Hence, by the relation c), there holds:

$$\big(\bar{Y}, (\bar{A}, \bar{B})\big) = \big((\bar{Y}, \bar{A}), \bar{B}\big) = (\bar{Y}, \bar{B}) = \bar{Y}.$$

We see that $\bar{Y}$ is a refinement of the decomposition $(\bar{A}, \bar{B})$.

Every common refinement of the decompositions A and $\bar{B}$ is, therefore, a refinement of their common refinement $(\bar{A}, \bar{B})$. Thus $(\bar{A}, B)$ is the greatest common refinement of $\bar{A}$ and $\bar{B}$.

3.6. Relations between the least common covering and the greatest common refinement of two decompositions

Let $\bar{A}$ and $\bar{B}$ stand for arbitrary decompositions on the set G.

It is easy to show that *between the least common covering* $[\bar{A}, \bar{B}]$ *and the greatest common refinement* $(\bar{A}, \bar{B})$ *of* $\bar{A}$, $\bar{B}$ *there hold the following equalities*:

$$[\bar{A}, (\bar{A}, \bar{B})] = \bar{A}, \quad (\bar{A}, [\bar{A}, \bar{B}]) = \bar{A}.$$

In fact, these equalities express the relations $\bar{A} \geqq (\bar{A}, \bar{B})$ and $[\bar{A}, \bar{B}] \geqq \bar{A}$ (3.4; 3.5).

3.7. Exercises

1. Deduce, for arbitrary decompositions $\bar{A}$, $\bar{B}$ of the set G, on the ground of $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$, $\mathbf{s}(\bar{a} \sqsubset \bar{B}) = \mathbf{s}(\bar{b} \sqsubset \bar{A}) = \bar{u}$, the relation $\bar{u} \in [\bar{A}, \bar{B}]$.
2. For any decompositions $\bar{A}, \bar{B}, \bar{X}$ on G, where $\bar{X} \geqq \bar{A}$, there holds a) $[\bar{X}, \bar{B}] \geqq [\bar{A}, \bar{B}]$, $(\bar{X}, \bar{B}) \geqq (\bar{A}, \bar{B})$; b) $(\bar{X}, [A, \bar{B}]) \geqq [\bar{A}, (\bar{X}, \bar{B})]$.
3. Find an example to show that, under the assumptions of the previous exercise, the equality in formula b) need not be valid.
4. Two decompositions *in* G always have the least common covering but need not have the greatest common refinement. For the least common coverings of the decompositions $\bar{A}$, $\bar{B}$, $\bar{C}$ in G there hold the formulae 3.4 a) b) c).

4. Special decompositions

In this chapter we shall deal with particular kinds of relations between decompositions in or on the set G.

4.1. Semi-coupled (loosely coupled) and coupled decompositions

Let $\bar{A}$ and $\bar{C}$ be decompositions in G.

The decompositions $\bar{A}, \bar{C}$ are called *semi-coupled* or *loosely coupled* if every element $\bar{a} \in \bar{A}$ is incident with, at most, one element of $\bar{C}$ and every element $\bar{c} \in \bar{C}$ with, at most, one element of $\bar{A}$ and if, moreover, at least for one pair of the elements $\bar{a} \in \bar{A}, \bar{c} \in \bar{C}$ the incidence really occurs. If $\bar{A}$ and $\bar{C}$ are semi-coupled, then the decomposition $\bar{A}$ ($\bar{C}$) is called semi-coupled or loosely coupled with the decomposition $\bar{C}$ ($\bar{A}$).

The decompositions $\bar{A}, \bar{C}$ are *coupled* if every element $\bar{a} \in \bar{A}$ is incident with exactly one element of $\bar{C}$ and every element $\bar{c} \in \bar{C}$ with exactly one element of $\bar{A}$. If $\bar{A}$ and $\bar{C}$ are coupled, then the decomposition $\bar{A}$ ($\bar{C}$) is said to be coupled with the decomposition $\bar{C}$ ($\bar{A}$).

We observe that two coupled decompositions in G are always semi-coupled.

