I. SETS

1. Basic concepts

We shall first introduce some basic concepts from the theory of sets and found the following considerations on these.

1.1. The notion of a set

A *set* is a number of particular things which are called the *elements* or *points* of the set. *Every set is uniquely determined by its elements.* Two sets consisting of the same elements are called *equal.*

All about us we can see examples of sets such as:

[1] the set consisting of the symbol a;

[2] the set consisting of all the words in this book;

[3] the set of all natural numbers.

In this book we shall often deal with sets of sets, that is to say, sets whose elements are again sets; for convenience we shall call them *systems of sets.*

In a system of sets there are elements of the system, namely sets, on the one hand, and elements of these sets, on the other hand. In such cases we generally use the terms elements of the systems and points of these elements.

A system of sets is, for example,

[4] the set whose elements are sets of natural numbers; one of these sets consists of all prime numbers 2, 3, 5, 7, 11, ..., another of all the products of two prime numbers, another of all the products of three prime numbers, ect.

1.2. Notation of sets

Sets will generally be denoted by Latin capitals, e.g. A, and the elements of sets by small Latin letters, e.g. a. But in case of systems of sets, both the systems and their elements would, by this rule, be denoted by Latin capitals; we shall therefore use the notation $\bar{A}$, $\bar{a}$; $\bar{B}$, $\bar{b}$, etc. for systems and their elements.

If a and b denote the same object, we say that a, b *are equal* and write $a = b$ or $b = a$. The opposite case, i.e., the inequality of a, b, is expressed by the formula $a \neq b$ or $b \neq a$. If the sets A, B consist of the same elements, then $A = B$ and, in the opposite case, $A \neq B$. If a is an element of A, we write $a \in A$.

If a set A consists of elements denoted $a, b, c, \ldots$, then we write $A = \{a, b, c, \ldots\}$. Thus $\{a\}$ and $\{1, 2, 3, \ldots\}$ are symbols of the above sets [1] and [3], respectively. Nevertheless, we shall not always stick to the chosen terminology literally but, if convenient, change it a little, if there is no danger of misunderstanding, of course. Instead of "the set A is the collection of the elements $a, b, c, \ldots$" we can say "the set A consists of the elements $a, b, c, \ldots$" or "the set A contains the elements $a, b, c, \ldots$ and no others"; instead of "a is an element of (in) the set A" we may say "a belongs to the set A", and similarly.

1.3. Further notions

As a set we also introduce the so-called *empty set*, characterized by the property that it has no elements. Since every set is uniquely determined by its elements, there exists only one empty set. We shall denote it by the symbol $\emptyset$. Later we shall see that the introduction of the empty set is useful as regards the formulation of our considerations in special cases.

Every set whose elements are certain symbols, e.g. letters, the meaning of which is not precisely determined, is called *abstract*; the above set [1], for instance, is abstract.

Every set consisting of a finite number of elements is called *finite*; in the opposite case it is *infinite*; e.g., the sets [1], [2] are finite, whereas [3], [4] are infinite.

By the *order* of a finite nonempty set we mean the number of its elements. The set [1], for example, has the order 1. It will also be useful to assign, to every infinite set, the order 0. The empty set has no order.

1.4. Subsets and supersets

Suppose A, B are sets. If each element of A is simultaneously an element of B, then we say that A is a *subset of* B or that B is a *superset of* A. We can also say that A is a *part of* B or that B *contains* A. Then we write $A \subset B$ or $B \supset A$, respectively. The empty set is considered to be a part of any set; in particular, $\emptyset \subset \emptyset$.

If $A \subset B$, then B may (but need not) contain elements that do not belong to A. If B includes at least one element that does not belong to A, then A is said to be a *proper subset* of B and B a *proper superset* of A. In the opposite case, A (B) is a *non-proper* subset (superset) of B (A) and we see that it equals B (A): $A = B$ ($B = A$).[1])

The set of all prime numbers, for example, is a proper subset of the set [3], for every prime number is an element of the set [3] and the latter also contains numbers that are not prime, e.g. 4. If A is a non-proper subset of B, then each element of A is an element of B and, at the same time, each element of B is also an element of A; that is to say, there simultaneously holds $A \subset B$ and $B \subset A$. It is clear that both these relations together express the equality $A = B$.

It is easy to see that each subset of B is either proper or equal to B. Note that the equality $A = B$ is equivalent to the relations $A \subset B, B \subset A$ in the sense that, if $A = B$, then $A \subset B$, $B \subset A$ and vice versa. Generally we can tell whether two sets are equal just by verifying that either of them is a subset of the other.

