may be connected in $(\bar{X}, \bar{B})$. Let, therefore, $\bar{a}_1, \bar{a}_2$ be such elements, so that $\bar{a}_1, \bar{a}_2 \subset \bar{x} \cap \bar{u}$. Since $\bar{A}, \bar{B}$ are complementary and $\bar{a}_1, \bar{a}_2$ lie in $\bar{u}$, there exists an element $\bar{b} \in \bar{B}$ which lies in $\bar{u}$ and is incident with $\bar{a}_1, \bar{a}_2$: $\bar{a}_1 \cap \bar{b} \neq \emptyset, \bar{a}_2 \cap \bar{b} \neq \emptyset$; as, moreover, $\bar{a}_1, \bar{a}_2$ lie in $\bar{x}$, we have $\bar{a}_1 \cap \bar{b} = \bar{a}_1 \cap (\bar{x} \cap \bar{b})$, $\bar{a}_2 \cap \bar{b} = \bar{a}_2 \cap (\bar{x} \cap \bar{b})$. It is easy to see that the elements $\bar{a}_1, \bar{a}_2$ are incident with $\bar{x} \cap \bar{b} \in (\bar{X}, \bar{B})$ so that the two-membered sequence $\bar{a}_1, \bar{a}_2$ is a binding $\{\bar{A}, (\bar{X}, \bar{B})\}$ from $\bar{a}_1$ to $\bar{a}_2$ and the proof is accomplished.

The above theorem cannot be converted. In fact, let us show that for two decompositions $\bar{A}_0, \bar{B}_0$ on the set G the following statement is correct: *if $\bar{B}_0$ is modular with regard to any covering of $\bar{A}_0$ and to $\bar{A}_0$ itself, then $\bar{A}_0, \bar{B}_0$ need not be complementary.*

Assuming the set G to consist of four elements: a_1, a_2, a_3, a_4, i.e., $G = \{a_1, a_2, a_3, a_4\}$, let $\bar{A}_0, \bar{B}_0$ be decompositions on G consisting of the elements:

$$\bar{a}_1 = \{a_1, a_2\}, \quad \bar{a}_2 = \{a_3, a_4\};$$
$$\bar{b}_1 = \{a_1\}, \quad \bar{b}_2 = \{a_2, a_3\}, \quad \bar{b}_3 = \{a_4\},$$

hence

$$\bar{A}_0 = \{\bar{a}_1, \bar{a}_2\}, \quad \bar{B}_0 = \{\bar{b}_1, \bar{b}_2, \bar{b}_3\}.$$

Then there holds $[\bar{A}_0, \bar{B}_0] = \{G\}$ and we see that, e.g., the elements $\bar{a}_1$ and $\bar{b}_3$ have no points in common; consequently, $\bar{A}_0, \bar{B}_0$ are not complementary. On the whole, there exist two coverings of $\bar{A}_0$, namely: $\bar{X}_1 = \bar{A}_0$, $\bar{X}_2 = \bar{G}_{\max}$ and $\bar{B}_0$ is modular with regard to both $\bar{X}_1$, $\bar{A}_0$ and $\bar{X}_2$, $\bar{A}_0$ (4.3).

From the above theorem we realize that the figures generated by the decompositions $\bar{X} \geqq \bar{A}$, $\bar{Y} \geqq \bar{B}$ have all the properties of modular decompositions described in (4.3). In particular, for

$$\mathring{A} = (\bar{X}, [\bar{A}, \bar{B}]) = [\bar{A}, (\bar{X}, \bar{B})],$$
$$\mathring{B} = (\bar{Y}, [\bar{B}, \bar{A}]) = [\bar{B}, (\bar{Y}, \bar{A})]$$

there hold the formulae (1), (2) given in 4.3. $\mathring{A}$, $\mathring{B}$ have even further properties based on the fact that $\bar{A}$, $\bar{B}$ are complementary. Let us just remark that *$\mathring{A}$, $\mathring{B}$ are complementary,* as the reader might verify by means of the formula $[\mathring{A}, \mathring{B}] = [\bar{A}, \bar{B}]$.

5.5. Local properties

Let again $\bar{A}$, $\bar{B}$ stand for complementary decompositions on G and $\bar{X}$, $\bar{Y}$ for coverings of $\bar{A}$, $\bar{B}$ so that $\bar{X} \geqq \bar{A}$, $\bar{Y} \geqq \bar{B}$. Let $\mathring{A}$, $\mathring{B}$ have the same meaning as in 5.4.

Let, moreover, $a \in G$ be an arbitrary point and $\bar{x} \in \bar{X}$, $\bar{a} \in \bar{A}$, $\bar{y} \in \bar{Y}$, $\bar{b} \in \bar{B}$ the elements of $\bar{X}$, $\bar{A}$, $\bar{Y}$, $\bar{B}$ containing a.