Example of coupled decompositions: If there holds, for the subset $X \subset G$ and the decomposition $\bar{Y}$ in G, the relation $X \cap \mathbf{s}\bar{Y} \neq \emptyset$, then the decompositions $X \sqsubset \bar{Y}$ and $\bar{Y} \sqcap X$ are coupled.

Let us now proceed to describe the properties of semi-coupled and coupled decompositions.

First, note that if the decompositions $\bar{A}$ and $\bar{C}$ are semi-coupled, then $\mathbf{s}\bar{A} \cap \mathbf{s}\bar{C} \neq \emptyset$. Indeed, in that case incidence occurs at least for one pair of the elements $\bar{a} \in \bar{A}, \bar{c} \in \bar{C}$ and we have $\mathbf{s}\bar{A} \cap \mathbf{s}\bar{C} \supset \bar{a} \cap \bar{c} \neq \emptyset$. To simplify the notation, we put $\mathbf{s}\bar{A} = A$, $\mathbf{s}\bar{C} = C$, so that $A \cap C \neq \emptyset$.

The decompositions $\bar{A}, \bar{C}$ are semi-coupled if and only if the intersections $\bar{A} \sqcap C$, $\bar{C} \sqcap A$ are equal: $\bar{A} \sqcap C = \bar{C} \sqcap A$.

Proof. a) Suppose $\bar{A}$ and $\bar{C}$ are semi-coupled. Then, with regard to $A \cap C \neq \emptyset$, we have: $\bar{A} \sqcap C \neq \emptyset \neq \bar{C} \sqcap A$. Let $\bar{a}' \in \bar{A} \sqcap C$ be an arbitrary element; evidently $\bar{a}' = \bar{a} \cap C$, $\bar{a}$ standing for a convenient element of $\bar{A}$. Since $\bar{a}' \subset C$, $\bar{a}$ is incident with at least one and therefore, by the above assumption, exactly one element $\bar{c} \in \bar{C}$; $\bar{a}$ is obviously the only element of $\bar{A}$ which is incident with $\bar{c}$. We see that: $\bar{a}' = \bar{a} \cap \bar{c} = \bar{c} \cap A \in \bar{C} \sqcap A$. Thus we have $\bar{A} \sqcap C \subset \bar{C} \sqcap A$. Naturally, there simultaneously holds the relation $\supset$ and, consequently, the equality of both decompositions.

b) Suppose $\bar{A} \sqcap C = \bar{C} \sqcap A$. Let $\bar{a} \in \bar{A}$ be an arbitrary element. The element $\bar{a}$ is either not incident with any element of $\bar{C}$ or is incident with at least one. If it is incident with the elements $\bar{c}_1, \bar{c}_2 \in \bar{C}$, then we have: $\bar{a} \cap (\bar{c}_1 \cup \bar{c}_2) \subset \bar{a} \cap C \in \bar{A} \sqcap C = \bar{C} \sqcap A$ and, consequently, there exists an element $\bar{c} \in \bar{C}$ for which there holds $\bar{a} \cap (\bar{c}_1 \cup \bar{c}_2) = A \cap \bar{c}$. Since any two different elements of a decomposition are disjoint, there follows $\bar{c}_1 = \bar{c}_2 = \bar{c}$. We see that every element of $\bar{A}$ is incident with at most one element of $\bar{C}$ and, obviously, there also holds that every element of $\bar{C}$ is incident with at most one element of $\bar{A}$. From $A \cap C \neq \emptyset$ it is clear that at least for one pair of the elements $\bar{a} \in \bar{A}$, $\bar{c} \in \bar{C}$ the incidence really occurs and the proof is accomplished.