1.5. The sum (union) of sets

By the *sum* or *union of the set A and the set B* we understand the set of all elements that belong to A or to B.

Since this definition determines all the elements that belong to the sum of A and B and every set is uniquely determined by its elements, there exists only one sum of A and B, denoted $A \cup B$. From the above definition there follows: $A \cup B = B \cup A$. Therefore we generally speak about the *sum of A and B* regardless of whether we mean the sum of A and B or B and A. We observe that the sum of A and B is the set of all the elements that belong to, at least, one of them. Either of the set A, B is a subset of $A \cup B$, for each element of, e.g., A belongs, at least, to one of the sets A, B, namely to A; so we can write: $A \subset A \cup B$, $B \subset A \cup B$. The sum of the set of all positive even numbers and the set of all positive odd numbers, for example, is the set [3] because there holds

$$\{2, 4, 6, \ldots\} \cup \{1, 3, 5, \ldots\} = \{1, 2, 3, \ldots\}.$$

The sum of the set consisting of a single word *and* and the set [2] is again the set [2].

The notion of the sum of two sets can easily be extended to the sum of systems of sets: by the *sum* or *union of anys ystem of sets*, $\bar{A}$, we mean the set of all the points belonging to, at least, one of the sets that are elements of $\bar{A}$.

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[1]) In this form we express: In the opposite case A is a non-proper subset of B an dwe see that it equals the set B: $A = B$; simultaneously, B is a non-proper superset of A and we see that it equals the set A: $B = A$.

A similar abbreviated form of expression will often be used throughout the book.

There holds, again, that the system $\bar{A}$ has exactly one sum and that every set which is an element of $\bar{A}$ is a subset of the sum of $\bar{A}$. The sum of $\bar{A}$ is generally denoted by $\boldsymbol{s}\bar{A}$; if the elements of $\bar{A}$ are denoted by $\bar{a}_1, \bar{a}_2, \ldots$, then the sum of $\bar{A}$ is denoted by $\bar{a}_1 \cup \bar{a}_2 \cup \ldots$, briefly $\bigcup \bar{a}$ or similarly, which is clear from the context.

1.6. The intersection of sets. Incident and disjoint sets

The *intersection of the set A and the set B* is the set of all the elements that belong to A as well as to B.

In a similar way as in the case of the sum, we can verify that there exists only one intersection of A and B; let us denote it by $A \cap B$. Moreover, we observe that $A \cap B = B \cap A$. So we generally speak about the *intersection of A and B* without paying any attention to whether we mean the intersection of A and B or that of B and A. It is obvious that the intersection of A and B is the set of all the elements that belong to both A and B. The intersection $A \cap B$ is a part of either A and B, for each element of $A \cap B$ belongs, e.g., to A. Note that, even if A and B have no common elements, the definition of the intersection of A and B applies because, in that case, $A \cap B$ is the empty set. And we realize that the notion of the empty set is of advantage; without it we could only speak about intersection in case of certain sets. Nevertheless, it is convenient to have special terms for sets that have common elements and for those that have not.

If A and B have common elements, they are called *incident* and A (B) is said to be *incident with* B (A). In the opposite case A and B are called *disjoint*. In the first case there holds: $A \cap B \neq \emptyset$, in the second: $A \cap B = \emptyset$.

Examples: The set consisting of the single word *and* and the set [2] are incident; their intersection is the former set. The set of all even natural numbers and the set of all odd natural numbers are disjoint; their intersection is obviously $\emptyset$.

The notion of the intersection of two sets can be extended to the intersection of a system of sets: The *intersection of any system of sets*, $\bar{A}$, is the set of all the points belonging to each of the sets that are elements of $\bar{A}$.

There again holds that $\bar{A}$ has exactly one intersection which is a subset of each element of $\bar{A}$. The intersection of $\bar{A}$ is denoted by $\boldsymbol{p}\bar{A}$; if the elements of $\bar{A}$ are denoted by $\bar{a}_1, \bar{a}_2, \ldots$, we write $\bar{a}_1 \cap \bar{a}_2 \cap \ldots$, briefly $\bigcap \bar{a}$, or similarly.

