Marta Pémová; Zita Sklenáriková

Karel Pelz an outstanding geometer of the 19th century

In: Martina Bečvářová (author); Christa Binder (author): Mathematics in the Austrian-Hungarian Empire. Proceedings of a Symposium held in Budapest on August 1, 2009 during the XXIII ICHST. (English). Praha: Matfyzpress, 2010. pp. 125–136.

Persistent URL: http://dml.cz/dmlcz/400825

KAREL PELZ AN OUTSTANDING GEOMETER OF THE 19th CENTURY

MARTA PÉMOVÁ, ZITA SKLENÁRIKOVÁ

Abstract: The second half of the 19th century was a crucial period for the formation of descriptive geometry as a science. This paper offers notes on the life and work of Karel Pelz, an outstanding mathematician in the field of both synthetic and constructive geometry, who belonged to the first generation of the best cultivators of descriptive geometry in the Austrian--Hungarian Empire. His name is closely connected with the highest development of descriptive geometry that was crowned by Emil Müller and his disciples in the first third of the 20th century. Pelz has excelled in the synthetic theory of conics, curves and surfaces (especially the quadratic ones) and has been interested also in other various contemporary problems. His contribution to the developing of the principles of orthogonal axonometry as a method of representation is also highly appreciated.

1 Curriculum Vitae and professional career

Karel Pelz, an outstanding Czech geometer, is a personage who is chronologically closing up the first strong generation of theoreticians of descriptive geometry in Austria-Hungary. This geometer of world importance was one of the most considerable representatives of the Czech geometrical school besides the Weyr brothers, Jan Sobotka, Vincenc Jarolímek and others. According to Gino Loria[1] *Karel Pelz won an honourable position among scientists who dedicated all their efforts to descriptive geometry. He was able to exploit the most determinative achievements reached during the 19th century in the geometry of position[2] for the further progress in her grades, particularly in the theory of conic sections. He succeeded to add to this discipline several important pages setting so an example which is desirable to follow by many other workers in interests of the development of knowledge.* [5]

Karel Pelz was born in Běleč at Křivoklát on the 2nd october of 1845. He grew up in the deep forests around Křivoklát, where his father was the dukes' gamekeeper. The father however was murdered by poachers, and so Pelz became orphan as far back as his childhood. After graduation from the Realschule in Rakovník he studied at the Institute of technology in Prague (1864–1869). In this period of crucial importance for the formation of descriptive geometry as a scientific discipline in the system of mathematical sciences, the favourable influence of the Prague department of descriptive geometry on students and young researchers was amplified by the activities of two significant personalities: František Tilšer, the second professor of the department of descriptive geometry for Czech lectures (after the unexpected death of Rudolf Skuherský)[3] and above all Wilhelm Fiedler, first professor of the department of descriptive geometry for German lectures (Fiedlers' lectures on newer geometry were at that time the first lectures on this topic in Austria).

[1] Gino Loria (19. 5. 1862 – 30. 1. 1954), a significant Italian mathematician and historician.

[2] *Geometry of position, newer/new geometry* = projective geometry.

[3] With the merit of Skuherský descriptive geometry on the polytechnic institute in Prague extricated from the role of an auxiliary discipline to an independent discipline with a twofold subsidy of lessons in comparison with the previous period. It is possible to find more details about the systematization of the departments of descriptive geometry in Austria – Hungary in [16].

In the first year of his stay in Prague Pelz studied descriptive geometry by Tilšer. In the second year he attended the same lectures by Fiedler, by him he elaborated laborious drawing complements to the lectures with such excellent results, that Fiedler requested them as exemplary models. Apart from that with active enthusiasm attended Pelz the optional lectures of Fiedler on *newer geometry*. The relationship between the teacher and the student grew into mutual affection; from the side of Fiedler it was support (material as well) and providenes, all his life Pelz held his teacher in the high respect and admiration for his ardour for projective and descriptive geometry and a contribution to these domains. It is evident from their lifelong correspondence.[4] Both soundness and conscientiousness were characteristic features of these two scientists in all their acts. ([20])