The decompositions $\bar{A}$, $\bar{C}$ are semi-coupled if and only if the closures $(\mathrm{H}\bar{A} =)\ \bar{C} \sqsubset \bar{A}$, $(\mathrm{H}\bar{C} =)\ \bar{A} \sqsubset \bar{C}$ *are coupled.*

Proof. a) Suppose $\bar{A}$, $\bar{C}$ are semi-coupled. Then, on taking account of $A \cap C \neq \emptyset$, we first have: $\mathrm{H}\bar{A} \neq \emptyset \neq \mathrm{H}\bar{C}$. Let us now consider an element $\bar{a} \in \mathrm{H}\bar{A}$. It is incident with at least one and, by the above assumption, exactly one element $\bar{c} \in \bar{C}$. The element $\bar{c}$ evidently belongs to the closure $\mathrm{H}\bar{C}$, hence $\bar{c} \in \mathrm{H}\bar{C}$, and is the only element of $\mathrm{H}\bar{C}$ which is incident with $\bar{a}$. It follows that every element of $\mathrm{H}\bar{A}$ is incident with exactly one element of $\mathrm{H}\bar{C}$. Since, analogously, every element of $\mathrm{H}\bar{C}$ is incident with exactly one element of $\mathrm{H}\bar{A}$, the closures $\mathrm{H}\bar{A}$ and $\mathrm{H}\bar{C}$ are coupled.

b) Suppose the closures $\mathrm{H}\bar{A}$, $\mathrm{H}\bar{C}$ are coupled. Then an arbitrary element $\bar{a} \in \bar{A}$ is either not incident with any element of $\bar{C}$ or is incident with at least one element of $\bar{C}$. In the latter case, $\bar{a}$ belongs to the closure $\mathrm{H}\bar{A}$ and, by the above assumption, it is incident with exactly one element $\bar{c} \in \mathrm{H}\bar{C}$. Except the elements of $\mathrm{H}\bar{C}$, no element of $\bar{C}$ is incident with $\bar{a}$. Consequently, every element of $\bar{A}$ is incident with at most one element of $\bar{C}$. For similar reasons, every element of $\bar{C}$ is incident with at most one element of $\bar{A}$. Therefore the decompositions $\bar{A}$, $\bar{C}$ are semi-coupled and the proof is complete.

The decompositions $\bar{A}$, $\bar{C}$ are coupled if and only if there simultaneously holds

$$\bar{A} \sqcap C = \bar{C} \sqcap A, \tag{1}$$

$$A = \mathbf{s}(C \sqcap \bar{A}), \quad C = \mathbf{s}(A \sqcap \bar{C}). \tag{2}$$

Proof. a) Suppose $\bar{A}$, $\bar{C}$ are coupled. Then every element of $\bar{A}$ and $\bar{C}$ is incident with at most and, at the same time, at least one element of $\bar{C}$ and $\bar{A}$, respectively. Consequently, on taking account of the above result, there holds (1) and, simultaneously, by 2.6.6, the first (second) equality (2).

b) Suppose the equalities (1), (2) are true. By means of the same theorems as in a), we can verify that $\bar{A}$, $\bar{C}$ are coupled.

If $\bar{A}$, $\bar{C}$ are coupled, then every element of $\bar{A}$ or $\bar{C}$ is incident with at least and, simultaneously, at most one element of $\bar{C}$ or $\bar{A}$, respectively. Consequently, there holds: $\bar{A} = \bar{C} \sqsubset \bar{A}$, $\bar{C} = \bar{A} \sqsubset \bar{C}$ (2.6.6).

Let us now assume that $\bar{A} = \bar{C} \sqcap \bar{A}$, $\bar{C} = \bar{A} \sqsubset \bar{C}$. Then, of course, our assumption: $A \cap C \neq \emptyset$ is satisfield as well.

Suppose $\bar{B}$ is an arbitrary common covering of the decompositions $\bar{A} \sqcap C$, $\bar{C} \sqcap A$ of the set $A \cap C$. By means of $\bar{B}$ we define, first, the decomposition $\bar{\bar{A}}$ $(\bar{\bar{C}})$ on $\bar{A}$ $(\bar{C})$ as follows: Each element of $\bar{\bar{A}}$ $(\bar{\bar{C}})$ consists of all the elements $\bar{a} \in \bar{A}$ $(\bar{c} \in \bar{C})$ that are incident with the same element of $\bar{B}$. Furthermore, by means of $\bar{\bar{A}}$ $(\bar{\bar{C}})$ we define the decomposition $\mathring{A}$ $(\mathring{C})$ in G: $\mathring{A}$ $(\mathring{C})$ is the covering of $\bar{A}$ $(\bar{C})$ enforced by $\bar{\bar{A}}$ $(\bar{\bar{C}})$.

