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LATTICE THEORY ITS BIRTH AND LIFE

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1 Introduction

In 1997 Gian-Carlo Rota [12] wrote the following words:

> *Never in the history of mathematics has a mathematical theory been the object of such vociferous vituperations as lattice theory. Dedekind, Jónsson, Kurosh, Malcev, Ore, von Neumann, Tarski, and most prominently Garrett Birkhoff have contributed a new vision of mathematics, a vision that has been cursed by a conjunction of misunderstanding, resentment, and raw prejudice.*

What are the reasons for using such strong expressions when talking about a mathematical theory? What is so special about lattice theory to attract such conflicting opinions? It is natural that any mathematician finds beauty in the field they are interested in and advocate its methods and results. But has lattice theory been really living in a world of such contradicting views? In my contribution I would like to present a short survey of its development and compare it with some phenomena that often occur in the history of a mathematical theory in general.

Beginnings and developments of theories

Though various areas and problems in mathematics have their unique development we can trace some features which are shared by more of them. As far as mathematical theories are concerned we can find some typical examples of the way a new theory appears and of how its development continues.

The beginning of a theory is not usually straightforward. It is often the case that there are more beginnings. The first ones, sometimes called pre-history or early history, are formed by more or less isolated results which are, despite the fact that they are leading towards the central idea of the theory, for some reasons, either missing the main point or/and finding none or too little response, remain forgotten or undeveloped.

After those first attempts to introduce new ideas some decades have to pass to arrive at the "second" beginning. This time the period is more riped for the incorporating of new methods and more mathematicians start to work on the same idea, very often in different fields of mathematics. The response to their research may be immediate or delayed for some short time, but soon the real development of the theory starts and depending on its importance it comes to the centre of attention in mathematics or in some areas.

We can find many well-known examples which fit similar pattern, let's mention for example non-Euclidean geometry or group theory. The first beginnings of non-Euclidean geometry can be viewed in attempts to prove the parallel postulate using the idea of its negation. Saccheri already at the end of the 17th century started reasoning leading to discovering non-Euclidean geometries, but not only that he made a mistake in his assumption, what is more important, his original thinking was not followed, with some exceptions like Lambert and Legendre. The second beginning came with independent work Gauss, János Bolyai and Lobachevsky, but still, there had to pass three decades to make this geometry understandable and mainly acceptable.

The development of group theory is a complex one, the first beginnings can be found in the work of mathematicians who came close to the concept of group, e.g. Euler, Gauss and mainly Ruffini. But the new beginning starts with the ideas of Galois who also introduced the term. However, it took again more than a decade before the importance of the notion was recognized and even longer before groups moved to the centre of mathematical investigation.

Further development of a theory can follow many schemes, but naturally there are two basic ones. Either its methods prove to be succesful and the theory yields a great number of results, problems and interesting insights, and thus it continues to grow in many directions, or, the optimism and hopes show to be overestimated and its importance does not reach a high level.

Lattices

A lattice is a partially ordered set in which for every two elements a and b the least upper bound (called join, denoted $a \vee b$) and the greatest lower bound (called meet, denoted $a \wedge b$) exist. Though lattice theory is built upon the concepts which are easy, they can be developed to a rich network of various properties with many applications. According to their properties lattices are divided into various types, the most basic ones being distributive, modular and complemented lattices.

The elements of a distributive lattice satisfy the distributive law:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

The elements of a modular lattice satisfy the modular law:

$$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c.$$

If a lattice has the greatest and the least elements and to each of its elements such an element exists that their join is the greatest element and their meet is the least element, it is called complemented. A complemented distributive lattice is called Boolean algebra.

Lattice theory – first beginnings

The birth of "lattices" follows the pattern of an uneasy start in more stages and in various areas of mathematics. The beginnings, however, were not isolated, the mathematicians interested in lattice structures learnt about each other's work, but their results were left without a wider response.

The first lattice structure appeared in the middle of the nineteenth century in the form of Boolean algebras. Thus the roots of lattice theory fall into the beginning of algebraic logic. G. Boole, Ch. S. Peirce and E. Schröder are the ones who contributed to the development of the notion lattice via their interests in algebraical logic. Another area which became an early source of ideas for lattices was number theory, namely the mathematician R. Dedekind.

