Chapter 5

Product integration in Banach algebras

A final treatment of Riemann product integration is given in the article [Mas] by Pesi Rustom Masani; it was published in 1947. Consider a matrix-valued function $f : [a,b] \to \mathbf{R}^{n\times n}$ and recall that Volterra defined the product integral of f as the limit of products

$$P(f,D) = \prod_{k=m}^{1} (I + f(\xi_k)\Delta x_k)$$

corresponding to tagged partitions D of interval $[a,b]$. This definition is also applicable to operator-valued functions $f : [a,b] \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is the space of all bounded linear operators on a Banach space X. It is just sufficient to replace multiplication by composition of operators in the definition of $P(f,D)$; the role of identity matrix is now played by the identity operator I.

Masani's intent was to define the product integral of a function $f : [a,b] \to X$ for the most general space X possible. Let X be a normed vector space equipped with the operation of multiplication. Assuming there is a vector $1 \in X$ such that $1 \cdot x = x \cdot 1 = x$ for every $x \in X$ and $\|1\| = 1$, we let

$$P(f,D) = \prod_{k=m}^{1} (1 + f(\xi_k)\Delta x_k),$$

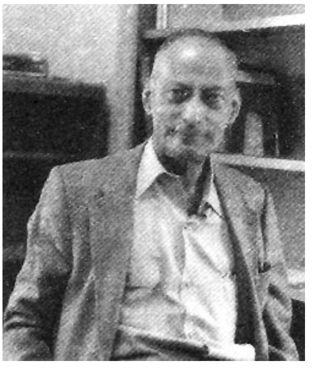
where D is an arbitrary tagged partition of $[a,b]$. We would like to define the product integral as the limit

$$\prod_a^b (1 + f(t)\,\mathrm{d}t) = \lim_{\nu(D)\to 0} P(f,D).$$

To obtain a reasonable theory it is necessary that the space X is complete, i.e. it is a Banach space.

Before giving an overview of Masani's result let's start with a short biography (see also [PRM, IMS]). Pesi Rustom Masani was born in Bombay, 1919. He obtained his doctoral degree at Harvard in 1946; the thesis concerned product integration in Banach algebras and it was supervised by Garrett Birkhoff [1]. During the years 1948–58 Masani held the chairs of professor of mathematics and science researcher in Bombay and then he returned to the United States. In the 1970's he accepted the position at the University of Pittsburgh. Masani was active in mathematics even after his retirement in 1989. He died in Pittsburgh on the 15th October 1999.

[1] G. Birkhoff also devoted himself to product integration, see [GB].



Pesi R. Masani[1]

Masani contributed to the development of integration theory, functional analysis, theory of probability and mathematical statistics. The appendix in [DF] written by Masani also concerns product integration. He collaborated with Norbert Wiener and edited his collected works after Wiener's death. Masani was also interested in history, philosophy, theology and politics.

5.1 Riemann-Graves integral

We begin with a brief recapitulation of facts concerning integration of vector-valued functions (see also [Mas]). The notion of Graves integral is a direct generalization of Riemann integral and was presented by Lawrence M. Graves in 1927.

Let X be a Banach space, $f : [a, b] \to X$. To every tagged partition $D : a = t_0 < t_1 < \cdots < t_m = b$ of interval $[a, b]$ with tags $\xi_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, m$ we assign the sum

$$S(f, D) = \sum_{i=1}^{m} f(\xi_i)\Delta t_i,$$

where $\Delta t_i = t_i - t_{i-1}$. We recall that if $T(D) \in X$ is a vector dependent on the choice of a tagged partition D, then

$$\lim_{\nu(D)\to 0} T(D) = T$$

means that to every $\varepsilon > 0$ there is $\delta > 0$ such that $\|T(D) - T\| < \varepsilon$ for every partition D of $[a, b]$ such that $\nu(D) < \delta$.

Definition 5.1.1. A function $f : [a, b] \to X$ is called integrable if

$$\lim_{\nu(D)\to 0} S(f, D) = S_f$$

[1] Photo from `http://www.york.ac.uk/depts/maths/histstat/people/`

for some $S_f \in X$. We speak of Riemann-Graves integral of function f on $[a, b]$ and denote $S_f = \int_a^b f(t)\,\mathrm{d}t$.

The following theorem provides additional two equivalent characterizations of integrable functions; recall that the notation $D' \prec D$ means that the partition D' is a refinement of partition D (see Definition 3.1.8).

Theorem 5.1.2. Let $f : [a, b] \to X$. The following statements are equivalent:

1) f is integrable and $\int_a^b f(t)\,\mathrm{d}t = S_f$.
2) Every sequence of partitions $\{D_n\}_{n=1}^{\infty}$ of $[a, b]$ such that $\nu(D_n) \to 0$ satisfies $\lim_{n\to\infty} S(f, D_n) = S_f$.
3) For every $\varepsilon > 0$ there is a partition D_ε of $[a, b]$ such that $\|S(f, D) - S_f\| < \varepsilon$ for every $D \prec D_\varepsilon$.

The proof proceeds in the same way as in the case when f is a real function. The rest of this section summarizes the basic results concerning the Riemann-Graves integral; again, the proofs can be carried out in the classical way.

Theorem 5.1.3. Let $f : [a, b] \to X$. Then the following statements are equivalent:

1) f is integrable.
2) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|S(f, D_1) - S(f, D_2)\| < \varepsilon$ whenever D_1 and D_2 are tagged partitions of $[a, b]$ satisfying $\nu(D_1) < \delta$, $\nu(D_2) < \delta$.

Theorem 5.1.4. If $f : [a, b] \to X$ is an integrable function, then it is bounded and

$$\left\| \int_a^b f(t)\,\mathrm{d}t \right\| \le (b - a) \sup_{t\in[a,b]} \|f(t)\|.$$

Theorem 5.1.5. Let $f : [a, b] \to X$. If the integral $\int_a^b f(t)\,\mathrm{d}t$ exists and if $[c, d] \subset [a, b]$, then the integral $\int_c^d f(t)\,\mathrm{d}t$ exists as well.

