Preface

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Preface

This publication is devoted to the theory of product integral, its history and applications. The text represents an English translation of my dissertation with numerous corrections and several complements.

The definition of product integral appeared for the first time in the work of Vito Volterra at the end of the 19th century. Although it is a rather elementary concept, it is almost unknown among mathematicians.

Whereas the ordinary integral of a function a provides a solution of the equation

$$y'(x) = a(x),$$

the product integral helps us to find solutions of the equation

$$y'(x) = a(x)y(x).$$

The function a can be a scalar function, but product integration is most useful when a is a matrix function; in the latter case, y is a vector function and the above equation represents in fact a system of linear differential equations of the first order.

Volterra was trying (on the whole successfully) to create analogy of infinitesimal calculus for the product integral. However, his first papers didn't meet with a great response. Only the development of Lebesgue integral and the birth of functional analysis in the 20th century was followed by the revival of interest in product integration. The attempts to generalize the notion of product integral followed two directions: Product integration of matrix functions whose entries are not Riemann integrable, and integration of more general objects than matrix functions (e.g. operator-valued functions).

In the 1930's, the ideas of Volterra were taken up by Ludwig Schlesinger, who elaborated Volterra's results and introduced the notion of Lebesgue product integral. Approximately at the same time, a Czech mathematician and physicist Bohuslav Hostinský proposed a definition of product integral for functions whose values are integral operators on the space of continuous functions.

New approaches to ordinary integration were often followed by similar theories of product integration; one of the aims of this work is to document this progress. It can be also used as an introductory textbook of product integration. Most of the text should be comprehensible to everyone with a good knowledge of calculus. Parts of Section 1.1 and Section 2.8 require a basic knowledge of analytic functions in complex domain, but both may be skipped. Sections 3.5 to 3.8 assume that the reader is familiar with the basics of Lebesgue integration theory, and Chapters 4 and 5 use some elementary facts from functional analysis.

Almost every text about product integration contains references to the works of V. Volterra, B. Hostinský, L. Schlesinger and P. Masani, who have strongly influenced the present state of product integration theory. The largest part of this

publication is devoted to the discussion of their work. There were also other pioneers of product integration such as G. Rasch and G. Birkhoff, whose works [GR] and [GB] didn't have such a great influence and will not be treated here. The readers with a deeper interest in product integration should consult the monograph [DF], which includes an exhausting list of references.

All theorems and proofs that were taken over from another work include a reference to the original source. However, especially the results of V. Volterra were reformulated in the language of modern mathematics. Some of his proofs contained gaps, which I have either filled, or suggested a different proof.

Since the work was originally written in Czech, it includes references to several Czech monographs and articles; I have also provided a reference to an equivalent work written in English if possible.