Chapter 7

Complements

This final chapter contains additional remarks on product integration theory. The topics discussed here complement the previous chapters; however, most proofs are omitted and the text is intended only to arouse reader's interest (references to other works are included).

7.1 Variation of constants

Product integral enables us to express solution of the differential equation

$$y'(x) = A(x)y(x), \quad x \in [a,b],$$

where $A : [a,b] \to \mathbf{R}^{n\times n}$, $y : [a,b] \to \mathbf{R}^n$. The fundamental matrix of this system is

$$Z(x) = \prod_a^x (I + A(t)\,\mathrm{d}t) = \begin{pmatrix} z_1^1(x) & \cdots & z_n^1(x) \\ \vdots & \ddots & \vdots \\ z_1^n(x) & \cdots & z_n^n(x) \end{pmatrix}$$

and its columns

$$z_i(x) = \begin{pmatrix} z_i^1(x) \\ \vdots \\ z_i^n(x) \end{pmatrix}, \quad i = 1, \ldots, n \tag{7.1.1}$$

thus provide a fundamental system of solutions.

We now focus our attention to the inhomogeneous equation

$$\begin{aligned} y'(x) &= A(x)y(x) + f(x), \quad x \in [a,b], \\ y(a) &= y_0. \end{aligned} \tag{7.1.2}$$

A method for solving this system using product integral (based on the well-known method of variation of constants) was first proposed by G. Rasch in the paper [GR]; it can be also found in the monograph [DF].

We assume that the functions $A : [a,b] \to \mathbf{R}^{n\times n}$ and $f : [a,b] \to \mathbf{R}^n$ are continuous, and we try to find the solution of (7.1.2) in the form

$$y(x) = \sum_{i=1}^n z_i(x)c_i(x), \tag{7.1.3}$$

where $c_i : [a,b] \to \mathbf{R}$, $i = 1, \ldots, n$ are certain unknown functions. If we denote

$$c(x) = \begin{pmatrix} c_1(x) \\ \vdots \\ c_n(x) \end{pmatrix},$$

then the equations (7.1.1) and (7.1.3) imply

$$y(x) = Z(x)c(x).$$

We obtain

$$y'(x) = Z'(x)c(x)+Z(x)c'(x) = A(x)Z(x)c(x)+Z(x)c'(x) = A(x)y(x)+Z(x)c'(x),$$

and using Equation (7.1.2)

$$f(x) = Z(x)c'(x).$$

Consequently

$$c'(x) = Z(x)^{-1}f(x),$$
$$c(a) = Z(a)^{-1}y(a) = y_0,$$

which implies

$$c(x) = y_0 + \int_a^x Z(t)^{-1}f(t)\,\mathrm{d}t.$$

The solution of the system (7.1.2) is thus given by the explicit formula

$$y(x) = Z(x)c(x) = Z(x)y_0 + Z(x)\int_a^x Z(t)^{-1}f(t)\,\mathrm{d}t =$$

$$= \prod_a^x (I + A(t)\,\mathrm{d}t)y_0 + \prod_a^x (I + A(t)\,\mathrm{d}t)\int_a^x \left(\prod_t^a (I + A(s)\,\mathrm{d}s)f(t)\right)\mathrm{d}t =$$

$$= \prod_a^x (I + A(t)\,\mathrm{d}t)y_0 + \int_a^x \left(\prod_t^x (I + A(s)\,\mathrm{d}s)f(t)\right)\mathrm{d}t.$$

We summarize the result: The solution of the inhomogeneous system (7.1.2) has the form

$$y(x) = \sum_{i=1}^n z_i(x)c_i(x),$$

where $z_1,\ldots,z_n : [a,b] \to \mathbf{R}^n$ is the fundamental system of solutions of the corresponding homogeneous equation, the functions $c_i : [a,b] \to \mathbf{R}$, $i = 1,\ldots,n$ are continuously differentiable and satisfy

$$\sum_{i=1}^n c_i'(x)z_i(x) = f(x), \quad x \in [a,b].$$

7.2 Equivalent definitions of product integral

Consider a tagged partition $D : a = t_0 < t_1 < \cdots < t_m = b$, $\xi_i \in [t_{i-1}, t_i]$, $i = 1,\ldots,m$. Ludwig Schlesinger proved (see Theorem 3.2.2) that the product integral of a Riemann integrable function $A : [a,b] \to \mathbf{R}^{n\times n}$ can be calculated not only as

$$\prod_a^b (I + A(t)\,\mathrm{d}t) = \lim_{\nu(D)\to 0} \left(\prod_{k=m}^{1} (I + A(\xi_k)\Delta t_k) \right),$$

but also as

$$\prod_a^b (I + A(t)\,\mathrm{d}t) = \lim_{\nu(D)\to 0} \left(\prod_{k=m}^{1} e^{A(\xi_k)\Delta t_k} \right).$$

The equivalence of these definitions can be intuitively explained using the fact that

$$e^{A(\xi_k)\Delta t_k} = 1 + A(\xi_k)\Delta t_k + O((\Delta t_k)^2),$$

and the terms of order $(\Delta t_k)^2$ and higher do not change the value of the integral. We have also encountered a similar theorem applicable to the Kurzweil and McShane integrals (see Theorem 6.2.4).