Theorem 5.1.6. Let $f : [a, c] \to X$, $a < b < c$. Suppose that the integrals $\int_a^b f(t)\,\mathrm{d}t$ and $\int_b^c f(t)\,\mathrm{d}t$ exists. Then the integral $\int_a^c f(t)\,\mathrm{d}t$ also exists and

$$\int_a^c f(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t + \int_b^c f(t)\,\mathrm{d}t.$$

The following two statements generalize the fundamental theorem of calculus to the case of vector-valued functions $f : [a, b] \to X$. The derivative of such a function is of course defined as

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

provided the limit exists (in the case when x_0 is one of the boundary points of $[a, b]$ we require only existence of the corresponding one-sided limit). Since X is a normed space, the last equation means that

$$\lim_{x\to x_0} \left\| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right\| = 0.$$

Theorem 5.1.7. Let $f : [a,b] \to X$ be integrable and put $F(x) = \int_a^x f(t)\,\mathrm{d}t$. If f is continuous at $x_0 \in [a,b]$, then $F'(x_0) = f(x_0)$.

Theorem 5.1.8. Let $F : [a,b] \to X$ and $F'(t) = f(t)$ for every $t \in [a,b]$. If f is an integrable function, then

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

Theorem 5.1.9. Let $f, g : [a,b] \to X$ be integrable functions, $\alpha, \beta \in \mathbf{R}$. Then

$$\int_a^b (\alpha f(t) + \beta g(t))\,\mathrm{d}t = \alpha \int_a^b f(t)\,\mathrm{d}t + \beta \int_a^b g(t)\,\mathrm{d}t.$$

The set of all integrable functions is thus a vector space; it is interesting to note that if the space X is equipped with the operation of multiplication (i.e. it is a Banach algebra, see the next section), then a product of two integrable functions need not be an integrable function. Another surprising fact concerning the Riemann-Graves integral is that every bounded function which is almost everywhere continuous is also integrable, but the converse statement is no longer true (it holds only in finite-dimensional spaces X).

5.2 Definition of product integral

Masani turns his attention to the product analogy of the Riemann-Graves integral. In the sequel we assume that X is a Banach algebra (Masani uses the term normed ring), i.e. that

1) X is a Banach space,
2) X is an associative algebra with a unit vector $1 \in X$, $\|1\| = 1$,
3) $\|x \cdot y\| \le \|x\|\|y\|$ for every $x, y \in X$.

The second condition means that for every pair $x, y \in X$ the product $x \cdot y \in X$ is defined, that the multiplication is associative and that there exists a vector $1 \in X$ such that $1 \cdot x = x \cdot 1 = x$ for every $x \in X$ and $\|1\| = 1$; we use the same symbol 1 to denote the unit vector of X as well as the number $1 \in \mathbf{R}$; the meaning should be always clear from the context.

Let $f : [a,b] \to X$. To every partition D of $[a,b]$ we assign the product

$$P(f,D) = \prod_{i=m}^{1} (1 + f(\xi_i)\Delta t_i) = (1 + f(\xi_m)\Delta t_m) \cdots (1 + f(\xi_1)\Delta t_1).$$

Definition 5.2.1. A function $f : [a,b] \to X$ is called product integrable if there is a vector $P_f \in X$ such that for every $\varepsilon > 0$ there exists a partition D_ε of $[a,b]$ such that

$$\|P(f,D) - P_f\| < \varepsilon$$

whenever $D \prec D_\varepsilon$. The vector P_f is called the (left) product integral of f and we use the notation $\prod_a^b (1 + f(t)\,\mathrm{d}t) = P_f$.

Remark 5.2.2. Masani also defines the right product integral as the limit of the products

$$P^*(f, D) = \prod_{i=1}^{m} (1 + f(\xi_i)\Delta t_i) = (1 + f(\xi_1)\Delta t_1) \cdots (1 + f(\xi_m)\Delta t_m),$$

which are obtained by reversing the order of factors in $P(f, D)$. Masani uses the symbols

$$\int_a^{b\,\frown} (1 + f(t)\,\mathrm{d}t), \quad \int_a^{b\,\smile} (1 + f(t)\,\mathrm{d}t)$$

to denote left and right product integrals. As he remarks, it is sufficient to study either the left integral or the right integral, respectively. This is because the following *principle of duality* holds:

To every Banach algebra X there is a dual algebra X^* which is identical with X except the operation of multiplication: We define the product $x \cdot y$ in X^* as the vector $y \cdot x$, where the last multiplication is carried out in X. Every statement C about Banach algebra X has a corresponding dual statement C^*, which is obtained by reversing the order of all products in C. Consequently, every occurence of the term "left product integral" must be replaced by "right product integral" and vice versa. A dual statement C^* is true in X^* if and only if C is true in X. In case C is true in every Banach algebra, the same can be said of C^*.

Theorem 5.2.3.[1] Let $f : [a, b] \to X$ be a bounded function. The following statements are equivalent:

1) f is product integrable and $\prod_a^b (1 + f(t)\,\mathrm{d}t) = P_f$.
2) Every sequence of partitions $\{D_n\}_{n=1}^{\infty}$ of interval $[a, b]$ such that $\nu(D_n) \to 0$ satisfies $\lim_{n\to\infty} P(f, D_n) = P_f$.
3) $\lim_{\nu(D)\to 0} P(f, D) = P_f$.

Proof. The equivalence of statements 2) and 3) is proved in the same way as in the case of ordinary integral. Assume that 3) holds, i.e. to every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P(f, D) - P_f\| < \varepsilon$ for every partition D of interval $[a, b]$ which satisfies $\nu(D) < \delta$. Let D_ε be such a partition. Then for every $D \prec D_\varepsilon$ we have $\nu(D) \le \nu(D_\varepsilon) < \delta$, and therefore $\|P(f, D) - P_f\| < \varepsilon$; thus we have proved the implication 3) $\Rightarrow$ 1). Masani gives only a brief indication of the proof of 1) $\Rightarrow$ 3), details are left to the reader; boundedness of f is important here. □

The following theorem represents a "Cauchy condition" for the existence of product integral.

Theorem 5.2.4. Let $f : [a, b] \to X$ be bounded. The following statements are equivalent:

[1] [Mas], p. 157–159

1) f is product integrable.
2) To every $\varepsilon > 0$ there is a partition D_ε such that $\|P(f,D) - P(f,D_\varepsilon)\| < \varepsilon$ whenever $D \prec D_\varepsilon$.
3) To every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P(f,D_1) - P(f,D_2)\| < \varepsilon$ whenever D_1, D_2 are partitions of $[a,b]$ satisfying $\nu(D_1) < \delta$, $\nu(D_2) < \delta$.