During the studies Pelz met the brothers Emil and Eduard Weyr; the relationship with Emil developed into a permanent lifelong friendship, even though fate didn´t grant them to live close to each other. Both Pelz and Emil Weyr loved Prague ardently; the desire to live in this city came to fruition only for Pelz in his declining years. Karel Pelz participated on the publication of the first scientific work of Emil Weyr by carrying out five sheets of drawings, which were an essential part of the publication.[5]

After graduating from the Institute of technology in 1869 Pelz assumed the post of a draughtsman at the central institute for meteorology and earth magnetism in Vienna. After a year he returned to Prague to the post of an assistant of descriptive geometry by professor Karel Küpper, who was appointed to the head of department of descriptive geometry after Fiedler left Prague. Pelz worked on this post until the year 1875, then he became the professor of a public *Realschule* (grammar school) in Těšín. At that time were already published Pelz's several scientific works, what gained him an invitation to the post of a professor at a technical college in Kamenica. Before he could accept this appointment, he was in 1876 appointed to the post of professor at a regional *Realschule* in Graz, domicile city of universities.

In the same time he entered his academic career in Graz as a highly regarded scientist. First he was a private docent at the local Institute of technology. His lectures evoked such interests, that he was appointed after two years to the post of the extraordinary professor of descriptive geometry. Firstly it was an expression of appreciation and recognition of Pelz's scientific and teaching work, but in the nearest future it brought a considerable increase of duties, through the extending his schoolmasterly work by his academic activities at the Institute of technology. However he did not remain long at that position. After the sudden death of Emil Koutný ([16], [17]) became Karel Pelz in 1881 a regular professor of descriptive geometry at the Institute of technology in Graz and he remained at that position fifteen prettiest and most fruitful years of his life.

In 1891 received Pelz a proposal for a professorial post of descriptive geometry at the University of technology in Vienna. Due to unacceptable rules on the Viennese side he has rejected the proposed post. In 1896 he repeatedly declined a proposal to Vienna, and in the same year he was appointed, on the basis of a public contest, a post of the ordinary professor at the Czech University of technology in Prague after the leaving of F. Tilšer to retirement. But the favourable activity of Karel Pelz at the University of technology in

---

4 Fiedler leaved in 1867 to the technical university in Zurich, where he occupied the post of professor of descriptive geometry. He preserved contact with Pelz practically to Pelz's last days; the teacher overlived his student more then four years.

5 The author of the publication is mentioning it with thanks in the introduction; he himself was not a good drawer and Pelz was practically the only one, from who was a correct and quick making out of this part of the work expectable. (Concerning the publication *Theorie der mehr-deutigen geometrischen Elementargebilde und der algebraischen Kurven und Flächen als deren Erzeugnisse*, 1869.)

Prague was already distressed by illness; in Prague he wrote only one scientific work devoted to the principals and characteristics of stereographic projection. However it is evident from his correspondence and the memoirs of his friends, that he was still mainly interested in structural problems, but it was not granted him to complete it to a level of publication. He stayed in written contact mainly with his former colleagues from Graz. He was on the best terms with Fr. Mertens, who also worked there (later Mertens obtained a professorial position at the University in Vienna). Pelz leaved Prague only unwillingly, he wanted to enjoy it as most as possible. He had several hobbies. His photography collection of outstanding mathematicians[6] of whose scientific merits and life stories he was able to talk interestingly and well-founded expanded to hundreds of pieces. He was at all an enthusiast of excellent memoirs, he liked to talk especially about Prague. Jan Sobotka wrote: *It is to regret that Pelz didn't write his memoirs – first of all about the scientific events and the cultural and social memorabilities in Prague. I am sure that he would be able to write as an expert and well-informed person eligible over others.* ([20]) In 1904 he was awarded a title and grade of the counselor of the court; however he did not live to enjoy the deserved departure to retirement.

Karel Pelz died on the 16th June in 1908 in Prague. He is entombed together with his wife Paulina (1854–1931) in Prague's memorial Slavín (Vyšehrad).