Accordingly, there holds $\bigcup \bar{a} \in \mathring{A}$ $(\bigcup \bar{c} \in \mathring{C})$ if and only if $\bigcup (\bar{a} \cap C) \in \bar{B}$ $\big(\bigcup (\bar{c} \cap A) \in \bar{B}\big)$.

Thus we have, by means of the decomposition $\bar{B}$, constructed certain coverings $\mathring{A}$ and $\mathring{C}$ of $\bar{A}$ and $\bar{C}$, respectively. The coverings $\mathring{A}$, $\mathring{C}$ are said to be *enforced* by the common covering $\bar{B}$ of the decompositions $\bar{A} \sqcap C$, $\bar{C} \sqcap A$. Note that the construction is based upon the relations $\bar{A} = \bar{C} \sqsubset \bar{A}$, $\bar{C} = \bar{A} \sqsubset \bar{C}$.

Obviously: $\mathbf{s}\mathring{A} = \mathbf{s}\bar{A}$ $(= A)$, $\mathbf{s}\mathring{C} = \mathbf{s}\bar{C}$ $(= C)$.

Now we shall prove that *the decompositions $\mathring{A}$, $\mathring{C}$ are coupled and intersect each other in the decomposition $\bar{B}$* so that $\mathring{A} \sqcap \mathring{C} = \bar{B}$.

Proof. The equality $\bar{A} = \bar{C} \sqsubset \bar{A}$ yields $\mathring{A} = \mathring{C} \sqsubset \mathring{A}$ and, similarly, $\mathring{C} = \mathring{A} \sqsubset \mathring{C}$. To prove the theorem, it is sufficient to verify that

$$\mathring{A} \sqcap C = \mathring{C} \sqcap A = \bar{B}.$$

Indeed, if these equalities are satisfied, then, by the above result, the decompositions $\mathring{A}$, $\mathring{C}$ are coupled and, on taking account of 2.3, we have: $\mathring{A} \sqcap \mathring{C} = (\mathring{A} \sqcap C) \sqcap (\mathring{C} \sqcap A) = \bar{B} \sqcap \bar{B} = \bar{B}$.

To every element $\mathring{a}' \in \mathring{A} \sqcap C$ there exist elements $\mathring{a} = \bigcup \bar{a}$, $\mathring{a} \in \mathring{A}$, $\bar{a} \in \bar{A}$ such that $\mathring{a}' = \mathring{a} \cap C = (\bigcup \bar{a}) \cap C = \bigcup (\bar{a} \cap C) \in \bar{B}$, whence $\mathring{A} \sqcap C \subset \bar{B}$. Conversely, every element $\bar{b} \in \bar{B}$ has the form: $\bar{b} = \bigcup (\bar{a} \cap C)$ where $\bar{a} \in \bar{A}$, $\mathring{a} = \bigcup \bar{a} \in \mathring{A}$ and there holds $\bar{b} = \bigcup (\bar{a} \cap C) = (\bigcup \bar{a}) \cap C = \mathring{a} \cap C \in \mathring{A} \sqcap C$, whence $\bar{B} \subset \mathring{A} \sqcap C$. So we have $\mathring{A} \sqcap C = \bar{B}$ and, for analogous reasons, even $\mathring{C} \sqcap A = \bar{B}$.

4.2. Adjoint decompositions

Suppose $\bar{A}$, $\bar{C}$ are decompositions and B, D subsets of G. Let $B \in \bar{A}$, $D \in \bar{C}$ and $B \cap D \neq \emptyset$. We shall again make use of the notation: $A = \mathbf{s}\bar{A}$, $C = \mathbf{s}\bar{C}$.

By the above assumptions there holds $B \in D \sqsubset \bar{A}$, $D \in B \sqsubset \bar{C}$ and, on taking account of $B \subset A$, $D \subset C$, we have

$$\emptyset \neq B \cap D \subset (B \cap C),\ (D \cap A).$$

Consequently (2.6.5),

$$D \sqsubset \bar{A} \sqcap C, \quad B \sqsubset \bar{C} \sqcap A$$

are decompositions in G.

