Witold Więsław

Old algebraical treatises

In: Eduard Fuchs (editor): Mathematics throughout the ages. Contributions from the summer school and seminars on the history of mathematics and from the 10th and 11th Novembertagung on the history and philosophy of mathematics, Holbaek, Denmark, October 28-31, 1999, and Brno, the Czech Republic, November 2-5, 2000. (English). Praha: Prometheus, 2001. pp. 21–36.

Persistent URL: http://dml.cz/dmlcz/401235

OLD ALGEBRAICAL TREATISES

WITOLD WIĘSŁAW

1 Introduction

The algebra as a tool of mathematics appeared already in the treatises of the Antiquity. Since only geometry was developing, necessarily all mathematical discoveries were formulated in geometrical language. We owe many algebraical dicoveries to EUCLID. It was honoured by the historians of mathematics, O. NEUGEBAUER and B. L. VAN DER WAERDEN. They introduced the notion of *geometric algebra*. However, only DIOPHANTUS built a domain now called *the algebra*.

Islamic scientists, mainly AL-KHWĀRIZMĪ, ABU KAMIL, and OMAR KHAYYAM, to mention just three of them, developed the ideas of Greeks from Alexandria in the following centuries, building a domain which can be called *verbal algebra*. It was already the algebra in the full extent, practised by words, without any symbols. Medieval Europe continued in this line, successively and slowly introducing first symbols. In this way, algebra started the period of developing its symbolism, and consequently, the period of the *symbolic*, or *literal, algebra*. It happened in the XVII century, but was often restrained for many reasons. At the end of the century, the algebra became practically modern algebra. Now we call it the *classical algebra*. In the XVII and XVIII centuries, algebra was a science of solving equations. EULER's attempts at finding a proof of the fundamental theorem of algebra as well as his efforts to prove that every equation of fifth degree is solvable in radicals show how strong and uncritical the belief in success of the new *literal method* was. However, that belief was justified. RENE DESCARTES proved that the idea of coordinates, known in fact already to EUCLID, realised by the Islamic astronomer AL-MARAKISHI in the XIV century, just lately to the universal symbolism was able to show its efficiency. However, the end of the XVIII century brought the first proof of *impossibility* in mathematics. The proof was obtained in 1799 by the professor of medicine and mathematics PAOLO RUFFINI. He proved that there exists no algorithm for solving algebraical equations of degree five and higher, using only arithmetical operations and extracting roots. Many mathematicians did not believe it, in spite of his five different proofs of the theorem. Finally, this result must have been accecpted. NIELS HENRIK ABEL gave another proof in 1826.

I describe below, necessarily very shortly, the development of beginning of algebra, based on selected algebraical treatises. I think that most of them played an important role in the periods discussed. For this reason I have omitted less important algebraical publications. This presentation will explain that, for a long time, until the end of the XVIII century, algebra was only a language for geometry, very useful in the applications, but only a language. It reminds one of the situation with set theory at the beginning of the XX century: set theory was then a good language in mathematics, but only few were convinced that it was, in fact, a new branch of mathematics.

2 Treatises of the Antiquity (Euclid, Diophantus)

It is well known that in EUCLID's *Elements* [1], elements of algebra appeared for the first time. EUCLID tells about proportions of magnitudes and he performs operations on them. EUCLID presents the well-known contemporary algorithms or formulae, e.g. the formula for the square of the sum (binomial formula) in geometric forms. EUCLID formulates and proves the properties of operations on ratios, i.e. fractions, only later, in the work $\triangle ATA$, saved only in the small part [2]. Evidently, the proofs are geometric, but the contents of theorems is purely algebraic. The range of algebra contained in *Elements* was enough for many centuries.

DIOPHANTUS' *Arithmetic* [3] brought the next step in the development of algebra. The treatise was probably written in the middle of the III century, perhaps later. The elements of algebra contained there are based on arithmetic, not on geometry, as was common among his mathematical predecessors. DIOPHANTUS introduced negative numbers, and applied literal symbols. Solutions of exercises in [3] prove that he well knew the fundamental properties of polynomials and he was able to make fundamental operations on them. He was solving algebraical equations by transforming them: one can add any magnitude to both sides of the equation, and one can also multiply the equation by it. There is no algebraical formalism there, but the description of the algorithms shows that it is indeed just algebra, although only verbal. Unfortunately, the new epoch was coming.