Proof. The equivalence of statements 1) and 2) is proved in the same way as in the case of ordinary integral. The statement 3) is clearly equivalent to the statement 3) of the previous theorem. □

5.3 Useful inequalities

We now present five inequalities which will be useful later. Masani didn't prove the first three; we have however met the first two in Chapter 3 – see the Lemmas 3.1.3 and 3.4.2. Although we have proved them only for matrices, the proofs are valid even for elements of an arbitrary Banach algebra X.

Lemma 5.3.1.[1] Let $x_k \in X$ for $k = 1, \ldots, m$. Then

$$\left\| \prod_{k=1}^{m} (1 + x_k) \right\| \leq \exp\left(\sum_{k=1}^{m} \|x_k\| \right).$$

Lemma 5.3.2.[2] Let $x_k, y_k \in X$ for $k = 1, \ldots, m$. Then

$$\left\| \prod_{k=1}^{m} (1 + x_k) - \prod_{k=1}^{m} (1 + y_k) \right\| \leq \exp\left(\sum_{k=1}^{m} \|x_k\| \right) \left(\exp \sum_{k=1}^{m} \|x_k - y_k\| - 1 \right).$$

Lemma 5.3.3.[3] Let $x_k \in X$ for $k = 1, \ldots, m$. Then

$$\left\| \prod_{k=1}^{m} (1 + x_k) - 1 \right\| \leq \exp\left(\sum_{k=1}^{m} \|x_k\| \right) - 1.$$

Proof. Elementary calculation yields

$$\left\| \prod_{k=1}^{m} (1 + x_k) - 1 \right\| = \left\| \sum_{j=1}^{m} \left(\sum_{1 \leq i_1 < \cdots < i_j \leq m} x_{i_1} \cdots x_{i_j} \right) \right\| \leq$$

$$\leq \sum_{j=1}^{m} \left(\sum_{1 \leq i_1 < \cdots < i_j \leq m} \|x_{i_1}\| \cdots \|x_{i_j}\| \right) \leq \sum_{j=1}^{m} \frac{1}{j!} \left(\sum_{i_1, \ldots, i_j = 1}^{m} \|x_{i_1}\| \cdots \|x_{i_j}\| \right) =$$

[1] [Mas], p. 153
[2] [Mas], p. 154
[3] [Mas], p. 153

$$= \sum_{j=1}^{m} \frac{1}{j!}(\|x_1\| + \cdots + \|x_m\|)^j \le \exp\left(\sum_{k=1}^{m} \|x_k\|\right) - 1.$$

□

Lemma 5.3.4.[1] Let $x_k \in X$ for $k = 1, \ldots, m$. Then

$$\left\| \prod_{k=1}^{m}(1 + x_k) - \left(1 + \sum_{k=1}^{m} x_k\right) \right\| \le \left(\exp \sum_{k=1}^{m} \|x_k\| - 1\right) \sum_{k=1}^{m} \|x_k\|.$$

Proof. The statement is a simple consequence of the inequality

$$\prod_{k=1}^{m}(1 + x_k) - \left(1 + \sum_{k=1}^{m} x_k\right) = \sum_{k=1}^{m} x_k \left(\prod_{j=k+1}^{m} (1 + x_j) - 1\right)$$

and Lemma 5.3.3. □

Lemma 5.3.5.[2] Let $m, n \in \mathbf{N}$, $u, v, x_j, y_k \in X$, $\|x_j\|, \|y_k\| \le 1/2$ for every $j = 1, \ldots, m$ and $k = 1, \ldots, n$. Then

$$\left\| \prod_{j=1}^{m}(1 + x_j) \cdot (u - v) \cdot \prod_{k=1}^{n}(1 + y_k) \right\| \ge \exp\left(-2\left(\sum_{j=1}^{m} \|x_j\| + \sum_{k=1}^{n} \|y_k\|\right)\right) \|u - v\|.$$

Proof. Define $f(t) = e^{2t} - te^{2t} - 1$. Then

$$f'(t) = e^{2t}(1 - 2t) \ge 0, \quad t \in [0, 1/2],$$

and therefore

$$e^{2t} - te^{2t} - 1 = f(t) \ge f(0) = 0, \quad t \in [0, 1/2].$$

We get

$$1 - t \ge e^{-2t}, \quad t \in [0, 1/2].$$

Now let $x, w \in X$, $\|x\| \le 1/2$. Then

$$\|w\| \le \|w + x \cdot w\| + \|x \cdot w\| \le \|(1 + x) \cdot w\| + \|x\| \cdot \|w\|,$$

which implies

$$\|(1 + x) \cdot w\| \ge \|w\|(1 - \|x\|) \ge \|w\| \exp(-2\|x\|). \tag{5.3.1}$$

For $y \in X$, $\|y\| \le 1/2$ we obtain in a similar way

$$\|w\| \le \|w + w \cdot y\| + \|w \cdot y\| \le \|w \cdot (1 + y)\| + \|y\| \cdot \|w\|,$$

[1] [Mas], p. 153
[2] [Mas], p. 152–153

$$\|w \cdot (1+y)\| \geq \|w\|(1-\|y\|) \geq \|w\| \exp(-2\|y\|). \tag{5.3.2}$$

To complete the proof it is sufficient to use m times the Inequality (5.3.1) and n times the Inequality (5.3.2). □

5.4 Properties of product integral

This section summarizes the basic properties of product integrable functions. We first prove that every product integrable function is necessarily bounded.

Lemma 5.4.1.[1] To every $\Delta : [a,b] \to (0,\infty)$ there exists a tagged partition $D : a = t_0 < t_1 < \cdots < t_m = b$, $\xi_i \in [t_{i-1}, t_i]$, such that $t_i - t_{i-1} \leq \Delta(\xi_i)$.