If there holds:

$$\mathbf{s}(D \sqsubset \bar{A} \sqcap C) = \mathbf{s}(B \sqsubset \bar{C} \sqcap A),$$

then the decompositions $\bar{A}$, $\bar{C}$ are said to be *adjoint with regard to the sets B, D*; we also say that $\bar{A}$ $(\bar{C})$ *is adjoint to* $\bar{C}$ $(\bar{A})$ *with regard to* B, D.

On taking account of the equalities

$$D \sqsubset \bar{A} \sqcap C = (D \cap A) \sqsubset (\bar{A} \sqcap C),$$
$$B \sqsubset \bar{C} \sqcap A = (B \cap C) \sqsubset (\bar{C} \sqcap A),$$

the formula (1) may be replaced by:

$$\mathbf{s}\big((D \cap A) \sqsubset (\bar{A} \sqcap C)\big) = \mathbf{s}\big((B \cap C) \sqsubset (\bar{C} \sqcap A)\big). \tag{1'}$$

For example, the decompositions $\bar{A}, \bar{C}$ are adjoint with regard to B, D if $\bar{A}$ is the greatest or the least decomposition of A.

Let us now assume that $\bar{A}, \bar{C}$ are adjoint with regard to B, D. Then:

$$\bar{A}_1 = C \sqsubset \bar{A}, \quad \bar{A}_2 = D \sqsubset \bar{A},$$
$$\bar{C}_1 = A \sqsubset \bar{C}, \quad \bar{C}_2 = B \sqsubset \bar{C}$$

are decompositions in G. Denote: $A_1 = \mathbf{s}\bar{A}_1$, $A_2 = \mathbf{s}\bar{A}_2$, $C_1 = \mathbf{s}\bar{C}_1$, $C_2 = \mathbf{s}\bar{C}_2$. Then we have:

$$\bar{A} \supset \bar{A}_1 \supset \bar{A}_2 \supset \{B\}, \quad A \supset A_1 \supset A_2 \supset B,$$
$$\bar{C} \supset \bar{C}_1 \supset \bar{C}_2 \supset \{D\}, \quad C \supset C_1 \supset C_2 \supset D.$$

We shall show that *there exist coupled coverings $\mathring{A}, \mathring{C}$ of the decompositions $\bar{A}_1, \bar{C}_1$ such that $A_2 \in \mathring{A}$, $C_2 \in \mathring{C}$.* These coverings are determined by the construction described in part a) of the following proof. *The sets A_2, C_2 are incident.*

Proof. a) Every element of $\bar{A}_1$ $(\bar{C}_1)$ lies in $\bar{A}$ $(\bar{C})$ and is incident with C (A) and, therefore, with some element of $\bar{C}$ $(\bar{A})$ incident with A (C); this element of $\bar{C}$ $(\bar{A})$ is, of course, contained in $\bar{C}_1$ $(\bar{A}_1)$. Hence:

$$\bar{A}_1 = \bar{C}_1 \sqsubset \bar{A}_1, \quad \bar{C}_1 = \bar{A}_1 \sqsubset \bar{C}_1.$$

It is also easy to realize that $A_1 \cap C_1 = A \cap C$. We observe that $A_1 \sqcap C_1$ $\bar{C}_1, \sqcap \bar{A}_1$ are decompositions on $A \cap C$. Let $\bar{U}$ be their least common covering so that $\bar{U} = [A_1 \sqcap \bar{C}_1, C_1 \sqcap \bar{A}_1]$. Now the decompositions $\mathring{A}, \mathring{C}$ are defined as the coverings of $\bar{A}_1, \bar{C}_1$, enforced by $\bar{U}$. So we have $\mathring{A} \sqcap \mathring{C} = \bar{U}$ and every element of $\mathring{A}$ $(\mathring{C})$ is the sum of all the elements of $\bar{A}_1$ $(\bar{C}_1)$ which are incident with one element of $\bar{U}$.