1.7. Sequences

By a *sequence on a* (non-empty) *set A*, briefly: a sequence, we mean the set A whose elements are numbered. Exactly one element is marked as the first, exactly one as the second, etc., each element of A being marked at least once. The element marked by the (natural) number γ is called the *γ-th member of* the sequence or the *member with index γ* or the *member of the rank γ*. The rank of a member is generally expressed by the adequate index; e.g., $a_1, a_2, \ldots$. Two different members of a sequence, for instance, a_1, a_2, may be the same element of A numbered once by 1 and another time by 2.

If the last member of a sequence is $\bar{a}_\alpha$, then the sequence is called *finite* or, more precisely, *α-membered* and α is its *length*. In that case there corresponds, to each number $\gamma = 1, 2, \ldots, \alpha$ exactly one member a_γ of the rank γ but the sequence does not comprise any members of a rank higher than α. Accordingly, such a sequence is denoted $(a_\gamma)_{\gamma=1}^{\alpha}$ or $(a_1, \ldots, a_\alpha)$ or in a similar way. If a sequence has no last member, we say that it is *infinite* or that its length is infinite. In an infinite sequence there corresponds, to each positive integer γ, precisely one member of the rank γ; notation $(a_\gamma)_{\gamma=1}^{\infty}$, $(a_1, a_2, \ldots)$, and similarly. If a sequence contains a finite number of different elements, then it is either finite or infinite; in the opposite case it is infinite.

Let the sequence $(a) = (a_1, a_1, \ldots)$ be either finite or infinite. Every sequence $(a_1', a_2', \ldots)$ generated from (a) by omitting some members a_γ is called a *partial sequence* or a *part of* (a). The sequence (a) is considered to be a part of itself. A partial sequence $(a_1, \ldots, a_\gamma)$ consisting of the first γ members of (a) is called the *γ-th main partial sequence* or the *γ-th main part of* (a); γ denotes a positive integer which, of course, in an α-membered sequence is not higher than α. If (a) consists of α members, then its main part $(a_1, \ldots, a_\gamma)$ has, for $\gamma = 1, \ldots, \alpha - 1$, exactly one *successor*, namely $(a_1, \ldots, a_\gamma, a_{\gamma+1})$; then the α-th main part of (a), naturally, coincides with (a). If (a) is infinite, then each of its main parts has exactly one successor.

The sequences $(a) = (a_1, a_2, \ldots)$, $(b) = (b_1, b_2, \ldots)$ are considered equal if and only if they have the same length and their members with the same indices are equal elements: $(a) = (b)$ means $a_1 = b_1, a_2 = b_2, \ldots$.

Now let us consider the above notions in case of sets of sequences.

Suppose $\mathcal{A}$ is a nonempty set consisting of finite, e.g., α-membered sequences. The main parts of the elements of $\mathcal{A}$, of length γ, where $1 \leqq \gamma \leqq \alpha$, form a nonempty set called the *γ-th set of the main parts belonging to $\mathcal{A}$*; notation: $\mathcal{A}_\gamma$. To the set $\mathcal{A}$ therefore belong the sets $\mathcal{A}_1, \ldots, \mathcal{A}_\alpha$; $\mathcal{A}_\alpha$, naturally, coincides with $\mathcal{A}$, i.e., $\mathcal{A}_\alpha = \mathcal{A}$. Furthermore, in case of $\gamma < \alpha$, there corresponds to each element $a^{(\gamma)} \in \mathcal{A}_\gamma$ a nonempty set of the successors of the element $a^{(\gamma)}$. This set consists of the main parts of all the elements of $\mathcal{A}$, of length $\gamma + 1$, beginning with $a^{(\gamma)}$; notation, e.g., $N(a^{(\gamma)})$. There evidently holds: $N(a^{(\gamma)}) \subset \mathcal{A}_{\gamma+1}$.

A remarkable example of a set consisting of α-membered sequences is an arbitrary nonempty set of points in an α-dimensional coordinate space. In that case, any point a is identical with a certain α-membered sequence $(a_1, \ldots, a_\alpha)$, the coordinates $a_1, \ldots, a_\alpha$ being real or complex numbers. The γ-th main part $(a_1, \ldots, a_\gamma)$, where $1 \leqq \gamma < \alpha$, is the "projection" of the point a into the γ-dimensional space given by the equations $a_{\gamma+1} = \cdots = a_\alpha = 0$.

1.8. The Cartesian product of sets. Cartesian powers

The *Cartesian product of the set A and the set B* is the set of all ordered pairs (a, b) such that a and b are elements of A and B, respectively. If either of the sets A, B is empty, then the Cartesian product of A and B is defined as the empty set.