Proof. The system of intervals $\{(t - \Delta(t)/2, t + \Delta(t)/2),\ t \in [a,b]\}$ forms an open covering of $[a,b]$ and the result follows from the Heine–Borel theorem. It is also a simple consequence of Cousin's lemma (see [Sch2], p. 55 or [RG], Lemma 9.2). □

Theorem 5.4.2.[2] Every product integrable function f is bounded and

$$\left\| \prod_a^b (1 + f(t)\,\mathrm{d}t) \right\| \leq \exp\left((b-a) \sup_{t\in[a,b]} \|f(t)\| \right).$$

Proof. Assume that f is not bounded. Choose $N \in \mathbf{N}$ and $\delta > 0$. Define

$$\Delta(x) = \begin{cases} \min(\delta, (2\|f(x)\|)^{-1}) & \text{if } \|f(x)\| > 0, \\ \delta & \text{if } f(x) = 0. \end{cases}$$

According to Lemma 5.4.1 there exists a tagged partition $D : a = t_0 < t_1 < \cdots < t_m = b$, $\xi_i \in [t_{i-1}, t_i]$, such that

$$t_i - t_{i-1} \leq \Delta(\xi_i). \tag{5.4.1}$$

Clearly $\nu(D) \leq \delta$. Since f is not bounded, we can find a sequence of points $\{x_n\}_{n=1}^\infty$ from $[a,b]$ such that $x_n \to x \in [a,b]$ and $\|f(x_n)\| \geq n$. There must be a point $y \in \{x_n\}_{n=1}^\infty$, which lies in the same interval $[t_{j-1}, t_j]$ as the point x and such that

$$\|f(y) - f(x)\| \geq \|f(y)\| - \|f(x)\| \geq N \cdot \left(\exp(-m) \cdot \min_{1\leq i\leq m} (t_i - t_{i-1}) \right)^{-1}.$$

Let D_1 and D_2 be tagged partitions that are obtained from D by replacing the tag ξ_j by x and y, respectively. Then, according to Lemma 5.3.5 and Inequality (5.4.1),

$$\|P(f,D_1) - P(f,D_2)\| \geq \exp\left(-2 \sum_{i\neq j} \|f(\xi_i)\|(t_i - t_{i-1}) \right) \|f(x) - f(y)\|(t_j - t_{j-1}) \geq$$

[1] [Mas], p. 162
[2] [Mas], p. 163

$$\geq \exp(-m)\|f(x)-f(y)\|(t_j - t_{j-1}) \geq N.$$

Since $\nu(D_1) = \nu(D_2) = \nu(D) \leq \delta$, the number δ can be arbitrarily small and N arbitrarily large, we arrive at a contradiction with Theorem 5.2.4. The second part of the theorem is easily proved using Lemma 5.3.1, which guarantees that

$$\|P(f,D)\| \leq \exp\left(\sum_{i=1}^{m} \|f(\xi_i)\|(t_i - t_{i-1})\right) \leq \exp\left((b-a)\sup_{t\in[a,b]}\|f(t)\|\right)$$

for every tagged partition D of $[a,b]$. □

Theorem 5.4.3.[1] Assume that $\prod_a^b(1+f(t)\,\mathrm{d}t)$ exists. If $[c,d] \subset [a,b]$, then $\prod_c^d(1+f(t)\,\mathrm{d}t)$ exists as well.

Proof. Denote $M = \sup_{t\in[a,b]}\|f(t)\| < \infty$. Let D_1, D_2 be tagged partitions of $[c,d]$, D_A a tagged partition of $[a,c]$ satisfying $\nu(D_A) < 1/(2M)$ and D_B a tagged partition of $[d,b]$ satisfying $\nu(D_B) < 1/(2M)$. Letting

$$D_1^* = D_A \cup D_1 \cup D_B, \quad D_2^* = D_A \cup D_2 \cup D_B,$$

we obtain (using Lemma 5.3.5)

$$\|P(f,D_1^*) - P(f,D_2^*)\| = \|P(f,D_B)\,(P(f,D_1)-P(f,D_2))\,P(f,D_A)\| \geq$$

$$\geq \exp\left(-2\sum_{D_A\cup D_B} f(\xi_i)(t_i - t_{i-1})\right)\|P(f,D_1)-P(f,D_2)\| \geq$$

$$\geq \exp(-2M(b-a))\|P(f,D_1)-P(f,D_2)\|,$$

therefore

$$\|P(f,D_1)-P(f,D_2)\| \leq \exp(2M(b-a))\|P(f,D_1^*)-P(f,D_2^*)\|.$$

Because f is product integrable, to every $\varepsilon > 0$ there is a tagged partition D_ε^* of interval $[a,b]$ such that

$$\|P(f,D^*) - P(f,D_\varepsilon^*)\| < \frac{\varepsilon}{\exp(2M(b-a))}$$

whenever $D^* \prec D_\varepsilon^*$. Without loss of generality assume that $D_\varepsilon^* = D_A \cup D_\varepsilon \cup D_B$, where D_A is a partition of $[a,c]$ satisfying $\nu(D_A) < 1/(2M)$, D_ε is a partition of $[c,d]$ and D_B is a partition of $[d,b]$ satisfying $\nu(D_B) < 1/(2M)$. If $D \prec D_\varepsilon$, we construct the partition $D^* = D_A \cup D \cup D_B$. Then

$$\|P(f,D) - P(f,D_\varepsilon)\| \leq \exp(2M(b-a))\|P(f,D^*) - P(f,D_\varepsilon^*)\| < \varepsilon.$$

□

[1] [Mas], p. 163–165

Theorem 5.4.4.[1] If $a < b < c$ and the integrals $\prod_a^b(1+f(t)\,\mathrm{d}t)$ and $\prod_b^c(1+f(t)\,\mathrm{d}t)$ exist, then the integral $\prod_a^c(1+f(t)\,\mathrm{d}t)$ exists as well and

$$\prod_a^c(1+f(t)\,\mathrm{d}t) = \prod_b^c(1+f(t)\,\mathrm{d}t)\cdot\prod_a^b(1+f(t)\,\mathrm{d}t).$$