b) There holds $A_2 \in \mathring{A}$ and $C_2 \in \mathring{C}$. In fact, as $B \in \bar{A}$, $D \in \bar{C}$, we have

$$C \cap B \in C \sqcap \bar{A}, \quad A \cap D \in A \sqcap \bar{C}$$

and, since $\bar{A}, \bar{C}$ are adjoint with regard to B, D, there holds (1'). So we have, by 3.7.1, $\bar{u} \in \bar{U}$ where $\bar{u}$ is the set (1'). The individual elements of $\mathring{A}$ and $\mathring{C}$, respectively, are the sums of all the elements of $\bar{A}_1$ and $\bar{C}_1$, incident with one element of $\bar{U}$. To prove the relations $A_2 \in \mathring{A}$ and $C_2 \in \mathring{C}$, we only need to show that A_2 and C_2 are the sums of all the elements of $\bar{A}_1$ and $\bar{C}_1$, respectively, incident with $\bar{u}$.

We see, first, that there holds:

$$\mathring{u} = \mathbf{s}(D \sqsubset \bar{A} \sqcap C) = \mathbf{s}(\bar{A}_2 \sqcap C) = A_2 \cap C.$$

An arbitrary element of $\bar{A}_1$ lies in $\bar{A}$ and is incident with the set C; it simultaneously lies in $\bar{A}_2$ if and only if it is incident with the set D and, consequently, with the set $A_2 \cap C = \mathring{u}$. Hence, it is exactly the elements of $\bar{A}_1$ which lie in $\bar{A}_2$ that are incident with $\mathring{u}$; their sum is, as we see, A_2. Similarly, from

$$\overline{u} = \mathbf{s}(B \sqsubset \bar{C} \sqcap A) = \mathbf{s}(\bar{C}_2 \sqcap A) = C_2 \cap A$$

there follows that the sum of the elements of $\bar{C}_1$ which are incident with $\mathring{u}$ is C_2.

c) From $\emptyset \neq B \cap D \subset A_2 \cap C_2$ we have $A_2 \cap C_2 \neq \emptyset$.

The notion of adjoint decompositions may be extended to adjoint chains of decompositions.

Suppose $(\emptyset \neq)\, B \subset A \subset G$, $(\emptyset \neq)\, D \subset C \subset G$ and let

$$\begin{aligned}&([\bar{K}] =)\ \bar{K}_1 \to \cdots \to \bar{K}_\alpha,\\ &([\bar{L}] =)\ \ \bar{L}_1 \to \cdots \to \bar{L}_\beta\end{aligned}$$

be chains of decompositions in G from A to B and from C to D.

The chains $[\bar{K}]$, $[\bar{L}]$ are called *adjoint* if: 1) their ends coincide, i.e., $A = C$, $B = D$; 2) every two members $\bar{K}_\gamma$, $\bar{L}_\delta$ are adjoint with regard to the sets $\mathbf{s}\bar{K}_{\gamma+1}$, $\mathbf{s}\bar{L}_{\delta+1}$; γ and δ run over $1, \ldots, \alpha$ and $1, \ldots, \beta$, respectively, and $\mathbf{s}\bar{K}_{\alpha+1} = B$, $\mathbf{s}\bar{L}_{\beta+1} = D$.

4.3. Modular decompositions

In this chapter we shall deal with special decompositions lying *on* G.

Suppose $\bar{X}$, $\bar{A}$, $\bar{B}$ are decompositions on G and let $\bar{X} \geqq \bar{A}$.

The reader has certainly noticed (see 3.7.2, 3) that the decomposition $(\bar{X}, [\bar{A}, \bar{B}])$ is a covering of the decomposition $[\bar{A}, (\bar{X}, \bar{B})]$ but that these two decompositions need not be equal.

If they are equal, i.e., if there holds

$$[\bar{A}, (\bar{X}, \bar{B})] = (\bar{X}, [\bar{A}, \bar{B}]),$$

then the decomposition $\bar{B}$ is called *modular with regard to* $\bar{X}$, $\bar{A}$ (in this order). If, e.g., $\bar{X} = \bar{A}$ or $\bar{X} = \bar{G}_{\max}$, then $\bar{B}$ is modular with regard to $\bar{X}$, $\bar{A}$.