This definition determines exactly one Cartesian product of the sets A and B, denoted by $A \times B$. A (B) is the first (second) *factor* of the Cartesian product $A \times B$ and a (b) is the first (second) *coordinate* of its element (a, b). From the above definition we see that, generally, $A \times B \neq B \times A$.

The *Cartesian second power* or the *Cartesian square of the set A* is the Cartesian product $A \times A$. The latter is, therefore, the set of all ordered pairs (a, b) such that a, b are elements of A. If the set A is empty, then the same applies to the Cartesian square $A \times A$. For example, the Cartesian square of the set [3] is the set of all ordered pairs formed by two equal or different positive integers; consequently, this Cartesian square is a set of points in the plane, both coordinates of which are positive integers.

The extension of the notion of the Cartesian product to more than two factors as well as the notion of the Cartesian second power of a set to higher Cartesian powers is easy and will be left to the reader. These, more general, notions will be omitted here, since we shall not make any use of them in the following considerations.

Note that the Cartesian products belong to sets consisting of finite sequences of elements.

1.9. α-grade structures

In the following study we shall come across even more complicated figures based on the concept of a set, in particular, the so-called α-grade structures.

Let α ($\geqq 1$) be a positive integer and $(A) = (A_1, \ldots, A_\alpha)$ a sequence of nonempty sets.

An *α-grade set structure with regard to the sequence* (A), briefly, an *α-grade structure* is a nonempty set $\tilde{A}$ of the following form: Each element $\bar{\bar{a}} \in \tilde{A}$ is an α-membered sequence $(\bar{a}) = (\bar{a}_1, \ldots, \bar{a}_\alpha)$ such that each of its members $\bar{a}_\gamma$ is a nonempty part of the set A_γ; $\gamma = 1, 2, \ldots, \alpha$.

We shall, in particular, meet with the case when $A_1, \ldots, A_\alpha$ consist of nonempty sets, so that every set A_γ is a system $\bar{A}_\gamma$ of nonempty sets; $1 \leqq \gamma \leqq \alpha$. Such α-grade structures $\tilde{A}$ are therefore of the following form: Every element $\bar{\bar{a}} \in \tilde{A}$ is an α-membered sequence, $(\bar{a}) = (\bar{a}_1, \ldots, \bar{a}_\alpha)$, each member $\bar{a}_\gamma$ of the latter being a nonempty sub-system of $\bar{A}_\gamma$; $\gamma = 1, 2, \ldots, \alpha$.

1.10. Exercises

1. $A \cup \emptyset = A$; $A \cup A = A$; $A \cap \emptyset = \emptyset$; $A \cap A = A$.
2. $A \cup (A \cap B) = A$; $A \cap (A \cup B) = A$.
3. If $A \subset B$, then $A \cup B = B$, $A \cap B = A$; conversely, if either of these equalities is correct, then $A \subset B$.
4. $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$.
5. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$; $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
6. A set of a finite number n ($\geqq 0$) of elements has 2^n subsets.
7. The Cartesian product of a set of m ($\geqq 0$) elements and a set of n ($\geqq 0$) elements consists of $m \,.\, n$ elements.
8. A part of the Cartesian product $A \times B$ is not necessarily the Cartesian product of a subset of A and a subset of B.

2. Decompositions (partitions) in sets

2.1. Decompositions in a set

Let G stand, throughout the book, for an arbitrary nonempty set.

A *decomposition* (*partition*) *in* G is a nonempty system of nonempty and mutually disjoint subsets of G.

This notion is one of the most important in this book and is, in fact, essential to the theory of groupoids and groups we intend to develop in the following chapters.

Every decomposition in G has therefore at least one element, each of its elements is a nonempty subset of G and, of course, the intersection of any two of its elements is empty.

A simple example of a decomposition in, let us say, the set of all positive integers is the system consisting of one single element, namely of the set of all even positive integers. More generally: the system consisting of a single element which is a nonempty subset of G is a decomposition in G. The system of sets [4] in part 1.1 is an example of a decomposition in the set of all positive integers $\geqq 2$.

2.2. Decompositions on a set

Let $\bar{A}$ be an arbitary decomposition in G. Any point of G may lie at most in one element of $\bar{A}$, since every two elements of $\bar{A}$ are disjoint; it may, of course, happen that it does not lie in any element of $\bar{A}$.