First, owing to the modularity of $\bar{A}$, $\bar{B}$, *the closures* $(\bar{x} \cap \bar{y}) \sqsubset \mathring{A}$, $(\bar{x} \cap \bar{y}) \sqsubset \mathring{B}$ *are coupled.*

Next, consider the following decompositions in G:

$$\bar{X}^a = \bar{x} \sqsubset \bar{A} \; (= \bar{A} \sqcap \bar{x}), \quad \bar{Y}^a = \bar{y} \sqsubset \bar{B} \; (= \bar{B} \sqcap \bar{y}).$$

We observe that the decomposition $\bar{X}^a$ lies on $\bar{x}$ and $\bar{a} \in \bar{X}^a$; analogously, the decomposition $\bar{Y}^a$ lies on $\bar{y}$ and $\bar{b} \in \bar{Y}^a$.

We shall prove that $\bar{X}^a$ *and* $\bar{Y}^a$ *are adjoint with regard to* $\bar{a}$, $\bar{b}$. To that purpose we must show that

$$\mathbf{s}(\bar{b} \sqsubset \bar{X}^a \sqcap \bar{y}) = \mathbf{s}(\bar{a} \sqsubset \bar{Y}^a \sqcap \bar{x}).$$

Indeed, let $\mathring{a} \in \mathring{A}$, $\mathring{b} \in \mathring{B}$ denote elements containing the point a. Since, by 5.3, the decompositions $\bar{A}$, $(X, \bar{B})$ are complementary and $\mathring{A}$ is their least common covering, we have (by 5.3)

$$\mathring{a} = \mathbf{s}\big((\bar{b} \cap \bar{x}) \sqsubset \bar{A}\big).$$

On taking account of $\bar{X} \geqq \bar{A}$, we see that the closure $(\bar{b} \cap \bar{x}) \sqsubset \bar{A}$ consists of exactly those elements of $\bar{A}$ that lie in $\bar{x}$ and are incident with $\bar{b}$ so that $(\bar{b} \cap \bar{x}) \sqsubset \bar{A} = \bar{b} \sqsubset \bar{X}^a$. Hence $\mathbf{s}(\bar{b} \sqsubset \bar{X}^a) = \mathring{a}$ and, moreover,

$$\mathbf{s}(\bar{b} \sqsubset \bar{X}^a \sqcap \bar{y}) = \mathbf{s}(\bar{b} \sqsubset \bar{X}^a) \cap \bar{y} = \mathring{a} \cap \bar{y} \in (\bar{Y}, \mathring{A}),$$

the last relation following from $\mathring{a} \cap \bar{y} \supset \{a\} \neq \emptyset$. Thus the set $\mathbf{s}(\bar{b} \sqsubset \bar{X}^a \sqcap \bar{y})$ is an element of $(\overline{Y}, \mathring{A})$ and, in fact, the element containing a. In a similar way we can verify that the set $s(\bar{a} \sqsubset \overline{Y} \sqcap \bar{x})$ is the element of $(\overline{X}, \mathring{B})$, containing a. From this and from $(\overline{X}, \mathring{B}) = (\overline{Y}, \mathring{A})$ there follows the equality we were to prove.

5.6. Exercises

1. If the decompositions $\bar{A}, \overline{B}$ are complementary, then the formulae $(\mathring{A}, \mathring{B}) = (\overline{X}, \mathring{B}) = (\overline{Y}, \mathring{A}) = ((\overline{X}, \overline{Y}), [\bar{A}, \overline{B}])$, valid for modular decompositions $\overline{X} \geqq \bar{A}, \overline{Y} \geqq \overline{B}$ (see 4.3. (2)), may be completed by $(\mathring{A}, \mathring{B}) = [(\overline{X}, \overline{B}), (\overline{Y}, \bar{A})]$. In that case the decompositions $(\overline{X}, \overline{B}), (\overline{Y}, \bar{A})$ are complementary as well.
2. Show that in a set of four elements there exist, beside the pairs consisting of a covering and a refinement, only the following pairs of complementary decompositions: a) pairs of decompositions consisting of two elements each of which comprises only two points of the set; b) pairs of disjoint decompositions each of which contains three elements.

6. Mappings of sets

The theory of decompositions in sets considered in the previous chapters is the set-basis of the theory of groupoids and groups we intend to develop. But the results we have hitherto arrived at are only one part of the means necessary to attain our object. The other part consists of the theory of the mappings of sets, dealt with in the following chapters. The reader will certainly welcome the fact that the preceding, at times rather complicated, deliberations will now again be replaced by simpler ones.

6.1. Mappings into a set

In everyday life we often come across phenomena connected with the mathematical concept of mapping. Such phenomena are, in the simplest case, of the following kind: We have two nonempty sets G, G^* and between their elements a certain relation by which there corresponds, to each element of G, exactly one element of G^*. For example:

[1] Between the spectators at a certain performance and the tickets issued for the latter there exists the relation that each of the spectators is present on the ground of exactly one ticket.