Let now $\bar{X}, \bar{Y}$ and $\bar{A}$, $\bar{B}$ stand for decompositions on G such that $\bar{X} \geqq \bar{A}$, $\bar{Y} \geqq \bar{B}$ and suppose $\bar{B}$ and $\bar{A}$ are modular with regard to $\bar{X}$, $\bar{A}$ and $\bar{Y}$, $\bar{B}$, respectively.

Then there holds:

$$\begin{aligned}(\mathring{A} =)\, [\bar{A}, (\bar{X}, \bar{B})] &= (\bar{X}, [\bar{A}, \bar{B}]),\\ (\mathring{B} =)\, [\bar{B}, (\bar{Y}, \bar{A})] &= (\bar{Y}, [\bar{B}, \bar{A}]),\end{aligned}$$

where the decompositions on either side of the first as well as the second formula are denoted $\mathring{A}$ and $\mathring{B}$, respectively.

We see, first, that *there holds*

$$\bar{X} \geqq \mathring{A} \geqq \bar{A}, \quad \bar{Y} \geqq \mathring{B} \geqq \bar{B},$$

so that the decompositions $\mathring{A}, \mathring{B}$ interpolate the decompositions $\bar{X}, \bar{A}$ or $\bar{Y}, \bar{B}$, respectively, in the sense of the above formulae.

Next, *there holds*:

$$[\mathring{A}, \mathring{B}] = [\bar{A}, \bar{B}], \quad [\bar{X}, \mathring{B}] = [\bar{X}, \bar{B}], \quad [\bar{Y}, \mathring{A}] = [\bar{Y}, \bar{A}], \tag{1}$$

$$(\mathring{A}, \mathring{B}) = (\bar{X}, \mathring{B}) = (\bar{Y}, \mathring{A}) = \big((\bar{X}, \bar{Y}), [\bar{A}, \bar{B}]\big). \tag{2}$$

These relations can easily be deduced from the properties of the least common covering and the greatest common refinement of two decompositions. For example, the first equality (1) by means of $(\bar{X}, \bar{B}) \leqq \bar{B} \leqq [\bar{B}, (\bar{Y}, \bar{A})]$, $(\bar{Y}, \bar{A}) \leqq \bar{A} \leqq [\bar{A}, \bar{B}]$ as follows:

$$\begin{aligned}[\mathring{A}, \mathring{B}] &= \big[[\bar{A}, (\bar{X}, \bar{B})], \quad [\bar{B}, (\bar{Y}, \bar{A})]\big] \\ &= \big[\bar{A}, [(\bar{X}, \bar{B}), [\bar{B}, (\bar{Y}, \bar{A})]]\big] = \big[\bar{A}, [\bar{B}, (\bar{Y}, \bar{A})]\big] \\ &= \big[[\bar{A}, \bar{B}], (\bar{Y}, \bar{A})\big] = [\bar{A}, \bar{B}].\end{aligned}$$

The other equalities may be deduced analogously.

The mentioned properties of modular decompositions can be specified as "global", since they concern decompositions as a whole without regard to the individual elements of which they consist. Besides these "global" properties, the modular decompositions also have the following "local" property, important to our purposes:

For any two incident elements $\bar{x} \in \bar{X}$, $\bar{y} \in \bar{Y}$ the closures $(\bar{x} \cap \bar{y}) \sqsubset \mathring{A}$, $(\bar{x} \cap \bar{y}) \sqsubset \mathring{B}$ are coupled.