Tadeusz Zieliñski wrote in his *History of the antique culture* (1920):

> *Ceasar Iustinian, with his religious eagerness, expelled the last teachers of the pagan Academy from Athens in the year 529, and eo ipso he closed the Academy, after more than nine centuries of its famous life.*

Boethius (475–526), the last great judge of the Greek literature, and also a mathematician, lived in Rome then. His influence on mathematics was not very significant. He wrote only some comments on Euclid's work. He probably wrote a text on arithmetic following some ideas of Diophantus, but the text has not survived.

Greek mathematics was forgotten for many centuries. Only the islamic science discovered and saved many ancient, mainly Greek treatises. Europe did this as late as in the XIV century. Many editions of Euclid, Archimedes, Apollonius, Ptolemy, and Diophantus appeared in Europe starting from the XV century.

3 Islamic algebra (Al-Khwārizmī, Omar Khayyam, al-Kashi)

Al-Khwārizmī rediscovered some algebraical knowledge of Diophantus. It seems that he did not know the treatises of Diophantus. Six centuries were over, Hellenistic civilization disappeared, and Greek was not so popular as in the centuries before. However, one cannot deny that al-Khwārizmī knew the achievements of Diophantus. The fundamental mathematical treatises of al-Khwārizmī are *Arithmetic* and *Al-jabr w'al muqâbala li-Muhammad ibn Musa al-Kwarizmi* [4] (see also [29]). His *Arithmetic*, although fundamental for the history of mathematics, is a standard text. It describes decimal system and fundamentals of arithmetic of integers. Similar texts existed in Indian mathematics two or three centuries earlier. His *Algebra* is much more important. It contains an algebraical part and a practical part (*The Book of Inheritances*).

In the algebraical part, al-Khwārizmī considers six types of equations, very important in his presentation:

1. Squares equal to roots: $ax^2 = bx$

2. Squares equal to number: $ax^2 = c$

3. Roots equal to number: $ax = c$

4. Squares and roots equal to number: $ax^2 + bx = c$

5. Squares and number equal to roots: $ax^2 + c = bx$

6. Roots and number equal to squares: $bx + c = ax^2$

In the equations, a, b, c denote positve numbers. AL-KHWĀRIZMĪ formulates the problems and algorithms verbally. Zero and negative solutions are not allowed by him. His algorithms are purely algebraic. He avoids geometrical methods. Every type of equation is illustrated by suitable examples. Algorithms for solving quadratic equations (4-6) reduce the equation to full squares by suitably complementing the equation. However, in some examples, he gives geometric interpretation.

For example, the equation x^2 is solved graphically by two methods. In both cases, he completes the left-hand side of the equation to a square. He adds to both sides of the equation the number $(\frac{10}{2})^2$ and obtains

$$x^2 + 2 \times \frac{10}{2}x + (\frac{10}{2})^2 = (x + \frac{10}{2})^2 = 39 + 25 = 64,$$

which implies that $x + 5 = 8$, i.e. $x = 3$.

The Persian poet, islamic mathematician, astronomer etc., OMAR KHAYYAM considered the equations of degree three. He gave their geometric interpretation ([6],[7]). OMAR KHAYYAM, following the ideas of AL-KHWĀRIZMĪ, divided the equations of degree three into 25 types. Thus the classification contains all types of equations of degree up to three. For example, in the case of the equation $x^3 + bx = a$, he proves geometrically that its solution x is the abscissa of the point of intersection of the circle $x(\frac{a}{b} - x) = y^2$ (i.e. the circle $(x - \frac{a}{2b})^2 + y^2 = (\frac{a}{2b})^2$) with the parabola $\sqrt{b}y = x^2$. In other words, finding (positive) roots of a polynomial of degree three is reduced to the solution of two eqautions, each of which represents a conic section. Three and a half centuries later, AL KASHI tried to find a similar classification for the equations of degree four. He wrote in [10] that there are 65 types of such equations, but in fact there are 70 of them. He probably also proved that the solutions can be obtained by finding the intersection of suitably choosen conic sections. However, we know neither his paper, nor his proofs.

The idea that the solution of an algebraic equation can be reduced to determination of the intersection points of suitably choosen algebraical curves, alived in Europe in the XVII and XVIII centuries.

4 Indian algebra (Mahaviracarya)

We can find the beginnings of algebra among the peoples of India in the treatise of Aryabhata *Aryabhatiya* (499) and in the works of Brahmagupta from the first half of the VII century. Algebra in the "adult" form is covered by Mahaviracarya in [5]. The treatise was written at the same time as *Al-jabr w'al muqbala*. There is small probability that one of the authors knew the work of the other. Although [5] deals with arithmetic and geometry, it contains also some elements of verbal algebra. In Chapter II the following operations are introduced successively: addition, division, squaring, extracting square roots, raising to third power, extractig cubic roots, addition, and subtraction. The author describes suitable algorithms and the order in which the operations are to be used. He describes also how to solve quadratic equations. Elements of algebra appear also in the formulae for the square of a sum of arbitrary many elements and for the sum of consecutive elements of arithmetic and geometric progressions. The formulae are presented verbally.