Proof. Masani's proof is somewhat incomplete; we present a modified version. The assumptions imply the existence of a tagged partition D_ε^1 of $[a,b]$ and a tagged partition D_ε^2 of $[b,c]$ such that

$$\left\|P(f,D^1)-\prod_a^b(1+f(t)\,\mathrm{d}t)\right\|<\varepsilon,$$

$$\left\|P(f,D^2)-\prod_b^c(1+f(t)\,\mathrm{d}t)\right\|<\varepsilon$$

whenever $D^1 \prec D_\varepsilon^1$ and $D^2 \prec D_\varepsilon^2$. Let $D_\varepsilon = D_\varepsilon^1 \cup D_\varepsilon^2$. Then every tagged partition $D \prec D_\varepsilon$ can be written as $D = D^1 \cup D^2$, where $D^1 \prec D_\varepsilon^1$ and $D^2 \prec D_\varepsilon^2$. We have $P(f,D) = P(f,D^2)\cdot P(f,D^1)$ and

$$\left\|P(f,D)-\prod_b^c(1+f(t)\,\mathrm{d}t)\cdot\prod_a^b(1+f(t)\,\mathrm{d}t)\right\| \le$$

$$\le \left\|P(f,D^2)\left(P(f,D^1)-\prod_a^b(1+f(t)\,\mathrm{d}t)\right)\right\| +$$

$$+\left\|\left(P(f,D^2)-\prod_b^c(1+f(t)\,\mathrm{d}t)\right)\prod_a^b(1+f(t)\,\mathrm{d}t)\right\| \le$$

$$\le \left(\left\|\prod_b^c(1+f(t)\,\mathrm{d}t)\right\|+\varepsilon\right)\varepsilon+\varepsilon\left\|\prod_a^b(1+f(t)\,\mathrm{d}t)\right\|,$$

which completes the proof. □

Statements similar to Theorem 5.4.4 have already appeared in the work of Volterra and Schlesinger. Their versions are however less general: They assume that f is Riemann integrable on $[a,c]$, which implies the existence of product integral on $[a,c]$ and the rest of the proof is trivial. Masani on the other hand proves that the existence of product integral on $[a,b]$ and on $[b,c]$ implies the existence of product integral on $[a,c]$. The same remark also applies to Theorem 5.4.3. In the following section we prove that the product integral exists if and only if the function is

[1] [Mas], p. 165

(Riemann-Graves) integrable; the proof of this fact is nevertheless based on the use of Theorem 5.4.3.

Lemma 5.4.5. Every $x \in X$ such that $\|x - 1\| < 1$ has an inverse element and

$$x^{-1} = \sum_{n=0}^{\infty} (1 - x)^n.$$

Proof. The condition $\|x - 1\| < 1$ implies that the infinite series given above is absolutely convergent; let x^{-1} be defined as the sum of that series. If

$$s_k = \sum_{n=0}^{k} (1 - x)^n,$$

then $s_{k+1} = 1 + (1 - x) \cdot s_k = 1 + s_k \cdot (1 - x)$.
Passing to the limit $k \to \infty$ we obtain

$$x^{-1} = 1 + (1 - x) \cdot x^{-1} = 1 + x^{-1} \cdot (1 - x),$$

i.e. $x^{-1} \cdot x = x \cdot x^{-1} = 1$. □

Theorem 5.4.6.[1] If $f : [a, b] \to X$ is a product integrable function, then $\prod_a^b (1 + f(t)\,\mathrm{d}t)$ is an invertible element of the Banach algebra X.

Proof. Denote $M = \sup_{t \in [a,b]} \|f(t)\| < \infty$. Choose $\delta > 0$ such that $\exp(M\delta) < 2$ and a partition $D : a = t_0 < t_1 < \cdots < t_m = b$ such that $\nu(D) \le \delta$. Then

$$\prod_a^b (1 + f(t)\,\mathrm{d}t) = \prod_{i=m}^{1} \prod_{t_{i-1}}^{t_i} (1 + f(t)\,\mathrm{d}t) \tag{5.4.2}$$

Lemma 5.3.3 implies that for every $i = 1, \ldots, m$

$$\left\| \prod_{t_{i-1}}^{t_i} (1 + f(t)\,\mathrm{d}t) - 1 \right\| \le \exp(M(t_i - t_{i-1})) - 1 < 1,$$

i.e. $\prod_{t_{i-1}}^{t_i} (1 + f(t)\,\mathrm{d}t)$ is (according to Lemma 5.4.5) an invertible element of the algebra X. As a consequence of (5.4.2) we obtain

$$\left(\prod_a^b (1 + f(t)\,\mathrm{d}t) \right)^{-1} = \prod_{i=1}^{m} \left(\prod_{t_{i-1}}^{t_i} (1 + f(t)\,\mathrm{d}t) \right)^{-1}.$$

□

[1] [Mas], p. 165–166

5.5 Integrable and product integrable functions

Masani now proceeds to prove an important theorem which states that the classes of integrable and product integrable functions coincide. The fact that the existence of Riemann integral implies the existence of product integral was already known to Volterra; the reverse implication appears for the first time in Masani's paper.

Lemma 5.5.1. Let $f : [a, b] \to X$ be a bounded function. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $[c, d] \subseteq [a, b]$, $d - c < \delta$ and D is a tagged partition of $[c, d]$, then

$$\|P(f, D) - (1 + S(f, D))\| \leq \varepsilon(d - c).$$

Proof. Denote $M = \sup_{t\in[a,b]} \|f(t)\|$. Choose $\delta > 0$ such that

$$(\exp(M\delta) - 1) < \varepsilon/M.$$

Then according to Lemma 5.3.4

$$\|P(f, D) - (1 + S(f, D))\| \leq (\exp(M(d - c)) - 1)M(d - c) \leq \varepsilon(d - c).$$

□

Definition 5.5.2. Let $Y \subseteq X$. The diameter of the set Y is the number

$$\operatorname{diam} Y = \sup\{\|y_1 - y_2\|;\ y_1, y_2 \in Y\}.$$

The convex closure of Y is the set

$$\operatorname{conv} Y = \left\{\sum_{i=1}^{k} \alpha_i y_i;\quad k \in \mathbf{N}, y_i \in Y, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1\right\}.$$

Theorem 5.5.3.[1] If $Y \subseteq X$, then

$$\operatorname{diam} \operatorname{conv} Y = \operatorname{diam} Y.$$

Proof. The proof is not difficult, although it's not included in Masani's paper. Since $Y \subseteq \operatorname{conv} Y$, it is sufficient to prove that

$$\operatorname{diam} \operatorname{conv} Y \leq \operatorname{diam} Y.$$

Let y^1, $y^2 \in \operatorname{conv} Y$,

$$y^1 = \sum_{i=1}^{l} \alpha_i y_i^1, \quad y^2 = \sum_{j=1}^{m} \beta_j y_j^2,$$