We now proceed to a more general theorem concerning equivalent definitions of product integral, which was given in [DF].

Definition 7.2.1. A function

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \tag{7.2.1}$$

is called admissible, if the series (7.2.1) has a positive radius of convergence $r > 0$ and

$$f(0) = c_0 = 1, \quad f'(0) = c_1 = 1.$$

For example, the functions $z \mapsto \exp z$, $z \mapsto 1 + z$ and $z \mapsto (1-z)^{-1}$ are admissible. For every matrix $A \in \mathbf{R}^{n\times n}$ such that $\|A\| < r$ we put

$$f(A) = \sum_{k=0}^{\infty} c_k A^k.$$

Theorem 7.2.2.[1] If f is an admissible function and $A : [a,b] \to \mathbf{R}^{n\times n}$ a continuous matrix function, then

$$\prod_a^b (I + A(t)\,\mathrm{d}t) = \lim_{\nu(D)\to 0} \left(\prod_{k=m}^{1} f(A(\xi_k)\Delta t_k) \right).$$

According to the previous theorem, the product integral of a function A can be defined as the limit

$$\lim_{\nu(D)\to 0} \left(\prod_{k=m}^{1} f(A(\xi_k)\Delta t_k) \right),$$

[1] [DF], p. 50–53

where f is an admissible function. Product integral defined in this way is usually denoted by the symbol $\prod_a^b f(A(t)\,\mathrm{d}t)$, e.g.

$$\prod_a^b (I + A(t)\,\mathrm{d}t), \quad \prod_a^b e^{A(t)\,\mathrm{d}t}, \quad \prod_a^b (I - A(t)\,\mathrm{d}t)^{-1}$$

etc. The integral $\prod_a^b e^{A(t)\,\mathrm{d}t}$ is taken as a primary definition in the monograph [DF]. We note that it is possible to prove an analogy of Theorem 7.2.2 even for the Kurzweil and McShane product integrals (see [JK, Sch1]).

7.3 Riemann-Stieltjes product integral

Consider two functions $f, g : [a,b] \to \mathbf{R}$. Then the ordinary Riemann-Stieltjes integral is defined as the limit

$$\int_a^b f(x)\,\mathrm{d}g(x) = \lim_{\nu(D)\to 0} \sum_{i=1}^m f(\xi_i)(g(t_i) - g(t_{i-1})), \tag{7.3.1}$$

where $D : a = t_0 < t_1 < \cdots < t_m = b$ is a tagged partition of $[a,b]$ with tags $\xi_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, m$ (provided the limit exists). This integral was introduced in 1894 by Thomas Jan Stieltjes (see [Kl], Chapters 44 and 47), who was working with continuous functions f and non-decreasing functions g. Later in 1909 Friedrich Riesz discovered that the Stieltjes integral can be used to represent continuous linear functionals on the space $\mathcal{C}([a,b])$. Also, if $g(x) = x$, we obtain the ordinary Riemann integral.

Assume that the function g is of bounded variation, i.e. that

$$\sup \left\{ \sum_{i=1}^m |g(t_i) - g(t_{i-1})| \right\} < \infty,$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_m = b$ of interval $[a,b]$. Then (see e.g. [RG]) the Riemann-Stieltjes integral exists for every continuous function f.

In particular, if f is continuous and g is a step function defined as

$$g = g_1\chi_{[t_0,t_1)} + g_2\chi_{[t_1,t_2)} + \cdots + g_{m-1}\chi_{[t_{m-2},t_{m-1})} + g_m\chi_{[t_{m-1},t_m]},$$

where $a = t_0 < t_1 < \cdots < t_m = b$, $g_1, \ldots, g_m \in \mathbf{R}$ and χ_M denotes the characteristic function of a set M, then

$$\int_a^b f(x)\,\mathrm{d}g(x) = f(t_1)(g_2 - g_1) + \cdots + f(t_{m-1})(g_m - g_{m-1}).$$

Now consider a matrix function $A : [a,b] \to \mathbf{R}^{n\times n}$. The product analogy of Riemann-Stieltjes integral can be defined as

$$\prod_a^b (I + \mathrm{d}A(t)) = \lim_{\nu(D)\to 0} \prod_{i=m}^{1} (I + A(t_i) - A(t_{i-1})) \tag{7.3.2}$$

(see e.g. [Sch3, GJ, Gil, DN]), or even more generally as

$$\prod_a^b (I + f(t)\mathrm{d}A(t)) = \lim_{\nu(D)\to 0} \prod_{i=m}^{1} (I + f(\xi_i)(A(t_i) - A(t_{i-1}))),$$

where $f : [a,b] \to \mathbf{R}$ (see the entry "Product integral" in [EM]). We now present some basic statements concerning the Riemann-Stieltjes product integral (7.3.2).