5 Medieval Europe (Leonardo Pisano, Jordanus Nemorarius)

They both died almost at the same age and lived in the XII/XIII century. Their main works, Leonardo Pisano's *Liber abaci* [8] and Jordanus Nemorarius' *De numeris datis* [9] were written at the beginning of the XIII century. Both treatises present elements of arithmetic, decimal system and element of elementary algebra: quadratic equations, linear equations and their systems. However, [8] has practical sense. It contains many exercises taken from real live. The best known exercise asks for the number of rabbits:

> *At first we have one pair of rabbits. How many pairs of them there will be after twelve months if every pair of them procreates another one in a month?*

The term *res* (a thing) denotes an unknown quantity in [8], and *census* (a fortune) denotes the square of the unknown. Leonardo used letters, but only abbreviated notation for numbers. He did not build any algebraical formalism. On examples, Leonardo described algorithms for solving systems of linear equations, quadratic and biquadratic equations. In the last case he not only gave an irrational solution but he also presented its rational approximation. Leonardo found it very precisely.

NEMORARIUS, contrary to PISANO, had no practical exercises. His presentation of algebra is similar as in [8], but not so long. He used letters, similarly as LEONARDO. NEMORARIUS called the unknown quantity as *numerus* (number), and the given number (any coefficient of an equation) he called *numerus datus* (given number). In his manuscript, NEMORARIUS described the algoritms for solving square equations on examples. He solved also some linear equations and mixed systems of two equations with two variables, one of which is quadratic.

6 Europe of XVI century (Stifel, Cardano, Tartaglia, Candella, Salignac, Viete)

Three hundred years later, MICHAEL STIFEL wrote his *Arithmetica Integra* [11]. Its title suggests that it is a textbook on arithmetic. However, the book contains also some elements of algebra. A lot of space in his book is occupied by the exposition of rational and irrational numbers. On page 104, we read: *Ordo fractorum inter 2 & 3.* Stifel orders the rational numbers between 2 and 3 in the following way:

$$2\frac{1}{2}\ 2\frac{1}{3}\ 2\frac{1}{4}\ 2\frac{3}{4}\ 2\frac{1}{5}\ 2\frac{2}{5}\ 2\frac{3}{5}\ 2\frac{4}{5}\ 2\frac{1}{6}\ 2\frac{5}{6}\ 2\frac{1}{7}\ 2\frac{2}{7}\ 2\frac{3}{7}.$$

And so consecutively to infinity.

In other words, STIFEL proves for the first time that there are countably many rational numbers in the interval $(2,3)$. It seems that he also could have been able to prove a more general statement that the set of rationals in any interval (a,b) with a and b rational is countable. Similarly, he shows that there exist infinitely many irrational numbers in $(2,3)$. Indeed, he states without proof that the irrational numbers

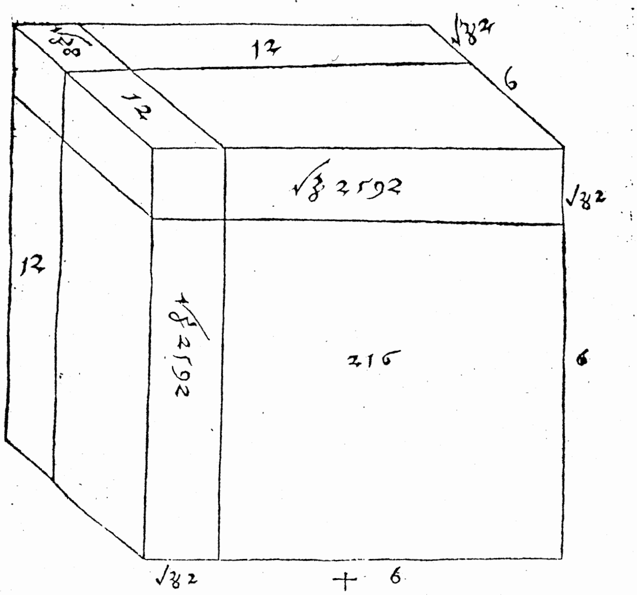
$$\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt[3]{9}, \sqrt[3]{10}, \sqrt[3]{11}, \sqrt[3]{12}, \sqrt[3]{13}, \sqrt[3]{14}, \sqrt[3]{15}, \sqrt[3]{16}, \sqrt[3]{17}, [\ldots],$$

$$\sqrt[3]{24}, \sqrt[3]{25}, \sqrt[3]{26}, \sqrt[4]{17}, \sqrt[4]{18}, [\ldots], \sqrt[4]{26}$$

all lie in $(2,3)$.