[1] [Mas], p. 159

where $y_i^1, y_j^2 \in Y$, $i = 1, \dots, l$, $j = 1, \dots, m$,

$$\sum_{i=1}^{l} \alpha_i = \sum_{j=1}^{m} \beta_j = 1.$$

Then

$$\|y^1 - y^2\| = \left\| \sum_{j=1}^{m} \beta_j \left(\sum_{i=1}^{l} \alpha_i y_i^1 \right) - \sum_{i=1}^{l} \alpha_i \left(\sum_{j=1}^{m} \beta_j y_j^2 \right) \right\| = \left\| \sum_{i=1}^{l} \sum_{j=1}^{m} \alpha_i \beta_j (y_i^1 - y_j^2) \right\| \leq$$

$$\leq \sum_{i=1}^{l} \sum_{j=1}^{m} \alpha_i \beta_j \|y_i^1 - y_j^2\| \leq \sum_{i=1}^{l} \sum_{j=1}^{m} \alpha_i \beta_j \operatorname{diam} Y = \operatorname{diam} Y.$$

□

Lemma 5.5.4.[1] Let $f : [a, b] \to X$ be a product integrable function. Then for every $\varepsilon > 0$ there is a partition $D : \ a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$ such that

$$\left\| \prod_{i=m}^{1} (1 + f(\xi_i)\Delta t_i) - \prod_{k=m}^{1} (1 + f(\eta_i)\Delta t_i) \right\| < \varepsilon$$

for every choice of $\xi_i, \eta_i \in [t_{i-1}, t_i]$, $i = 1, \dots, m$.

Proof. Follows from Theorem 5.2.4. □

Remark 5.5.5. Masani notes that the reverse implication is not valid; his counterexample is

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1/x & \text{if } x \in (0, 1/2), \\ -2 & \text{if } x \in [1/2, 1]. \end{cases}$$

Taking the partition $t_0 = 0 < t_1 = 1/2 < t_2 = 1$ we obtain

$$\prod_{i=2}^{1} (1 + f(\xi_i)\Delta t_i) = 0$$

for every choice of $\xi_i \in [t_{i-1}, t_i]$, but f is not product integrable (because it is not bounded).

Lemma 5.5.6.[2] Consider function $f : [a, b] \to X$. Assume that for every $\varepsilon > 0$ there is a partition $D_\varepsilon : \ a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ such that

$$\left\| \sum_{i=1}^{n} f(\xi_i)\Delta t_i - \sum_{i=1}^{n} f(\eta_i)\Delta t_i \right\| < \varepsilon$$

[1] [Mas], p. 160–161
[2] [Mas], p. 159–160

for every choice of $\xi_i, \eta_i \in [t_{i-1}, t_i]$. Then f is an integrable function.

Proof. If we introduce the notation

$$\sum_{i=1}^{n} f([t_{i-1}, t_i])\Delta t_i = \left\{ \sum_{i=1}^{n} f(\xi_i)\Delta t_i;\ \xi_i \in [t_{i-1}, t_i] \right\},$$

then the assumption of the lemma might be written as

$$\operatorname{diam}\left(\sum_{i=1}^{n} f([t_{i-1}, t_i])\Delta t_i \right) < \varepsilon.$$

To prove that f is integrable it is sufficient to verify that for every partition $D \prec D_\varepsilon$ which consists of division points

$$t_{i-1} = t_0^i < t_1^i < \cdots < t_{m(i)}^i = t_i, \quad i = 1, \ldots, n$$

and for every choice of $\xi_j^i \in [t_{j-1}^i, t_j^i]$, $\eta_i \in [t_{i-1}, t_i]$ we have

$$\|P(f, D) - P(f, D_\varepsilon)\| = \left\| \sum_{i=1}^{n} \sum_{j=1}^{m(i)} f(\xi_j^i)\Delta t_j^i - \sum_{i=1}^{n} f(\eta_i)\Delta t_i \right\| < \varepsilon.$$

But

$$\sum_{i=1}^{n} \sum_{j=1}^{m(i)} f(\xi_j^i)\Delta t_j^i = \sum_{i=1}^{n} \sum_{j=1}^{m(i)} \frac{\Delta t_j^i}{\Delta t_i} f(\xi_j^i)\Delta t_i \in \operatorname{conv}\left(\sum_{i=1}^{n} f([t_{i-1}, t_i])\Delta t_i \right),$$

and the proof is completed by using Theorem 5.5.3. □

Theorem 5.5.7.[1] Every product integrable function $f : [a, b] \to X$ is integrable.

Proof. We verify that the assumption of Theorem 5.5.6 is fulfilled. According to Lemma 5.5.1 it is possible to choose numbers $a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b$ in such a way that

$$\|P(f, D_k) - (1 + S(f, D_k))\| \leq \frac{\varepsilon}{3} \frac{(s_k - s_{k-1})}{(b - a)}$$

for every tagged partition D_k of interval $[s_{k-1}, s_k]$. Since f is product integrable on $[s_{k-1}, s_k]$, there exists (according to Lemma 5.5.4) a partition

$$s_{k-1} = t_0^k < t_1^k < \cdots < t_{m(k)}^k = s_k$$

such that

$$\left\| \prod_{i=m(k)}^{1} (1 + f(\xi_i^k)\Delta t_i^k) - \prod_{i=m(k)}^{1} (1 + f(\eta_i^k)\Delta t_i^k) \right\| < \frac{\varepsilon}{3} \frac{(s_k - s_{k-1})}{(b - a)}$$

[1] [Mas], p. 167–169

for every choice of $\xi_i^k, \eta_i^k \in [t_{i-1}^k, t_i^k]$. For such ξ_i^k, η_i^k we have

$$\left\| \sum_{i=1}^{m(k)} f(\xi_i^k)\Delta t_i^k - \sum_{k=1}^{m(k)} f(\eta_i^k)\Delta t_i^k \right\| \le \left\| \left(1 + \sum_{i=1}^{m(k)} f(\xi_i^k)\Delta t_i^k\right) - \prod_{i=m(k)}^{1} (1 + f(\xi_i^k)\Delta t_i^k) \right\| +$$

$$+ \left\| \prod_{i=m(k)}^{1} (1 + f(\xi_i^k)\Delta t_i^k) - \prod_{i=m(k)}^{1} (1 + f(\eta_i^k)\Delta t_i^k) \right\| +$$

$$+ \left\| \prod_{i=m(k)}^{1} (1 + f(\eta_i^k)\Delta t_i^k) - \left(1 + \sum_{k=1}^{m(k)} f(\eta_i^k)\Delta t_i^k\right) \right\| < \frac{\varepsilon(s_k - s_{k-1})}{(b-a)}.$$