Proof. Suppose $\bar{x} \in \bar{X}$, $\bar{y} \in \bar{Y}$ are arbitrary incident elements. Consider an element $\mathring{a} \in (\bar{x} \cap \bar{y}) \sqsubset \mathring{A}$ and show that it is incident with exactly one element $\mathring{b} \in (\bar{x} \cap \bar{y}) \sqsubset \mathring{B}$. In fact, since the element $\mathring{a} \in \mathring{A}$ is incident with the set $\bar{x} \cap \bar{y}$ and, according to the assumption, there holds $\bar{X} \geqq \mathring{A}$, we have: $\mathring{a} \subset \bar{x}$, $\bar{y} \cap \mathring{a} \neq \emptyset$. Hence, in particular, $\bar{y} \cap \mathring{a}$ is an element of the decomposition $(\bar{Y}, \mathring{A})$. As $(\bar{X}, \mathring{B}) = (\bar{Y}, \mathring{A})$, there exists an element $\mathring{b} \in \mathring{B}$ such that $\bar{x} \cap \mathring{b} = \bar{y} \cap \mathring{a}$. We see that $\mathring{b}$ is incident with $\mathring{a}$ so that $\mathring{b} \cap \mathring{a} \neq \emptyset$. As $\mathring{b}$ is also incident with $\bar{x} \cap \bar{y}$, we have $\mathring{b} \in (\bar{x} \cap \bar{y}) \sqsubset \mathring{B}$. Consequently, the element $\mathring{a}$ is incident at least with the element $\mathring{b}$ of the closure $(\bar{x} \cap \bar{y}) \sqsubset \mathring{B}$. But in the latter there are no further elements incident with $\mathring{a}$ because every element incident with $\mathring{a}$ forms a part of $\bar{y}$, cuts the set $\bar{y} \cap \mathring{a} = \bar{x} \cap \mathring{b}$ and therefore coincides with $\mathring{b}$. For analogous reasons, every element of $(\bar{x} \cap \bar{y}) \sqsubset \mathring{B}$ is incident with exactly one element of $(\bar{x} \cap \bar{y}) \sqsubset \mathring{A}$ and the proof is accomplished.

4.4. Exercises

1. Two finite coupled decompositions have the same number of elements.
2. On taking account of the last theorem of 4.3, show that there holds:

$$((\bar{x} \cap \bar{y}) \sqsubset \mathring{A}) \sqcap \mathbf{s}((\bar{x} \cap \bar{y}) \sqsubset \mathring{B}) = ((\bar{x} \cap \bar{y}) \sqsubset \mathring{B}) \sqcap \mathbf{s}((\bar{x} \cap \bar{y}) \sqsubset \mathring{A})$$
$$= (\bar{x} \cap \bar{y}) \sqcap [\bar{A}, \bar{B}].$$

5. Complementary (commuting) decompositions

Further particular situations generated by decompositions on the set G arise from the so-called complementary or commuting decompositions. As the latter play an important part in the following deliberations, we shall discuss them in a special chapter.

5.1. The notion of complementary (commuting) decompositions

Let $\bar{A}$, $\bar{B}$, $\bar{C}$ stand for arbitrary decompositions on G.

By the definition of the least common covering $[\bar{A}, \bar{B}]$, every element $\bar{u} \in [\bar{A}, \bar{B}]$ is the sum of certain elements $\bar{a} \in \bar{A}$ and, at the same time, the sum of certain elements $\bar{b} \in \bar{B}$. The *decomposition* $\bar{A}$ *is* called *complementary to* or *commuting with the decomposition* $\bar{B}$ if every element $\bar{a} \in \bar{A}$ is incident with each element $\bar{b} \in \bar{B}$ that lies in the same element $\bar{u} \in [\bar{A}, \bar{B}]$ as $\bar{a}$.

If. for example, $\bar{A}$ is a covering of $\bar{B}$, then $\bar{A}$ is complementary to $\bar{B}$. The new notion generalizes the concept of a covering.

There holds:

a) $\bar{A}$ *is complementary to* $\bar{A}$.
b) *If* $\bar{A}$ *is complementary to* $\bar{B}$, *then* $\bar{B}$ *is complementary to* $\bar{A}$.

Indeed, a) is obviously true. To prove b), let us accept the assumption but reject the assertion. Then there exists an element $\bar{b} \in \bar{B}$, lying in a certain element $\bar{u} \in [\bar{B}, \bar{A}]$, which is not incident with every element of $\bar{A}$ that lies in $\bar{u}$. Consequently, $\bar{b}$ is not incident with an element $\bar{a} \in \bar{A}$ lying in $\bar{u}$. Hence, $\bar{a}$ is not incident with all the elements of $\bar{B}$ lying in $\bar{u}$, which contradicts our assumption that $\bar{A}$ is complementary to $\bar{B}$ and the proof is accomplished.