Henrik Kragh Sørensen

Transformation and continuity of mathematics in the 1820s

In: Eduard Fuchs (editor): Mathematics throughout the ages. Contributions from the summer school and seminars on the history of mathematics and from the 10th and 11th Novembertagung on the history and philosophy of mathematics, Holbaek, Denmark, October 28-31, 1999, and Brno, the Czech Republic, November 2-5, 2000. (English). Praha: Prometheus, 2001. pp. 178–185.

Persistent URL: http://dml.cz/dmlcz/401248

NIELS HENRIK ABEL:

TRANSFORMATION AND CONTINUITY OF MATHEMATICS IN THE 1820s.

Henrik Kragh Sørensen

Abstract

My PhD project concern the mathematics of the early 19th century, in particular the works of the Norwegian mathematician Niels Henrik Abel (1802–1829). This presentation is a preliminary version of the introduction to be contained in my thesis and contains no references or footnotes. The thesis will be handed in to the Faculty of Science, University of Aarhus in the beginning of 2002. Comments and discussion will be much welcome.

In the aftermath of the French Revolution of 1789, the political and scientific scenes in Paris underwent radical change. Social and educational reforms introduced the first massive instruction in mathematics at the *Ecole Polytechnique*; and mathematics, itself, changed and erupted into the form recognizable to modern mathematicians. In the beginning of the 19th century, the neohumanist movement greatly influenced Prussian academia introducing mathematics into a very prominent position in the curriculum of secondary schools. At the university level, mathematics gained a certain autonomy and started to evolve along a distinctly theoretical line focusing less on applications and mathematical physics.

The thesis focuses on one of the main innovative figures in mathematics in the 1820s, the Norwegian Niels Henrik Abel (1802–29), and describes his contribution to and influence on the fermentation of the mathematical discipline in the early 19th century. Born at the periphery of the mathematical world and with a lifespan of less than 27 years, Abel nevertheless contributed importantly to the subdisciplines which he studied. The overall outline of the thesis is recapitulated in the following three sections which introduce Abel's professional background and training, the mathematics of his works, and the illustrated themes of development in mathematics in the first half of the 19$^{\text{th}}$ century. Throughout, Abel's mathematics is seen in its historical tradition, and the influences of mathematicians like Cauchy, Gauss, Lagrange, and Legendre is traced and described. This approach facilitates both a discussion of aspects of continuity and novel transformation in Abel's works.

1 The historical and geographical setting of Abel's life

Niels Henrik Abel lived in a politically turbulent time during which his birthplace, Finnøy, belonged to three different monarchies. When Abel was born in 1802, it belonged to the Danish-Norwegian twin monarchy but in the wake of the Napoleonic Wars, the province of Norway was ceeded to Sweden after a short spell of independence. Education in the twin monarchy was centered in Copenhagen, and only in 1813 was the university in Christiania (now Oslo) opened. The scientific climate was beginning to ripe, but mathematics was not studied at a high level.

As was common practice for the sons of a minister, Abel attended grammar school in Christiania and soon got the young Bernt Michael Holmboe (1795–1850) as a mathematics teacher. Holmboe was the first to notice Abel's affinity with and skills in mathematics and they began to study the works of the masters in special lessons. In 1821, after graduating from grammar school, Abel enrolled at the university but continued his private studies of the masters of mathematics. In 1824, he applied for a travel grant to go to the Continent and he embarked on his European tour in 1825. It brought him to Berlin and Paris where he had the opportunities to meet some of the most prominent mathematicians of the time and frequent the well equipped continental libraries. More importantly, Abel came into contact with August Leopold Crelle (1780–1855) in Berlin, who became Abel's friend and published most of Abel's works in the newly founded *Journal für die reine und angewandte Mathematik*. When Abel returned to Norway in 1827 he found himself without a job and with no family fortune to cover his expences, he took up tutoring mathematics. He had suffered from a lung infection during the tour, and in 1829 he succumbed to turberculosis.

Abel's geographical background thus dictated his approach to mathematics; it forced him to study the masters and do individual original work. In his short life span he carefully studied works of the previous generation and went beyond those. During the months abroad, he came into contact with the newest trends in mathematics, and immediately engaged in new research. Almost all his publications were written during or after the tour. The presentation of historical and biographical background serves to provide a framework for tracing ideas, influences, and connections concerning ABEL's work.

2 The mathematical topics involved

Theory of equations. The essentials of mathematics in the 18th century often come down to the work of a single brilliant mind, LEONHARD EULER (1707–83). Through a lifelong devotion to mathematics which spanned most of the century, EULER reformulated the core of the subject in many profound ways. Inspired by his attempts to prove that any polynomial of degree n had n roots (the so-called Fundamental Theorem of Algebra), EULER introduced another important mathematical question: Can any root of a polynomial be expressed in the coefficients by radicals, i.e. only by use of basic arithmetic and the extraction of roots? This question concerned the algebraic solvability of equations and to EULER it was almost self-evident. However, mathematicians strived to supply even the evident with proof, and JOSEPH LOUIS LAGRANGE (1736–1813) developed an entire theory of equations based on permutations to answer the question. Though a believer in generality, LAGRANGE came to recognize that the effort required to solve even the general fifth-degree equation might exceed the humanly possible. In LAGRANGE's native country, Italy, an even more radical perception of the problem had emerged; around the turn of the century, PAOLO RUFFINI (1765–1822) made public his conviction that the general quintic equation could *not* be solved by radicals and provided his claim with lengthy proofs.

