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SOME REMARKS ON HILBERT'S AXIOMATIC METHOD AND THE UNITY OF SCIENCE

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In this talk,[1] I try to look at HILBERT's axiomatic method as a response to the challenges of an ever increasing specialization within the mathematical and physical sciences, a response that is specific both individually to HILBERT and historically to the challenges of his time. Such specialization made a competent understanding of all of mathematics and the neighbouring sciences ever more difficult and, in particular, challenged attempts of conceiving a unity of the diversified subbranches of the mathematical and physical sciences. HILBERT studied the diverse branches of mathematics as much as he could and in all conceivable detail. To be sure, he was not alone in this endeavor but, to some extent, undertook these efforts first together with his younger friend MINKOWSKI and under the guidance of their elder friend HURWITZ, later with his Göttingen colleagues and assistants. In an oft-quoted passage by HILBERT, written at the occasion of the HURWITZ obituary, he wrote about the Königsberg hikes with MINKOWSKI and HURWITZ:

> *Auf zahllosen, zeitweise Tag für Tag unternommenen Spaziergängen haben wir damals während acht Jahren wohl alle Winkel mathematischen Wissens durchstöbert.* [1, p. 371]

As a fruit also of these early studies, HILBERT in 1900 was able to identify a number of fundamentally important problems from various different areas of mathematics. He concludes the presentation of his famous 23 problems of future mathematical research asking

> *whether mathematics is doomed to the fate of those other sciences that have split up into separate branches, whose representatives scarcely understand one another and whose connection becomes ever more loose.* [2, p. 282]

[1] Shortened version of a paper presented at the workshop "The Use of Axiomatics in the Exact Sciences", Göttingen, 11 June 1999, and at the 10$^{\text{th}}$ Novembertagung on the History of Mathematics, Holbaek, Denmark, 10 October 1999. A comprehensive version of the paper will be published elsewhere.

For our modern ears this worry sounds somewhat old-fashioned because we are used to a degree of specialization which implies that usually even office neighbours in mathematical research institutes or university departments find it difficult to communicate about more than the most common grounds of their respective science, let alone discuss concepts of unity of the diversity of their specialized fields. In HILBERT's days, around 1900, let's say, this was clearly different. HILBERT not only felt that he himself had visited "jeden Winkel mathematischen Wissens" but also institutionally the field of mathematics at the average German university was still taught by only one or two full professors. It was indeed quite uncommon that Göttingen would be granted a third ordinary professorship for mathematics in 1902, to be taken by HILBERT's friend MINKOWSKI, and another one in 1904.

Nevertheless, HILBERT perceived the present state of mathematics of his days as a challenge. It seemed to him a field of knowledge that was very hard to understand comprehensively for a single researcher but nevertheless a field that could still be mastered if you had some talent and if you would study it hard.

In the following, I shall look at HILBERT's axiomatic method as one particular position in reaction to the perceived state of mathematics, as a conceptual means to bring together, to integrate, or to unify different branches of the diversified mathematical sciences. I believe that HILBERT himself understood the axiomatic method in this way. Specifically, these concerns seem to be particularly evident when HILBERT is talking about how physics should be dealt with. Moreover, this motivation of unification for the axiomatic method is found in HILBERT's writings from the early Paris problems until late in his life. Along these lines, I shall first try to characterize how HILBERT perceived the "disunified" state of the mathematical sciences. I will then take a look at how HILBERT believed that the axiomatic method could help in the situation. And then confront this with the way he actually put his axiomatic method to work in his work on the foundation of physics.

HILBERT by 1900 certainly had a profound and comprehensive knowledge of mathematics and his mathematical achievements comprised and covered rather diverse mathematical subdisciplines such as invariant theory, algebraic number theory, and geometry. It seems to me, that many of his achievements were results of some kind of integrative work: by reinterpreting old problems from a different perspective, by employing mathematical techniques from remote fields for the treatment of problems in hitherto unrelated branches, and by synthesizing a full body of knowledge of algebraic number theory in his comprehensive report of the "Zahlbericht". Thus, I should like to suggest that Hilbert's mathematical achievements prior to 1900 were due *essentially* to his *comprehensive knowledge* of the diverse branches of mathematics. As one bit of evidence, let me mention that one of the main results of the *Grundlagen der Geometrie* was the proof of the consistency of the geometric axioms by reducing the consistency of the geometric axioms to those of arithmetic, that is precisely by setting up a conceptual connection between two fields or branches of mathematics.