2 Scientific work

The name of Karel Pelz is closely connected with the top development of descriptive geometry crowned by *Emil Müller*[7] and his disciples in the first third of the 20th century. His scientific works were quoted, highly regarded and applied almost in all publications devoted to this mathematical discipline. At the turn of the 19th century Prague has contributed in a high degree to the development of new geometric methods. From the second half of the sixties of the 19th century it went through a whole *epoch* of successful expansion in geometry (the arrival of *W. Fiedler* to the Prague polytechnic institute, the rapidly growing high repute of the *Weyr* brothers, the arrival of *Jan Sobotka* to the Prague University, and many others, among whom Pelz had a characteristic and distinguished position. The scientific work of K. Pelz (altogether 34 papers[8]) has been mainly related to two main branches of descriptive geometry: 1. the synthetic theory of conic sections, curves and surfaces; 2. the scientific building of methods of representations.

2.1 Synthetic theory of conic sections, curves and surfaces

One of the first problem Pelz has been interested in, was the study of the so called *brilliant points* of a circle and an ellipse (under the condition that both the center of illumination and the eye of the observer lay in the plane of the given curve). The first problem he has solved by the construction of the tangents of the given circle that were the tangents of an auxiliary parabola as well, and the second one by the construction of the intersection points of the given ellipse and an auxiliary circular cubic. The elegance of the procedures and the achieved results inspired the Dutch geometer *P. H. Schouten* (1846–1913), who has e. g. proved, that an algebraic plane curve of the degree m has $m(2m-1)$ brilliant points that lay on a curve of the degree $2m$ passing through ideal points of the given curve.

[6] The collection is the property of the Czech Academy of Science. See [20].
[7] On the life and work of Emil Müller, an outstanding representative of the geometrical school of Vienna, is possible to learn more in [15].
[8] The list of the papers can be found in [19], [20].

A great deal of Pelz's work is devoted to *quadratic surfaces* (quadrics), to their properties and representation (the contours of quadrics, the contours of their projections, the construction of the axes of quadrics, the construction of the axes and foci of the contours of their projections, problems related to the illumination, i.e. the construction of the shadows of quadrics in parallel and central illumination, etc.) The solution of these problems is connected with the problems about conic sections. In many of them we can meet a very ingenious use and generalization of the following theorem of Steiner:[9] *The tangent and the normal in a point of a central conic section are together with the axes of the conic section tangents of a parabola*; *the tangent point of this parabola with the normal is the center of curvature of the given conic section in a corresponding point.* Pelz named this parabola *Steiner's parabola* and proved the following theorem:

Theorem 1. Let k be an arbitrary central conic section, P an arbitrary point not laying on it and p its polar line with respect to the conic section k. Let Q be an arbitrary point of the line p and q its polar line with respect to the conic section k, thus holds true: for all points Q of the line p the set of polar lines q' conjugate to the polar line q (in relation to the conic section k), which are perpendicular to the line q, is an envelope of a parabola[10].

Note 1. The axes ${}^{i}o$ (i = 1, 2) of the conic section k, its normal lines ${}^{i}n$ in the points ${}^{i}T \in p \cap k$ (i = 1, 2), the line p and the bisectors ${}^{i}r$ of the angles of tangents ${}^{i}t = \overleftrightarrow{{}^{i}TP}$ (i = 1, 2) are also the tangents of the parabola from the theorem 1. The directrix of this parabola is the diametral line PS of the given conic section k. Some of the mentioned tangents points we can see in Fig. 1.

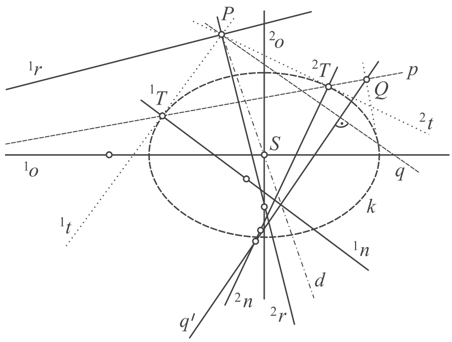

Fig. 1

One of the most relevant problems of the theory of quadrics is to determine the axes of a quadratic surface. Pelz has investigated this problem in [7] especially for the quadratic conic surface (it is possible to reduce the construction of the axes of any other quadric to this case). M. Chasles gave two solutions of the problem (without proof) in his Aperçu historique.[11] In the introduction of his paper Pelz proved the correctness of both Chasles's constructions in his own way, simply and compendiously, and consequently deduced a new, elegant and simpler solution moreover. We give a brief sketch of the main idea of it.