In Chapters IX-XI (Liber II, 122–132), he gives examples of arithmetical operations on numbers of the form $a+b\sqrt{c}$ with a,b,c rational. In fact, he proves that, in our contemporary algebraic terminology, such numbers form a field for a given c. Instead of the binomial formulae for $(x+y)^n$ $(n=2,3)$, he applies two-dimensional picture in the case $n=2$ (PYTHAGORAS) and three-dimensional picture (a cube with a side

ARITHMETICAE LIBER III. 130.



Binomium autem illud contractum ad cubum, cernitur sub istis particulis compositionis.

√ʒ8	12	√ʒ2592	216
	12	√ʒ2592	
	12	√ʒ2592	

Cōpositurꝰ igit̃ regulā de extractionibꝰ cubicis ex binomijs & residuis, respiciat ad distractionē particularꝝ positam, & sciat esse ꝓportionalitatē continuā inter 4 supremas particulas, scilicet inter √ʒ8 & 216, sunt 12 & √ʒ2592, duo media ꝓportionalia, sciatq; particulas mediales ad se additas, perficere portionē

K ij binomij.

Page 130 from *Arithmeticae Liber*

$x + y$ divided by suitable planes) discovered for the first time probably either by HERON OF ALEXANDRIA or even earlier (*Arithmeticae Liber*, page 130). The idea of calculating $(a + b)^3$ by cutting the cube with sides $(a + b)$ by planes parallel to its bases and summing the volumes of rectangular parallelepipeds obtained in this way goes back to HERON OF ALEXANDRIA (I century). HERON applied this idea to calculating cubic roots. The idea was used by the Italian mathematicians GIROLAMO CARDANO and NICCOLO TARTAGLIA ([12]–[14]). If e.g. $a = x + b$, then $(x+b)^3 = x^3+3x^2b+b^3 = x^3+3bx(x+b)+b^3$, i.e. $x^3+3abx+(b^3-a^3) = 0$. Comparing this with the equation of degree three in the general form x^3+px+Q $(p, q > 0)$, it is not difficult to find the formulae for its roots. This was exactly the idea of CARDANO and TARTAGLIA (loc. cit.). Since they applied only positive numbers, they had to consider also other types of equations, e. g. $x^3 + px = q$. They both used very complicated notations. Their notations came from abbreviations of suiable words, but were essentially different. CARDANO wrote in Latin, TARTAGLIA in Italian and thus, their notations were also different. TARTAGLIA [14] wrote binomial coefficients in a triangle form. They are now unjustly called Newton coefficients and the triangle Pascal triangle.

Algebraical symbolism started to appear. For example, FRANCISCO CANDALLA [15] used the old terminology of ARCHIMEDES, but his notations are new. CANDALLA denoted the equality $A : B = C : D$ by the symbol $ABCD$. $ACBD$ is *Permutata ratio*, $BADC$ is *conversa ratio*, and *composita ratio* means $AB\,B\,CD\,D$, that is the ratio $(A + B) : B = (C + D) : D$.

BERNARD SALIGNAC [16] distinguished already precisely arithmetic from algebra: *ARITHMETICA est ars numerorum (aritmetic is the art of numbers)*; but: *ALGEBRA est numerorum figuratorum Arithmetica (algebra is the arithmetic of figurative numbers).* He used symbols of addition and subtraction (+ and −) and abbreviations q (*square* from *quadratus*), c (*cube* from *cubus*), bq (*the fourth power – biquadratus*) and so on. The symbol $10bq + 4l - 4$, for example, denotes the polynomial $10x^2 + 4x - 4$. In the second part of the book SALIGNAC writes: *Secunda pars Algebrae AEquationem Algebraicam docet (the second part of the algebra teaches about equations).* He did not have the equality and the multiplication symbols yet.