The success of his investigations on the foundations of geometry became the guidepost for treating other mathematical problems as well. The reduction of one field to another, the recognition of similarities, analogies, etc. between different parts is, according to Hilbert, a characteristic feature of mathematics. Hence, he answers his own question whether mathematics is doomed to split up into separate, disconnected branches like this:

> *I do not believe this nor wish it. Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts.* [2, p. 282]

And the passage immediately following this statement may well reflect Hilbert's own research experience in the various mathematical subdisciplines. He says:

> *For with all the variety of mathematical knowledge, we are still clearly conscious of the similarity of the logical devices, the relationship of the ideas in mathematics as a whole and the numerous analogies in its different departments. We also notice that, the farther a mathematical theory is developed, the more harmoniously and uniformly does its construction proceed, and unsuspected relations are disclosed between hitherto separate branches of the science.* [2, p. 282]

These are vague notions. A little later, Hilbert goes on about the progress of mathematical science:

> ... *let me point out how thoroughly it is ingrained in mathematical science that every real advance goes hand in hand with the invention of sharper tools and simpler methods which at the same time assist in understanding earlier theories and cast aside older more complicated developments. It is therefore possible for the individual investigator, when he*

> *makes these sharper tools and simpler methods his own, to find his way more easily in the various branches of mathematics than is possible in any other science.* [2, p. 282]

In this passage, HILBERT does not use the word axiomatic but I read it between lines. The "sharper tools" and the "simpler methods" are identified at least and arrived at following the axiomatic method.

On the background of these remarks I would like to reread HILBERT's sixth problem. It asks for the mathematical treatment of the axioms of physics and it explicitly mentions the investigations on the foundations of geometry as a model for such a treatment. Specifically, it asks for the axiomatic treatment of those physical disciplines in which already today mathematics plays a major role. And those are mechanics and probability theory. In the explanatory sections of the sixth problem, HILBERT is more explicit about what the axiomatic method — called for following the model of geometry — actually is supposed to do. HILBERT says:

> *If geometry is to serve as a model for the treatment of physical axioms, we shall try first by a small number of axioms to include as large a class as possible of physical phenomena, and then by adjoining new axioms to arrive gradually at the more special theories.* [2, pp. 257f.]

The heuristic value of the axiomatic method here is to capture a field of physical phenomena by a small number of axioms. This small number of axioms is to represent the wealth of physical phenomena. One may survey some field of physics by identifying its defining small and finite set of axioms. The adjunction of new axioms specializes the theory and thus narrows the field of phenomena represented. This is the first feature of the axiomatic method as applied to the physical sciences as diverse and independent fields of knowledge. The representation of some class of phenomena by a set of axioms identifies it and gives structure to it, makes it analyzable by mathematical means. In a situation of a highly specialized science with mathematics separated from physics and both fields being differentiated to a large degree within themselves, HILBERT thus hoped, I would say, that the axiomatic formulation of a subdiscipline of physics spares him the perhaps tedious, or at least time-consuming investigation of all the details by, say, working through the textbook and research literature or even by experimental laboratory work—and still get a handle on it. The axiomatic method is a means of overcoming the non-communication of specialized scientists whose respective fields of expertise have drifted too far away as to let them continue to communicate. We may talk about a field that we do not specialize in by talking about its possible axiomatization. The second feature is a consequence of the first. If two separate fields of science can both be surveyed by their axiomatic formulation one can analyze whether the theories of both fields are consistent or not. Also, if within a field, we specialize by adjoining new axioms, we can test the internal consistency by testing the consistency of the axioms.