Adding these inequalities for $k = 1, \ldots, n$ and using the triangle inequality leads to

$$\left\| \sum_{k=1}^{n} \sum_{i=1}^{m(k)} f(\xi_i^k)\Delta t_i^k - \sum_{k=1}^{n} \sum_{k=1}^{m(k)} f(\eta_i^k)\Delta t_i^k \right\| < \varepsilon.$$

This means that the partition D can be chosen as

$$a = t_0^1 < t_1^1 < \cdots < t_{m(1)}^1 = t_0^2 < \cdots < t_{m(n-1)}^{n-1} = t_0^n < t_1^n < \cdots < t_{m(n)}^n = b.$$

$\square$

We now follow Masani's proof of the reverse implication which says that every integrable function is also product integrable and that the product integral might be expressed using the Peano series. As we know, the history of the theorem can be traced back to Volterra (in the case $X = \mathbf{R}^{n\times n}$). Masani was probably the first one to give a rigorous proof.

We will be working with tagged partitions $D : a = t_0 < t_1 < \cdots < t_{m(D)} = b$, $\xi_i \in [t_{i-1}, t_i]$. For every $n \le m(D)$ we define

$$T_n(f, D) = \sum_{m(D)\ge i_1 > i_2 > \cdots > i_n \ge 1} f(\xi_{i_1}) \cdots f(\xi_{i_n})\Delta t_1 \cdots \Delta t_n$$

and

$$T(f, D) = T_1(f, D) + \cdots + T_{m(D)}(f, D).$$

We state the following lemma without proof; see Remark 2.4.4 for the proof in the finite-dimensional case (the difficulty in the general case is hidden in the fact that the product of two integrable functions need not be integrable).

Lemma 5.5.8.[1] Let $f : [a, b] \to X$ be an integrable function, $n \in \mathbf{N}$. Then the limit $T_n(f) = \lim_{\nu(D)\to 0} T_n(f, D)$ exists and

$$T_n(f) = \int_a^b \int_a^{t_1} \cdots \int_a^{t_{n-1}} f(t_1) \cdots f(t_n)\, \mathrm{d}t_n \cdots \mathrm{d}t_1.$$

[1] [Mas], p. 174–176

Masani refers to the following lemma as to the extension of Tannery's theorem.

Lemma 5.5.9.[1] Consider function $f:[a,b]\to X$ and assume that the following conditions are satisfied:

1) There exists $T_n(f)=\lim_{\nu(D)\to 0} T_n(f,D)$ for every $n\in\mathbf{N}$.
2) $M_n=\sup_D \|T_n(f,D)\|<\infty$ for every $n\in\mathbf{N}$, where the supremum is taken over all partitions D of interval $[a,b]$ which consist of at least n division points.
3) The series $\sum_{n=1}^\infty M_n$ is convergent.

Then

$$T(f)=\lim_{\nu(D)\to 0} T(f,D)=\sum_{n=1}^\infty T_n(f).$$

Proof. The series $T(f)=\sum_{n=1}^\infty T_n(f)$ is convergent, because $\|T_n(f)\|\le M_n$ for every $n\in\mathbf{N}$. We will prove that $T(f)=\lim_{\nu(D)\to 0}T(f,D)$. Choose $\varepsilon>0$. There exists a number $n(\varepsilon)\in\mathbf{N}$ such that

$$\sum_{k=n(\varepsilon)+1}^\infty M_k<\varepsilon/3.$$

According to the first assumption, there exists a $\delta>0$ such that

$$\|T_k(f,D)-T_k(f)\|<\frac{\varepsilon}{3n(\varepsilon)},\quad k=1,\dots,n(\varepsilon),$$

for every tagged partition D of $[a,b]$ that satisfies $\nu(D)<\delta$. Without loss of generality we assume that δ is so small that D consists of at least $n(\varepsilon)$ division points, i.e. $T_1(f,D)$, …, $T_{n(\varepsilon)}(f,D)$ are well-defined. Now for every tagged partition D that satisfies $\nu(D)<\delta$ we estimate

$$\|T(f,D)-T(f)\|=\left\|\sum_{k=1}^{m(D)}T_k(f,D)-\sum_{k=1}^\infty T_k(f)\right\|\le\sum_{k=1}^{n(\varepsilon)}\|T_k(f,D)-T_k(f)\|+$$

$$+\sum_{k=n(\varepsilon)+1}^{m(D)}\|T_k(f,D)\|+\sum_{k=n(\varepsilon)+1}^{\infty}\|T_k(f)\|<n(\varepsilon)\,\frac{\varepsilon}{3n(\varepsilon)}+2\sum_{k=n(\varepsilon)+1}^\infty M_k<\varepsilon.$$

□

Theorem 5.5.10.[2] Let $f:[a,b]\to X$ be an integrable function. Then f is also product integrable and

$$\prod_a^b(1+f(t)\,\mathrm{d}t)=1+\sum_{n=1}^\infty T_n(f).$$

[1] [Mas], p. 189–191
[2] [Mas], p. 176–177

Proof. Denote $M = \sup_{t\in[a,b]} \|f(t)\| < \infty$. For every partition D of interval $[a,b]$ we have

$$P(f,D) = 1 + T(f,D),$$

$$\prod_a^b (1 + f(t)\,\mathrm{d}t) = 1 + \lim_{\nu(D)\to 0} T(f,D),$$

$$\|T_n(f,D)\| \le \frac{(b-a)^n M^n}{n!},$$

$$\sum_{n=1}^{\infty} \frac{(b-a)^n M^n}{n!} = \exp(M(b-a)) - 1 < \infty.$$

The statement of the theorem is therefore a consequence of the preceding two lemmas. □

We have proved that a function is product integrable if and only if it is integrable. Thus, in the rest of this chapter we use the terms "integrable" and "product integrable" as synonyms.