Only VIETE ([17]–[18]) consequently started to use literal notations, calling DIOPHANTUS his predecessor. In the paper [17], VIETE writes: *Magnitudinum Scalarium prima est Latus, seu Radix. 2. Quadratum. 3. Cubus. 4. Quadrato-quadratum* [...] *(The first scalar magnitude is*

LA SESTA PARTE DEL
GENERAL TRATTATO
DE' NVMERI, ET MISVRE,
DI NICOLO TARTAGLIA;
NELLA QVALE SE DELVCIDA QVELLA ANTICA
PRATICA, SPECVLATIVA DE L'ARTE MAGNA,
DETTA IN ARABO ALGEBRA, ET ALMVCABALA, OVER
REGOLA DELLA COSA TROVATA DA MAVMETH
FIGLIVOLO DI MOISE ARABO,
LA QVALE SE PVO DIRE LA PERFETTA ARTE DEL
calculare, perche la supplisse, & serue, per risoluere infiniti casi, ouer
questioni, si in Geometria, come in Arithmetica, che alcuna
delle altre regole (fin'hora dette) non potria seruire.

AGIONTOVI IN FINE MOLTI QVESITI RISOLTI
per Algebra, si in Arithmetica, come in Geometria.

IN VENETIA PER CVRTIO TROIANO. M. D. LX.

Title page of Tartaglia's
La sesta parte del general trattato

14 HIERONYMI CARDANI

Capitula primitiua carentia deriuatiuis.

Numerus ęqualis rebus, uel numerus æqualis q̃dratis, uel numerus æqualis cubis, uel numerus æqualis q̃d' q̃dratorum, uel numerus æqualis nomini seu relato primo, ac ita deinceps comparando numerum cuicunq; denominationi.

Numerus & q̃drata æqualia rebus, uel numerus & cubus æq̃lia rebus, uel numerus & cubus ęqualia quadratis, uel numerus & q̃d' quadrata æqualia rebus, uel numerus & quadrati quadrata æqualia quadrato, uel numerus & q̃d' quadrata ęqualia cubis, uel numerus & nomẽ primũ æq̃lia reb. aut q̃dratis aut cubis & sic absq; fine.

Numerus & positio, & ignota quantitas.

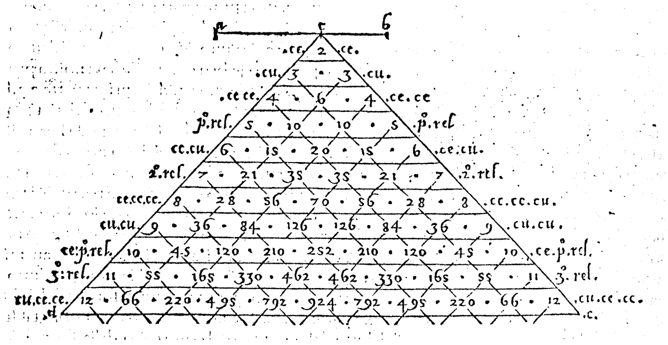
Numerus et q̃dratũ positionis, & ignota quantitas, seu numerus & q̃dratũ quãtitatis ignotę & positio, seu numerus cũ q̃drato positionis quantitatis ignotæ, seu numerus & productũ ex positione in quantitatem ignotam, cũ altera earũ, uel cum quadrato unius earũ.

Capitula primitiua.		Capitula deriuatiua.
1 Numerus ęqualis quadrato & rebus.	1	Numerus æqualis q̃d' q̃d' & q̃d.
	2	Numerus æqualis cu' quad' & cub'.
2 Numerus & res æqualia quadrato.	3	Numerus & quad' æquales q̃d' q̃d'.
	4	Numerus & cub' æquales cub' q̃d'.
3 Numerus & q̃d' æqualia rebus.	5	Numerus & q̃d' q̃d' ęqualia quad'.
	6	Numerus & cub' quad' ęqualia cub'.
4 Numerus æqualis cubo & rebus.	7	Numerus ęqualis quad' & cub' q̃d'.
	8	Numerus ęqualis cub' & cubo cubi.
5 Numerus & res æqualia cubis.	9	Numerus & quad' ęqualia cub' q̃d'.
	10	Numerus & cub' ęqualia cub' cub'.
6 Numerus & cub. æq̃l' reb. æquatio prima.	11	Numerus & cub' q̃d' ęq̃l' q̃d' ęq̃° pri^a.
	12	Num. & cub' cub' ęq̃lia cub' ęq̃° pri^a.
7 Numerus & cub' æq̃alia rebus æq̃° secũda.	13	Num. & cub' q̃d' ęq̃lia q̃d' ęq̃° secũda.
	14	Nu. & cub' cub' ęq̃lia cub' ęq̃° secũd^a.
8 Numerus ęqualis quadrato & cubo.	15	Num. ęqualis q̃d' q̃d' & cub' q̃drat^i.
	16	Num. ęqualis cub' quad' et cub' cub^i.
9 Numerus & quadratũ æqualia cubo.	17	Num. & q̃d' q̃d' ęqualia cub' q̃drat^i.
	18	Num. et cub' quad' ęqualia cub' cubi.
10 Numerus & cub' ęq̃lia q̃drato ęq̃° prima.	19	Num. et cub' q̃d' ęq̃lia q̃d' q̃d' ęq̃° pri^a.
	20	Nu. & cub' cub' ęq̃lia cub' q̃d' ęq̃ pri^a.
11 Numerus & cub' ęq̃lia quadrato ęq̃° secũda.	21	Nu. & cub' q̃d' ęq̃l' q̃d' q̃d' ęq̃° secũda.
	22	Nu. & cub' cub' ęq̃l' cu' q̃d' ęq̃° secũd.
12 Numerus ęq̃lis rebus quadrato & cubo.	23	Nu. ęq̃lis q̃d' & q̃d' q̃d' & cub' quad'.
	24	Nu. ęq̃lis cub' & cub' q̃d' & cub' cu'.