The "synthesis of the whole," as FELIX KLEIN had called it, becomes possible by expressing the various subdisciplines by means of respective small sets of axioms. And in so far as this is possible for those neighbouring disciplines like physics, those neighbouring disciplines may be integrated as well. But to the extent that the axioms are analyzed by logical and mathematical means, the axiomatic method makes mathematics the foundation of physics as well. Thus, HILBERT wrote in the very last paragraph of his Paris lecture:

> *The organic unity of mathematics is inherent in the nature of this science, for mathematics is the foundation of all exact knowledge of natural phenomena.* [2, p. 282]

In 1900, HILBERT was expressing a program. Among his 23 mathematical problems he had listed also the task of the "mathematical treatment of the axioms of physics". In the following years, and particularly so in the period from 1911 onwards, HILBERT devoted much of his time in pursueing this very task. HILBERT's work in physics will be documented by two volumes of the editorial project of publishing a selection of HILBERT's writings on the Foundations of Mathematics and the Natural Sciences.[2] HILBERT's work in physics prior to 1911 extended to lecture courses in mechanics and some little bit of research in continuum mechanics. After 1911, he started an intense and comprehensive studying of various fields of theoretical physics, electron and radiation theory, kinetic theory, atomic and the early quantum theory, relativity. He was fifty by that time, and he had people around him, personal assistents, colleagues, advanced students, as well as guests, whom he asked about what was going on in physics. These people, to some extent, if I may use this expression, predigested a certain field of physics for HILBERT so that his own considerations could begin with central issues of a field, attempting to come up with some kind of set of axioms for that area.

In November 1915, HILBERT presented the first note on the foundations of physics to the Göttingen Academy [3], in which he presented

[2] See http://www.gwdg.de/~uhwg/hilbertedition.html for further details.

generally covariant gravitational and electromagnetic field equations in terms of a variational principle. The gravitational equations were equivalent to the ones EINSTEIN published at the same time in his final step towards General Relativity if the respective energy-momentum tensors are identified. Moreover, HILBERT's first note contained a special case of Noether's second theorem and the first attempt at a unified theory of the electromagnetic and the gravitational field.

HILBERT's famous work on the foundations of physics from 1915 started from such a "predigested" account by BORN of GUSTAV MIE's theory of matter. This theory, in BORN's version, was a generalized Maxwellian electrodynamics based on a variational formulation with a Lagrangian depending on the electromagnetic four-potential and its first derivatives. It generalized Maxwellian electrodynamics by postulating that the Lagrangian only be Lorentz invariant but not necessarily gauge-invariant. It could hence depend explicitly and non-linearly on the electromagnetic potential. The corresponding differential equations would then be non-linear generalizations of Maxwell's equations and one might hope that their solutions would properly describe the electron as the fundamental entity of matter.

The other ingredient of HILBERT's theory was EINSTEIN's theory of general relativity as expounded in a major review published in spring 1914. At this time EINSTEIN was still believing in non-generally covariant field equations, published first in spring 1913. But except for the gravitational field equations, the theory of 1914 already contained all ingredients of the final theory of late 1915: the metric tensor as the representation of the gravitational potential, the mathematical apparatus of the "absolute differential calculus" of RICCI and LEVI-CIVITA and generally covariant equations of motion. The 1914 exposition of EINSTEIN's theory also presented a derivation of the gravitational field equations based on a variational formulation. A major part of this exposition was indeed devoted to determining the specific non-invariant Lagrangian that would give the specific gravitational field equations.

In late 1915, the core idea of HILBERT's work of his first note on the foundations of physics was to combine these two ingredients. He postulated a variational integral with a Lagrangian that would depend on the components of the metric tensor, its first and second derivatives, as well as on the electromagnetic potential and its first derivatives. Lagrangian differentiation with respect to the metric would then give the graviational equations and Lagrangian differentiation with respect to the components of the electromagnetic potential would then give the generalized Maxwell equations. Specialization of the latter equations to the case of a flat Lorentz metric would finally recover MIE's original special relativistic theory.