---

[9] Jacob Steiner (1796–1863), professor of geometry at the University in Berlin, one of the most outstanding German geometers of the 19th century.

[10] We call this parabola *Steiner-Pelz parabola* pertaining to point P and the given conic section k.

[11] *Michele Chasles* (1793–1880), one of the greatest French mathematicians of the 19th century (*Aperçu historique sur l'origine et le dèveloppement des méthodes en géometrie*, II. Ed., Paris, 1875).

Let $\boldsymbol{K}$ be a conical surface given by a conic section k in the plane α and by its vertex V. As almost the whole solution of the problem will be made in the plane α, so we can determine the vertex V by its orthogonal projection V' into this plane and by the distance $d = |V,\ \alpha\ | = |VV'|$. By the axes x, y, z of the surface $\boldsymbol{K}$ we mean three mutually perpendicular lines for which holds that any of them is a conjugate polar line to the plane given by the remaining two one's (with respect to the surface $\boldsymbol{K}$).[12] If a triple of lines x, y, z of the required properties does exist, the trihedral given by them is an autoconjugate one with respect to the surface $\boldsymbol{K}$. Followingly, the triangle XYZ ($X = x \cap \alpha$, $Y = y \cap \alpha$, $Z = z \cap \alpha$) is an autoconjugate (polar) triangle (with respect to the conic section $k = \boldsymbol{K} \cap \alpha$) with the point V' being its orthocenter[13] (Fig. 2b). Pelz has stated the necessary and sufficient conditions for the existence of such triangle. He has demonstrated, in a simple way, that all its sides are tangents of the parabola p (followingly the axes ^{i}o (i = 1, 2) of the conic section k, the polar line of the point V' and the bisectors ^{i}h (i = 1, 2) of the angles of the straight lines $^{1}FV'$, $^{2}FV'$, – the points ^{i}F (i = 1, 2) being the foci of the k – are also the tangents of p). The focus P of the parabola p is a diagonal point of a quadrilateral given by pairs of tangents ^{i}o, ^{i}h (i = 1, 2) (Fig. 2a). The lines $V'X$, $V'Y$, $V'Z$ are the conjugate polar lines to the lines YZ, XZ, XY respectively, each one orthogonal to the corresponding polar line. As the envelope of parabola p is the set of all conjugate polar lines to the lines of a pencil of the straight lines with the center V', which are perpendicular to the corresponding lines of the pencil, the vertices of the polar triangle have to lie on the polar figure to this envelope. This point set is an equiangular hyperbola k_1, the asymptotes ^{i}a (i = 1, 2) of which are parallel to the axes of conic section k; the hyperbola k_1 passes through the center S of k, and also through the point V' (the tangent t of the hyperbola in the point V' is a conjugate polar line to the polar of point V', perpendicular to it). Hyperbola k_1 is given; let us sign its center S_1 and let the second end point of the diameter of the hyperbola passing through the point V' be Q (moreover the points S, P, Q are collinear).[14]

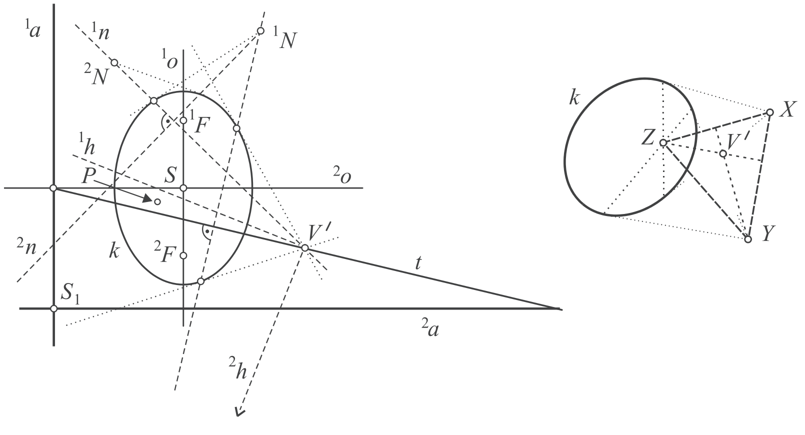

Fig. 2a, b

---

[12] The number of such triples may be infinite, e.g. in the case of a conic surface of revolution.