13 Nume-

A page from Cardano's book

Artis Magnae, sive de regvlis algebraicis



Tartaglia's presentation of Pascal triangle

a side i.e. radius, the second one is square, the third a cube & c). Next he writes that the Latin letters $A, B, C, D, \ldots$ will denote the known quantities and letters $\ldots, W, X, Y, Z$ will denote indeterminates. In the book [18], Viete writes: $A2.B3.C4.D6.$, which means that $A : B = C : D$ for the given values. In another place (loc. cit. p. 55) he remarks that *$Aq - Dp$ est propos estre egal a $Gq - BA$,* [...] *et restera $Aq + BA$ egal a $Gq + Dp$.* It means that the equality $A^2 - D = G^2 - BA$ implies the equality $A^2 + BA = G^2 + D$. The proof occupies 12 lines in the text!

7 Europe of XVII century (Peletier, Stevin, des Cartes, Wallis)

Peletier's book [19] is very similar to Stifel's treatise [11]. Some parts of [19] are identical with respective parts of [11]. One even can say that [19] is a free translation of [11]. Stevin [20] used geometrical terminology of Viete as well as his literal notations. He used the common notation for polynomials of one variable and for rational numbers written in the decimal notation. Many authors followed his notation. He also studied the arithmetic of quadratic irrationalities. In many books from the first half of the XVII century we can find many examples and many properties of such numbers.

However, Descartes' treatise *Géométrie*, written in 1637, became a revolution in mathematics. It had many reeditions with numerous comments, e.g. [21]. Schooten was its main commentator. He prepared many editions of *Géométrie*. The treatise has only 106 pages. The other 836 pages (the edition from 1683) are comments and modifications of

Géométrie by other authors. We can mention here that DESCARTES introduces equations in concrete geometrical situations, applying coordinates, but not necessarily orthogonal. Moreover, the coordinates do not have to be the functions of the point. Only SCHOOTEN in the second half of the XVII century used coordinates in the form which we often attribute to DESCARTES. Algebraical notations of DESCARTES are almost the same as today. Only the equality symbol was essentially different. Although DESCARTES' book deals with geometry, it contains also about 30 pages of algebra, i.e. the algebra of polynomials of one variable. On a few examples, he remarked that a is a root of a polynomial f, if and only if $x - a$ divides f in the ring of polynomials. The theorem is unjustly called *Bezout theorem*, although BEZOUT rediscovered the theorem only a hunderd years later, also without any proof. DESCARTES remarked that if f is a monic polynomial with integer coefficients then its rational roots are divisors of $f(0)$. This theorem is also attributed to other persons.

The author of modern algebra is JOHN WALLIS. It can be proved by examining his treatises [22] and [23]. His treatise on algebra [23] is very interesting. This monumental work contains all algebraical knowledge up to his epoch. WALLIS introduces operations on polynomials. He defines inequalities and proves their fundamental properties. Moreover, he defines ratios between magnitudes, following the Greek tradition. He uses the symbol of the eqaulity, but in the case of proportions he writes, as for centuries, the equality of ratios in the form :: . His *Algebra* contains also the *infinitary arithmetic (arithmetica infinitorum).* Infinitary arithmetic presents different kinds of expressions of numbers, such as infinite products, series and continued fractions. WALLIS recalls by occasion some classical notions of the limit in the geometrical language, called as *Methodus Exhaustionum*, and CAVALIERI's *Methodus Indivisibilium.* WALLIS also finds an approximate rectification of an arc of a circle. He uses power series expansions of functions in his construction. Algebra enetered the XVIII century with rather wide knowledge.