Product integrals of the type (7.3.2) are encountered in survival analysis (when working with the cumulative hazard $A(t) = \int_0^t a(s)\,\mathrm{d}s$ instead of the hazard rate $a(t)$; see Example 1.4.1) and in the theory of Markov processes (when working with cumulative intensities $A_{ij}(t) = \int_0^t a_{ij}(s)\,\mathrm{d}s$ for $i \neq j$ and $A_{ii}(t) = -\sum_{j\neq i} A_{ij}(t)$ instead of the transition rates $a_{ij}(t)$; see Example 1.4.2).

A sufficient condition for the existence of the limit (7.3.2) is again that the variation of A is finite. A different sufficient condition (see [DN]) says that the product integral exists provided A is continuous and its p-variation is finite for some $p \in (0,2)$, i.e.

$$\sup\left\{\sum_{i=1}^{m} \|A(t_i) - A(t_{i-1})\|^p\right\} < \infty,$$

where the supremum is again taken over all partitions $a = t_0 < t_1 < \cdots < t_m = b$ of interval $[a,b]$.

If $A : [a,b] \to \mathbf{R}^{n\times n}$ is a step function defined as

$$A = A_1\chi_{[t_0,t_1)} + A_2\chi_{[t_1,t_2)} + \cdots + A_{m-1}\chi_{[t_{m-2},t_{m-1})} + A_m\chi_{[t_{m-1},t_m]},$$

where $a = t_0 < t_1 < \cdots < t_m = b$ and $A_1, \ldots, A_m \in \mathbf{R}^{n\times n}$, then

$$\prod_a^b (I + \mathrm{d}A(t)) = (I + A_m - A_{m-1}) \cdots (I + A_2 - A_1). \tag{7.3.3}$$

Thus, if $A_{k-1} - A_k = I$ for some $k = 2, \ldots, m$, then

$$\prod_a^b (I + \mathrm{d}A(t)) = 0,$$

i.e. the product integral need not be a regular matrix. Equation (7.3.3) also suggests that the indefinite product integral

$$Y(x) = \prod_a^x (I + \mathrm{d}A(t)), \quad x \in [a,b],$$

need not be a continuous function.

If $A : [a,b] \to \mathbf{R}^{n\times n}$ is a continuously differentiable function, it can be proved that

$$\prod_a^b (I + \mathrm{d}A(t)) = \prod_a^b (I + A'(t)\,\mathrm{d}t).$$

As we have seen in the previous chapters, the Riemann product integral provides a solution of the differential equation

$$\begin{aligned} y'(x) &= A(x)y(x), \\ y(a) &= y_0, \end{aligned}$$

or the equivalent integral equation

$$y(x) - y_0 = \int_a^x A(t)y(t)\,\mathrm{d}t.$$

Similarly, the Riemann-Stieltjes product integral can be used as a tool for solving the generalized differential equation

$$\begin{aligned} \mathrm{d}y(x) &= \mathrm{d}A(x)y(x), \\ y(a) &= y_0, \end{aligned}$$

or the equivalent integral equation

$$y(x) - y_0 = \int_a^x \mathrm{d}A(t)y(t).$$

The details are given in the paper [Sch3].

There exists a definition of product integral (see [JK, Sch1, Sch3]) that generalizes both the Riemann and Riemann-Stieltjes product integrals: Consider a mapping V that assigns a square matrix of order n to every point-interval pair $(\xi, [x,y])$, where $[x,y] \subseteq [a,b]$ and $\xi \in [x,y]$. We define

$$\prod_a^b V(t, \mathrm{d}t) = \lim_{\nu(D)\to 0} \prod_{i=m}^{1} V(\xi_i, [t_{i-1}, t_i]),$$

provided the limit exists. The choice

$$V(\xi, [x,y]) = I + A(\xi)(y-x)$$

leads to the Riemann product integral, whereas

$$V(\xi, [x,y]) = I + A(y) - A(x)$$

gives the Riemann-Stieltjes product integral.

We note that it is also possible to define the Kurzweil-Stieltjes and McShane-Stieltjes product integrals (see [Sch3]), whose definitions are based on the notion of Δ-fine M-partitions and K-partitions (see Chapter 6); these integrals generalize the notion of Riemann-Stieltjes product integral.