ABEL's first and lasting romance with mathematics was with this topic, the theory of equations; his first independent steps out of the shadows of the masters were unsuccesful when in 1821 he believed he had obtained a general solution formula for the quintic equation. Provoked by the necessity to elaborate, he realized that his argument was in err, and by 1824 he gave a proof that no such solution formula could exist. The proof, which was based on an elaborated theory of permutations and a classification of possible solutions, reached world (i.e. European) publicity in 1826 when it appeared in the first volume of CRELLE's *Journal für die reine und angewandte Mathematik*. But as so often happens, solving one question only leads to posing another. Realizing that the general fifth-degree equation could not be solved by radicals, Abel set out on a mission to investigate which equations could and which equations could not be solved algebraically. Despite his efforts — which were soon distracted to another subject — Abel had to leave it to the younger French mathematician Evariste Galois (1811–32) to describe the criteria for algebraic solvability.

Elliptic functions. Since the invention of the calculus towards the end of the 17$^{\text{th}}$ century, the mathematical discipline of analysis had been able to treat an increasing number of geometrical curves. In his textbook *Introductio in analysin infinitorum* of 1748, Euler elevated the concept of function to the central object of analysis. Concrete functions were studied through their power series expansions and the brilliant calculator Euler obtained series expansions for all known functions including the trigonometric and exponential ones. However, Euler did not stop there but ventured into the territory of unknown functions of which he tried to get hold. One important type of function whose analysis had struggled to treat on a par with the rest was the so-called elliptic integrals that can measure the length of an arc of an ellipse.

Mathematicians such as Leonhard Euler and Adrien-Marie Legendre (1752–1833) felt and spoke of an unsatisfactory restriction of analysis for only being able to treat a limited set of elementary transcendental functions. Admitting new functions into analysis meant obtaining the kind of knowledge about these functions that would allow them to be given as *answers*. If a function today is nothing more than a mapping of one set into another, the knowledge of a function then included tabulation of values, series expansions and other representations, differential relations, functional relations, and much more.

When Abel turned his attention towards elliptic integrals as his main research topic, much knowledge concerning these objects had already been established. An algebraic inspiration which had profound influence on Abel was Gauss' study of the division problem for the circle (construction of regular n-gons) in the *Disquisitiones arithmeticae* (1801). Gauss had hinted that his approach could be applied to the lemniscate integral, and Abel took it upon himself to provide the claim with a proof. By a praised new idea, Abel inverted the study of elliptic integrals into the study of elliptic functions: Instead of considering the value of an integral as a function of its upper limit, he considered the upper limit as a function of the value of the integral (compare arcsin and sin). Through formal substitutions and certain additional formulae, ABEL obtained elliptic functions of a complex variable. By this inversion of focus, he managed to place the entire theory of elliptic integrals on a new and much more fertile footing. Initiated by a fierce competition between ABEL and the German mathematician CARL GUSTAV JACOB JACOBI (1804–51), the new theory gained almost immediate momentum and became one of the central pillars of and main motivations for 19th century advances in mathematics. To the professional mathematical community of the 1820s, ABEL's algebraic approach to elliptic functions must have seemed both classical and profoundly new. Many of the questions asked were formulated by the mid-18th century, but among ABEL's methods we find some of the gems of the new attitude to mathematics, which were incomprehensible to some of his most prominent contemporaries.

Although ABEL had presented the crucial idea of inverting elliptic integrals into elliptic functions, his impact on the further development of the theory stemmed as much from a vast generalization of the addition formulae which he had handed in to the Parisian *Académie des sciences* in 1826 (not published until 1841). In this paper, ABEL treated an even broader class of integrals generalizing the elliptic integrals and — again using primarily algebraic means — proved that the sum of any number of similar integrals could be reduced a certain number of integrals (depending on the form of the integral) and known algebraic and logarithmic terms. The quest of later mathematicians to reapply ABEL's daring inversion of elliptic integrals to this broader class of integrals led to much of the important development in complex analysis and topology.

Rigor. Although the theory of equations was closest to ABEL's heart, and the theory of elliptic functions brought him fame in the 19th century, his mathematical legacy remembered in the 20th century is just as much about his intense perception of CAUCHY's new rigor. Picking up from the theory of functions of LAGRANGE, CAUCHY had placed concepts such as continuity and convergence in the foreground and founded these concepts on a new interpretation of *limits*. Equally importantly, CAUCHY had shown a way of working with these concepts to deduce properties of *classes* of objects (e.g. continuous functions or convergent series) rather than a tedious studies of specific objects.