The two ingredients of HILBERT's theory survive in it as two distinct axioms. Introducing EINSTEIN's metric tensor and the electromagnetic four-potential, the first axiom introduces a "world function", i.e. a Lagrangian that would depend on the components of the metric tensor and of electromagnetic four potential as well as on its derivatives, and says that the physical phenomena are determined by this world function in such a way that the variation of the action integral with respect to each of its arguments vanish. HILBERT calls this axiom "Mie's axiom of the world function." The second axiom then brings in EINSTEIN's idea of general relativity by postulating that the world function be invariant with respect to any transformation of the coordinates. HILBERT calls this axiom the "axiom of general invariance."

While these two axioms refer to an electromagnetic field theory of matter on the one hand and a tensorial relativistic field theory of gravitation on the other hand, they also bring in at the same time two branches of mathematics. Axiom I postulates a variational formulation and hence calls for the calculus of variations. Axiom II postulates general invariance and hence brings in invariant theory. If one forgets, for a moment, the physical meaning of these axioms then the theory should result in a combination of these two different branches of mathematics. Indeed, one of the key results of the paper is an early special version of EMMY NOETHER's second theorem, formulated and proven in full generality by EMMY NOETHER in 1918. This theorem formulates invariance properties of variational problems. In HILBERT's note, it is formulated for the special case at hand saying that of the fourteen differential equations resulting as Lagrangian derivatives of the action four are a consequence of the other 10 in the sense that four relations between the Lagrangian expressions and their derivatives are identically satisfied. HILBERT now brings in the physical meaning of the two axioms and interprets this mathematical result in the sense that it is the four generalized Maxwell equations that are a consequence of the ten gravitational field equations. The result, however, is not true in full generality. Another problem of the theory is its presupposition of an electromagnetic theory of matter based on a not necessarily gauge-invariant action integral. Nevertheless, HILBERT was rather self-confident about his theory. He says at the end of his paper:

> *Wie man sieht genügen bei sinngemäßer Deutung die weni-*

> *gen einfachen in den Axiomen I und II ausgesprochenen Annahmen zum Aufbau der Theorie: durch dieselbe werden nicht nur unsere Vorstellungen über Raum, Zeit und Bewegung von Grund aus in dem von Einstein dargelegten Sinne umgestaltet, sondern ich bin auch der Überzeugung, daß durch die hier aufgestellten Grundgleichungen die intimsten bisher verborgenen Vorgänge innerhalb des Atoms Aufklärung erhalten werden und insbesondere allgemein eine Zurückführung aller physikalischen Konstanten auf mathematische Konstanten möglich sein muß — wie denn überhaupt damit die Möglichkeit naherückt, daß aus der Physik im Prinzip eine Wissenschaft von der Art der Geometrie werde: gewiß der herrlichste Ruhm der axiomatischen Methode, die hier wie wir sehen die mächtigsten Instrumente der Analysis, nämlich Variationsrechnung und Invariantentheorie in ihre Dienste nimmt.*

HILBERT seems to have believed that with this work his sixth problem of axiomatizing physics, in principle, had been solved. The work that remained to be done was to find an appropriate "world function" that would properly describe the electron in Mie's sense. Historically, however, this is the weak point of HILBERT's theory. From our modern point of view, Mie's theory was a rather idiosyncratic theory, an interesting idea at best. This may have been different for scientists at the time, particularly to those devoted to the program of reducing all of physics to Maxwellian electromagnetism like the Göttingen physicist MAX ABRAHAM. MAX BORN in his early days also was sympathetic of this electromagnetic world view, and generally one could argue that Göttingen was in fact a stronghold of this program. Therefore it may have seemed a convincing theory to HILBERT who had to rely to some extent on those people around him that would tell him about what was going on in physics.

HILBERT thought that with a small group of experts he could succeed in bringing about an axiomatic foundation of mathematics including at least some parts of theoretical physics as a first step towards a conceptually integrated, unified science of nature. In the specific historical situation of the turn of the century and for mathematics, this may have been a reasonable hope. But applied to those neighbouring sciences, the axiomatic method was far less successful. As an integrative, unifying method as envisaged by HILBERT, it found its limitations already in the neighbouring sciences of physics. It nevertheless had immense heuristic value for HILBERT and in this respect also proved very fruitful in his work.

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