The first step on the journey towards the concept of lattice was taken by George Boole when he tried to formalize propositional logic in the style of algebra. In his *The Mathematical Analysis of Logic*, 1848 he presented the list of laws which are satisfied by various algebras, e.g. calculus of logic. The rules he set concerned three binary operations on a structure, the first two satisfied associative, commutative and distributive laws, the third one corresponded to the creation of complements. The first appearance of lattice structures thus came into existence in the specialized form which continued to be generalized in the future process of establishing the concept.

The work of BOOLE did not find many followers, yet, there were two who contributed greatly to the development of algebraic logic and at the same time to lattice structures. CHARLES SANDERS PEIRCE made great improvements to BOOLE's calculus of logic and especially to its axiomatization. An important step towards studying lattice structures was, however, taken by ERNST SCHRÖDER who proved that the distributive law in the set of rules for BOOLE's calculus is independent of the others. (PEIRCE thought the opposite.) As a result SCHRÖDER started to study two systems of algebraic logic, and therefore distinguished two lattice structures: "identity calculus" as the specialized Boolean algebra and "logical calculus with groups" as a more general system which did not satisfy the distributive law.

RICHARD DEDEKIND was interested in algebraic number theory, and he investigated properties of structures called "dual groups" which were lattices from our point of view – modules of rings with the operations of taking the largest common divisor and the smallest common multiple. He introduced the modular law and studied the structures with distributive property, non-distributive structures with modular property and non-modular structures. Thus DEDEKIND arrived at even more general structures than SCHRÖDER because modularity is a weakened form of distributivity.

However, the study of abstract structures was not in fashion in the second half of the ninteenth century, and therefore the systems of SCHRÖDER and DEDEKIND did not provoke further investigation. It was not until the 1930's that a research of this kind obtained a wider response.

Lattice theory – second beginnings

Lattice structures started to be studied again at the end of 1920's – this time in still another area of mathematics. KARL MENGER presented the set of axioms characterizing projective geometries which are in fact complemented modular lattices. His investigations did not attract immediate attention, however, lattice structures soon appeared also in the field of formal logic (FRITZ KLEIN who gave lattices its German name: "Verband") and mainly algebra (ROBERT REMAK, OYSTEIN ORE). The biggest merits in the early developments of lattice theory belong to GARRETT BIRHOFF who also approached it from the side of algebra and united its various applications. In his first article about lattice structures [4] he rediscovered, apart from others, DEDEKIND's results, and only after its publishing it was revealed that the studies of dual groups are identical with BIRKHOFF's approach. G. BIRKHOFF also introduced the English word "lattice", which is not the translation of its German equivalent, but was inspired by the image of some Hasse diagrams presenting lattices.

GARRETT BIRKHOFF says how he became interested in lattices [11]:

> *I had thought a good deal about the subgroups and normal subgroups of groups, and about Remak's papers on the structure of groups. Having read van der Waerden and Remak, I became convinced of the importance of lattices for understanding the structure of groups.*

He expresses his enthusiasm concerning lattices [5]:

> *Actually I was probably the first person to conceive of lattices as a basic tool in algebra ... My lattice-theoretic arguments seemed to me so much more beautiful, and to bring out so much more vividly the essence of the considerations involved, that they were obviously the 'right' proofs to use.*

For G. BIRKHOFF lattice theory became the one playing the role of universal algebra in the sense of VAN DER WAERDEN. In the 1930's the approach of *modern algebra* came to the centre of attention in mathematics, and therefore it was also the suitable time for the development of lattice theory. The development didn't take place only in lattice theory proper or in its connection with algebra, but also in the field of geometry, topology, logic, probability and functional analysis. All the results and new views contributed to the optimism of mathematicians interested in lattices, and mainly G.BIRKHOFF predicted it a great future.

At the end of 1930's the first summarizing works on lattices started to appear. The biggest success and influence had BIRKHOFF's famous *Lattice Theory* [1] from 1940. It did not only present all the notions of lattice theory, but it also set it in the context of other areas of mathematics and showed its applications and contributions there.