In the memorable and often quoted letter dated 1826 (first published 1839), ABEL expressed his convertion to CAUCHYism and gave the new rigor its dogmatic manifesto. Apparently more radical than CAUCHY himself, ABEL helped to determine the formulation of the new rigor through his interpretative readings of Cauchy. In the process of refounding analysis on rigorous grounds, the central concepts were specified and *changed* (stretched) to an extent where they showed behaviour that was deemed abnormal. The encounter and resolution of these abnormalities, *exceptions* as they were often called, is an integrated part of the rigorization process; such counter examples shed interesting light on the role and use of concepts in mathematics in the early 19^{th} century.

3 Themes from early 19^{th} century mathematics

The early 19^{th} century marks a period of transition and fermentation in mathematics which involves most layers of the discipline, external as well as internal. Fixing the boundaries at 1790 and 1840, a definite change in the way mathematics was performed and presented is evident; research mathematicians began working in institutions set up for instruction in mathematics and started presenting their results in professional periodicals with substantial circulation. However, the change even effected the internals of the discipline: how mathematics was done, what mathematics was, and which mathematical questions were interesting. Gradually, *concepts* and relations between concepts took an increasingly central position in mathematics research; although the concern for concrete objects never seized completely.

Concept based mathematics. Concepts such as *function*, *continuity* of functions, *irreducibility* of equations, and *convergence* of series attained central importance in mathematical research in the transitional period. Cauchy's contribution to the rigorization of the calculus lay as much in *working* with technical definitions of concepts to prove theorems as with providing the definitions, themselves. Generalization in the 1820s turned the attention from specific objects to *classes* of objects, which themselves were then investigated. This shift of attention towards collections of individual objects had a very direct influence on the style of presenting mathematical research. In the 'old' tradition, mathematical papers could easily be concerned with explicit derivations (calculations) pertaining to single mathematical objects, e.g. functions. Although this presentational style far from seized to fill periodicals, a less explicit style gained impetus in the first half of the 19^{th} century. By deriving properties of classes instead of individual objects, the arguments became more abstract and often more comprehensible by lowering the load of calculations and simplifying the mathematical notation. The transition is evident in ABEL's works which show deep traces of the calculation based approach to doing mathematics. On the other hand, ABEL's works were markedly conceptual at times; his 1826 paper on the binomial theorem is a fascinating mixture of both approaches.

Abstract definitions and coming to know mathematical objects. In many evolving fields of mathematics in the early 19$^{\text{th}}$ century, new concepts were specified using abstract definitions based on previous proofs, intentions, and intuition. In the approach which I term *concept based mathematics*, the concepts were *defined* in the modern sense that there is nothing more to a concept than its definition. However, when abstract definitions determine the extent of a concept, representations and demarcation criteria are required in order to get hold of properties of objects, and this quest for understanding, *coming to know*, the objects is an important aspect of early 19$^{\text{th}}$ century mathematics. In many ways, analogies may be drawn to the effort of coming to know geometrical objects, i.e. *curves*, in the 17$^{\text{th}}$ century. To mathematicians of the 17$^{\text{th}}$ century, a curve meant more than any single given piece of information. In particular, an equation (or a method of constructing any number of points on the curve) was not considered sufficient to accept the curve as *known*. Similarly, in the 19$^{\text{th}}$ century, knowledge of an elliptic function meant more than just a formal definition and included various representations, basic properties, and even tabulation of values.

The question of coming to know a mathematical object can be traced to the problem of accepting the object as solutions to problems. The reduction of properties of curves to questions pertaining those *basic* curves which were considered well known was important in the 17$^{\text{th}}$ century. However, certain properties were not expressible in basic curves (or functions) but required higher transcendentals such as elliptic integrals. Thus, much of EULER's research on elliptic integrals in the 18$^{\text{th}}$ century can be seen as an effort to make these integrals *basic* in the sense of acceptable solutions to problems. This research programme was continued and reformulated in the 19$^{\text{th}}$ century during which the foundations, definitions, and framework of elliptic functions underwent repeated revolutions.

Critical revision. The critical mode of thought, rooted in the Enlightenment, had a profound impact on mathematics. Together with the demand for wider instruction in mathematics, the critical attitude brought about a deeply sceptical reading of the masters which focused on the foundations. In geometry, some mathematicians began to believe in the possibility of a non-Euclidean version, and in analysis, the longstanding problem of the foundation of the calculus was made an important *mathematical* research topic.

CAUCHY's definition of the central concept of limits was itself a novelty, but of equal importance was the outlook for a concept based version of the calculus. CAUCHY's new foundation for the calculus was *arithmetical* and introduced the arithmetical concept of equality. In the wake of the change of foundations of the calculus, certain objects and methods could no longer be allowed into analysis, and it became a quest to prop up parts of the mathematical complex recently made insecure. In particular, CAUCHY had to abolish from analysis all divergent series which had formerly been interpreted by a *formal* concept of equality. However, divergent series had provided new insights to mathematicians which they were reluctant to abolish and it became a legitimate, albeit difficult, mathematical problem to investigate how problematic or outright unjust procedures led to correct results.

Henrik Kragh Sørensen
History of Science Department
University of Aarhus
Denmark
e-mail: hkragh@imf.au.dk