8 Europe of XVIII century (Gvisnee, de l'Hospital, Newton, Sounderson, Euler)

Mathematicians of the XVIII century fixed and extended the results of their predecessors. GVISNEE [24] extended the ideas of DESCARTES, giving the theory of conic sections in a very elegant form. He proved that trisection of angle leads to the construction of a root of equation of degree three, so he claims that *it cannot be reduced to equations of degree two, consequently the problem is a space problem.* Thus for the first time P.-L. Wantzel theorem was formulated (1837), yet without proof, but already well motivated. Gvisnee also proved that trisection can be done with the help of a parabola.

The treatise [25] was published after de l'Hospital's death and is written in a very elegant style, similarly to [24]. Although [25] is geometrical treatise, the last part of the book deals with algebra.

De l'Hospital proves geometrically that algebraical equations of degree $5, 6, 7, 8$, and 9 can be solved by conic sections and some curves called parabolas by him. Every solution of such equation is either abcissa or ordinate of the intersection point of the above mentioned curves: conic sections and (generalized) parabolas (curves with equations of the form $y = x^m$ for some positive integer m). In this way European mathematics came back to the idea of Omar Khayyam (compare [6]), but in a much more general context.

Newton's treatise [26] from the year 1707 is known rather well (see [30], [32]). Since its contents is described in available sources I recall only that [26] is the textobook of arithmetic, algebra and analytical geometry, using rather archaic (even in his epoch) notation.

Nicholas Saunderson, blended mathematician from Cambridge, forty-one years younger than Newton, wrote the monumental treatise [27], published one year after his death. The textbook of classical algebra [27] (748 pages) is written in a lengthy way, but can be read without difficulties today. Very detailed text contains algebra of polynomials of one variable, algorithms for releasing an irrationality in denominator, solving equations of degree two, three, and four, decomposition of rational functions into primary fractions etc. The proofs and applications of algebra invoke geometry. In particular, the book contains a portion of analytic geometry.

Euler's treatise [28] was written in a different style. The first volume presents fundamentals of arithmetic and algebra. The arithmetic is taken as the ground for the presentation. For the first time, no geometrical arguments were used in the proofs of theorems from arithmetic and algebra. The second volume, contrary to the convention in the first one, is far from being elementary. It contains the theorems of classical algebra (solving equations of degree 3 and 4) as well as interesting theorems from number theory. Euler uses complex numbers of special types in proofs of this part (some rings of algebraic numbers, e.g. the complex numbers of the form $n + im$ with n, m integer). Short fragments of [28] in Polish can be found in [32].

9 Final remarks

A short description of the oldests algebraical treatises presented here is only a contribution to the history of algebra. However, it presents a possibility to look at the development of algebra slightly differently than before. Up to the XVIII century algebra was only the language of geometry, notwithstanding many efforts to be independent of geometry. In the XVIII century algebra finally became independent, although the connections betweeen the two domains remained and are still strong. Algebra and geometry penetrated each other starting from that time. It is still very useful for both domains.

References below are arranged in chronological order. This bibliography is far from being complete. In particular, I quote only selected editions of the described treatises even in the case when there were many editions of the book during centuries. My main idea was to show how some notions of algebra were developing. I did not plan to give the full list of classical treatises from algebra. For rather complete bibliography, see Rider [31].

The titles of the books are cited in the same way as in their original editions, with some abbrevations denoted in the text by [...].

References

[1] Euclid. *Elements.* (about 300 B.C.) ¿From more than fifty editions I will mention only a few: EUCLIDES, edita opera M.J.Barrow, Londini 1675. EUCLIDIS. *Elementorum libri XV Accessit liber XVI de quinque solidorum regularium inter se compartione.* Köln 1607. *Euklidesa Początków Geometryi Ksiag Ośmioro, toiest sześć pierwszych, iedenasta i dwunasta z dodanemi przypisami* [...] przez Józefa Czecha, Wilno 1817. [edition II].

[2] Evclidis data, Opvs ad vetervm geometriae, lvtetiae Parisiorvm, Impensis Melchioris Mondiere, Anno MDCXXV.

[3] Diophantus. *Arifmetika i kniga o mnogougolnych čislach.* Moskwa 1974. [Russian]

[4] Muhammad ibn Musa al Chorezmi, *Matematičeskie Traktaty.* Tashkent 1983. [Russian]

[5] The Ganita-Sara-Sangraha of Mahaviracarya with English translation and notes by M. Rangacarya, Madras 1912.