The first general symposium on lattice theory was already held on April 15, 1938 in conjunction with a regular meeting of the American Mathematical Society, and lattice theory was described there as a "vigorous and promising younger brother of group theory" [6]. It was expected to become very important in mathematics.

Lattice theory – development

BIRKHOFF's book [1] became an impulse for further development. In 1940's lattice theory became an accomplished part of modern algebra, its terminology and notation unified (terms "lattice" in English, "Verband" in German and "treillis" in French) and the number of articles devoted to lattices grew. Most of the development followed the lines suggested by BIRKHOFF's monograph: theorems about homomorphism and isomorphism, congruence lattices, lattices of subalgebras, free lattices and applications of lattices.

In 1948 the second edition of *Lattice Theory* [2] was published. It was a revised edition, G. BIRKHOFF to a great extend modified and enlarged the first book. The results of the previous years allowed to present lattice theory as a more self-contained theory than in the first edition. The author also included main results of areas close to lattice theory, e.g. ordered sets, and thus the monograph continued to be the source of ideas for other fields as well.

In the preface to the second edition, G. BIRKHOFF makes a favourite comparison of lattice theory to group theory and views its progress in the following way [2]:

> *Like its elder sister group theory is a fruitful source of abstract concepts* [...] *it was this which convinced me from the first that lattice theory was destined to play – indeed, already did play implicitly – a fundamental role in mathematics. Though its importance will probably never equal that of group theory, I do believe that it will achieve a comparable status.*

The quotation suggests that though G. BIRKHOFF still believes in the importance of lattice theory, his early predictions that lattices will play the central role as universal algebra have not been fulfilled.

In the 1950's lattice theory proper as well as its applications grew in the contents and each area saw its development. However, the optimism of the early years calmed down. ROTA even remembers [12] the words of O. ORE: "I think lattice theory is played out." Such a pessimism of one the founders of the concept lattice is surprising. Others did not express such scepticism. They only pointed out that the emphasis and areas of research in lattice theory had changed since the 1930's, and that although lattice theory provided a useful framework for many topics and developed "into a full-fledged member of the algebraic family with an extensive body of knowledge and a collection of exciting problems all of its own," the great hopes had not been realized [6].

The year 1967 saw the publication of the third, new edition of *Lattice Theory*. This time G. BIRKHOFF admits in the preface that lattice theory does not attract so much attention which it deserves. He says [3]: "lattice-theoretic concepts pervade the whole of modern *algebra*, though many books on algebra fail to make this apparent."

At the beginning of the 1960's another mathematician - GEORGE GRÄTZER, together with E. T. SCHMIDT was thinking about writing "a work on lattice theory that would treat the subject in depth" [10]. However, they felt there still need to be done some work on certain areas of lattice theory to make their project successful. Many such research breakthroughs were completed in the 1960's and GRÄTZER finished his task of writing the book in the 1970's, he started with *Lattice Theory: first concepts and distributive lattices* [8] and then he completed the monograph in *General Lattice Theory* [9].

As many important results from the 1960's and 1970's (some of which solved the problems stated already in the 1940's) opened doors to many directions of investigation, lattice theory made tremendous developments in the last decades. Some of its chapters became so large that they form specialized areas of their own. The progress can be seen in the second edition of GRÄTZER's *General Lattice Theory* [10] from 1998 where the author comments on the solution or the development of the problems stated in the first edition, and several essays concerning recent evolutions of the main areas of lattice theory proper are included. Lattices have also continued to play important roles in its applications, e.g. in theoretical computer science or quantum mechanics.

Conclusion

Although lattice theory has gone through various stages of development with changing approaches and expectations, it has been growing considerably in each decade since its birth in the first half of the 1930's. This continuous development does prove the fact that if the theory finds importants applications and itself is a fruitful source investigation, its position in mathematics cannot be overlooked. This can be the reason why the idea about the importance of lattice theory is repeated again, after more than sixty years of its existence, by ROTA [12]:

> *These developments* [. . .] *are a belated validation of Garrett Birkhoff's vision* [. . .] *and they betoken Professor Gelfand's*

oft-repeated prediction that lattice theory will play a leading role in the mathematics of the twenty-first century.

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