Theorem 5.5.11.[1] Let $f : [a,b] \to X$ be an integrable function. Suppose that $f(x)\cdot f(y) = f(y)\cdot f(x)$ for each pair $x, y \in X$. Then

$$\prod_a^b (1 + f(t)\,\mathrm{d}t) = \exp\left(\int_a^b f(t)\,\mathrm{d}t\right).$$

Proof. A simple consequence of Theorem 5.5.10 and the equality

$$\int_a^b \int_a^{t_1} \cdots \int_a^{t_{n-1}} f(t_1)\cdots f(t_n)\,\mathrm{d}t_n \cdots \mathrm{d}t_1 = \frac{1}{n!}\left(\int_a^b f(t)\,\mathrm{d}t\right)^n$$

(see Lemma 2.4.2). □

5.6 Additional properties of product integral

This section is devoted to Masani's versions of the fundamental theorem of calculus, the uniform convergence theorem, and the change of variables theorem.

Theorem 5.6.1.[2] Let $f : [a,b] \to X$ be an integrable function. Denote

$$Y(x) = \prod_a^x (1 + f(t)\,\mathrm{d}t), \quad x \in [a,b].$$

[1] [Mas], p. 179
[2] [Mas], p. 178

Then

$$Y(x) = 1 + \int_a^x f(t)Y(t)\,\mathrm{d}t, \quad x \in [a,b]. \tag{5.6.1}$$

Proof. Using Theorem 5.5.10 we obtain

$$Y(t) = 1 + \int_a^t f(t_1)\,\mathrm{d}t_1 + \int_a^t \int_a^{t_1} f(t_1)f(t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 + \cdots. \tag{5.6.2}$$

Since

$$\left\| \int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} f(t_1)\cdots f(t_n)\,\mathrm{d}t_n \cdots \mathrm{d}t_1 \right\| \le \frac{(b-a)^n M^n}{n!},$$

the series (5.6.2) is uniformly convergent. Because f is bounded, the series

$$f(t)Y(t) = f(t) + \int_a^t f(t)f(t_1)\,\mathrm{d}t_1 + \int_a^x \int_a^{t_1} f(t)f(t_1)f(t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 + \cdots$$

is also uniformly convergent and might be integrated term by term on $[a,x]$; performing this step leads to Equation (5.6.1). □

Corollary 5.6.2.[1] If $f : [a,b] \to X$ is a continuous function, then

$$Y'(x)Y(x)^{-1} = f(x)$$

for every $x \in [a,b]$.

The previous corollary represents an analogy of the formula

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$$

(see Theorem 5.1.7). Also the Newton-Leibniz formula

$$\int_a^b f'(x)\,\mathrm{d}t = f(b) - f(a)$$

(see Theorem 5.1.8) has the following product analogy (whose proof we omit).

Theorem 5.6.3.[2] Assume that $Z : [a,b] \to X$ satisfies $Z'(x)Z(x)^{-1} = f(x)$ for every $x \in [a,b]$. Then

$$\prod_a^b (1 + f(t)\,\mathrm{d}t) = Z(b)Z(a)^{-1}$$

provided the function f is integrable.

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1 [Mas], p. 181
2 [Mas], p. 182

The next theorem establishes a criterion for interchanging the order of limit and product integral, i.e. for the formula

$$\lim_{n\to\infty} \prod_a^b (1 + f_n(t)\,\mathrm{d}t) = \prod_a^b (1 + \lim_{n\to\infty} f_n(t)\,\mathrm{d}t).$$

We have already encountered such a criterion in Chapter 3 when discussing the Lebesgue product integral; Schlesinger's statement represented in fact a product analogy of the Lebesgue dominated convergence theorem. Masani's theorem concerns the Riemann product integral and requires uniform convergence to perform the interchange of limit and integral.

Theorem 5.6.4.[1] Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of integrable functions which converge uniformly to function f on interval $[a, b]$. Then

$$\prod_a^b (1 + f(t)\,\mathrm{d}t) = \lim_{n\to\infty} \prod_a^b (1 + f_n(t)\,\mathrm{d}t).$$

Proof. The existence of $\prod_a^b (1+f(t)\,\mathrm{d}t)$ follows from the fact that the limit of a uniformly convergent sequence of integrable functions is again an integrable function. For an arbitrary tagged partition D we can use Lemma 5.3.2 to estimate

$$\|P(f, D) - P(f_n, D)\| \le \exp(M(b-a)) \cdot \left(\exp\left(\sum_i \|f(\xi_i) - f_n(\xi_i)\|\Delta t_i\right) - 1\right),$$

where $M = \sup_{t\in[a,b]} \|f(t)\|$. Choose $\varepsilon > 0$ and find a corresponding $\varepsilon_0 > 0$ such that

$$\exp(M(b-a)) \cdot (\exp(\varepsilon_0(b-a)) - 1) < \varepsilon/3.$$

Let $n_0 \in \mathbf{N}$ be such that $\|f(t) - f_n(t)\| < \varepsilon_0$ for every $t \in [a, b]$ and $n \ge n_0$. The partition D can be chosen so that the inequalities

$$\left\|P(f, D) - \prod_a^b (1 + f(t)\,\mathrm{d}t)\right\| < \varepsilon/3, \quad \left\|P(f_n, D) - \prod_a^b (1 + f_n(t)\,\mathrm{d}t)\right\| < \varepsilon/3$$

hold. Then for every $n \ge n_0$ we have

$$\left\|\prod_a^b (1 + f_n(t)\,\mathrm{d}t) - \prod_a^b (1 + f(t)\,\mathrm{d}t)\right\| \le \left\|\prod_a^b (1 + f(t)\,\mathrm{d}t) - P(f, D)\right\| +$$

$$+ \left\|P(f, D) - P(f_n, D)\right\| + \left\|P(f_n, D) - \prod_a^b (1 + f_n(t)\,\mathrm{d}t)\right\| < \varepsilon.$$

$\square$

[1] [Mas], p. 171

Masani also proved a generalized version of the change of variables theorem for the product integral (compare to Theorem 2.5.10); we state it without proof.

Theorem 5.6.5.[1] Let $f : [a, b] \to X$ be an integrable function, $\varphi : [\alpha, \beta] \to [a, b]$ increasing, $\varphi(\alpha) = a$, $\varphi(\beta) = b$. If φ' exists and is integrable on $[a, b]$, then

$$\prod_a^b (1 + f(t)\,\mathrm{d}t) = \prod_\alpha^\beta (1 + f(\varphi(u))\varphi'(u)\,\mathrm{d}u).$$

1 [Mas], p. 187–188