[6] Omar Khayyam. Matematičeskie Traktaty. *Istoriko-matematičeskie issledovania*, vypusk VI (1953), 11–168. (Russian) [the treatise written in 1064–1071]

[7] Omar Khayyam. Piervyj ałgebraičeskij traktat, *Istoriko-matematičeskie issledovania*, vypusk XV (1963), 445–472. (Russian)

[8] Leonardo Pisano. *Liber Abbaci* 1202. [in: *Scritti di Leonardo Pisano*, volume I, publicati di Baldassare Boncompagni, Roma 1857].

[9] Jordanus Nemorarius. *De Numeris Datis. Arithmetica in Decem Libris Demonstrata.* [XII/XIII century]

[10] Džemšid Gijas ad-Din al-Kaši, *Kluč Arifmetiki* [1427], Moskwa 1956. [Russian]

[11] Arithmetica integra, Authore Michaele Stifelio. Cum praefatione Philippi Melanchtchonis, Norimbergae apud Iohan. Petreium. Anno Christi M. D. XLIIII.

[12] Hieronymi Cardani, *Artis Magnae, sive de regvlis algebraicis,* M. D. XLV.

[13] Hieronymi Cardani, *Operis perfecti svi sive algebraicae logisticae* [. . .]

[14] General trattato di nvmeri, et misvre di Nicolo Tartaglia, M D LVI. [. . .] La sesta parte del General Trattato de nvmeri [. . .] detta in arabo algebra, et almvcabala [. . .] M. D. LX.

[15] Francisco Flussate Candalla, *Evclidis Megarensis Mathematici clarissimi Elementa geometrica,* Libris XV [. . .] Parisiis [. . .] 1566.

[16] Bernardi Salignaci Bvrdegalensis *Arithmeticae libri duo, et algebrae totidem, cum demonstrationibus.* Francofurti. M D LXXX.

[17] Francisci Vietae in Artem analyticem isogage. Seorsim excussa ab Opere restitutae Mathematicae Analyseos, Seu, Algebr nouv. TVRONIS, Anno 1591.

[18] Francisci Vietae, *Introdvction en l'art analytic, ov Novvelle algrbre de Francois Viete* [. . .] Paris [. . .] Anno M. DC. XXX.

[19] L'Alagebre de Iaques Peletier. Departie endeux liures: Par Iean de Tovrnes. M. DC XX.

[20] Les oeuvres Mathematiques de Simon Stevin de Bruges. A Leyde. Anno MDCXXXIV.

[21] Francisci a Schooten *Principia matheseos vniversalis, sev introductio ad geometriae methodum Renati des Cartes* [. . .] Lvdg. Batav. MDCLI. Renati Des-Cartes Geometria. Editio tertia. [. . .] M DC LXXXIII.

[22] Johannis Wallis S.T.D. [. . .] *Opera mathematica Tribus Voluminibus contenta.* OXONIAE MDCXIX.

[23] Johannis Wallis, *De algebra tractatus* [. . .] OXONIAE MDCXCIII.

[24] Application de l'algebre a la geometrie ov methode de de'montrer par l'algebre [...] Par Gvisne'a [...] A PARIS MDCCV.

[25] Trait analytique des sections coniques et de leur usage pour la resolution des equations [...] Ouvrage posthume De M. le Marquis de L'Hospital [...] A Paris [...] M. DCC. XX.

[26] Isaac Newton, *Aritmetica Universalis, sive de compositione et resolutione liber,* Londini 1707.

[27] Nicholas Saunderson, *The elements of algebra in ten books*: [...] CAMBRIDGE [...] MDCCXL.

[28] Leonhard Euler. *Vollständige Anleitung zur Algebra.* St. Petersburg 1770.

Other references:

[29] J.Baštinec, Z.Kubištová, *Muhammad ibn Musa al-Chorezmi*, in Dějiny matematiky, svazek 11, Matematika v proměnách věků I, Praha 1998, pp. 125-141.

[30] J.Bečvář, *Algebra v 16. a 17. století*, in Dějiny matematiky, svazek 12, Matematika v 16. a 17. století, Praha 1999, pp. 161-235.

[31] R. E. Rider. *A bibliography of early modern algebra, 1500-1800.* Berkeley Papers in History of Mathematics, University of Califiornia, Berkeley, 1982.

[32] Witold Więsław. *Matematyka i jej historia.* Wydawnictwo NOWIK, Opole 1997.

Witold Więsław
Instytut Matematyczny
Uniwersytetu Wrocławskiego
plac Grunwaldzki 2/4
50-384 Wrocław
Poland
e-mail: wieslaw@math.uni.wroc.pl