[13] *Note*: If in the following text will be the reference to polarity/polar conjugated figures (without additional information), we will mean the polarity/polar conjugated figures with respect to the conic section $k \subset \alpha$.

[14] Point Q can not, evidently, be indicated in Fig. 2a; it is the point for which: $(V'QS_1) = -1$.

Further on Pelz is proving that the circumscribed circle l about the triangle XYZ passes through the fixed points P and Q. Every circle of the family of circles (that passes through the points P, Q) has in general case three other points with the hyperbola k_1, except the point Q, in common. These points are the vertices of the polar triangle (with respect to the given conic section k) with the orthocenter V'. In conclusion we need to choose a circle of the pencil, for which the lines passing through the vertex of the conic surface $\boldsymbol{K}$ and through one vertex of the polar triangle (corresponding to this circle) are mutually perpendicular. This condition is fulfilled if and only if the circle l passes through the anti-pole O of the diametral line of hyperbola k_1 perpendicular to the line $V'S_1$ (with respect to the distance circle k^d of vertex V.[15] Then $l \cap k_1 = \{Q, X, Y, Z\}$ and the lines VX, VY, VZ are the solution of the problem.

The next original contribution of Karel Pelz to the solution of problems related to the constructions of contours of the projections of quadrics and an analogous problem, related to the construction of their shadows in the parallel, as well as in the central illumination, was the generalization of a significant theorem of *Quetelet* and *Dandeline* (Q-D Theorem). Pelz has stated the necessary and sufficient conditions for the solution of the problems introduced above for all regular quadrics except the one-sheet hyperboloid of revolution and the hyperbolic paraboloid, for which he had to use another method of solution. The generalization of Pelz sounds:

Theorem 2. The contour of the projection of a quadratic surface into a plane belonging to the system of the circular plane sections of the surface in the parallel, as well as in the central projection, is a conic section. The foci of this conic section are the projections of the end points of a diameter of the quadratic surface conjugate with the system of planes parallel to the plane of projections (related to the given surface).[16]

Pelz has returned to this theorem also in his last work devoted to the stereographic projection.[17] This paper has illustrated how – quite spontaneously – all characteristics of this projection were derived of the Quetelet–Dandeline theorem. Pelz remarked: *Just a few theorems of solid geometry are playing such an important role in the tuition of descriptive geometry as this theorem – for its numerous and multilateral use in the discipline in question.*

With respect to the extent and object of this paper we will not analyze further papers of Pelz from this field. We will mention only two next theorems, denoted as the Theorems of Pelz ([2], [20]), with the help of which it is possible to prove the Theorem of Chasles quoted without proof in his note XXVI in Aperçu historique.

Theorem 3. The foci of the projections of parallel plane sections of a given quadric with planes parallel to the plane of projection, are lying on two conic sections. These conic sections are confocal with the contour of projection of the quadric.[18]

Theorem 4. The orthogonal projection of the foci of the two-sheet hyperboloid of revolution, the prolate ellipsoid of revolution and the paraboloid of revolution are the

---

[15] The circle k^d with the center V' and diameter d ($d = |V, \alpha|$) is a circle of the plane α. The circle l is anti-inverse to the nine-point circle of the triangle XYZ; the center of anti-inversion is the point V' and its coefficient is equal $-d^2$ ($|V'O| \,.\, |V'S_1| = d^2$). Pelz has introduced in his paper [7] *very detailed proofs* of all arguments. The part of the proof related to the circle l the author illustrated on a case, in which the conic section k was a hyperbola.

[16] The theorem on the contour of the projection of a quadric of revolution (except the one-sheet hyperboloid of revolution) is a simple consequence of this theorem. The focuses in this case are the projections of the vertices of the surface.

[17] *The main theorems of stereographic projection as consequences of the Q-D Theorem*, Věstník královské české společnosti nauk v Praze, 1898.

[18] It is possible to prove, that in the case of parallel projection this theorem is in vigor for an arbitrary system of planes not including the plane of projection.

foci of the contours of projections of these surfaces. The contours of orthogonal projections of the one-sheet hyperboloid of revolution and of the oblate ellipsoid of revolution are confocal with the projection of a circle, which is generated by the rotation of a focus of the meridian of the surface about the axis of revolution of a quadric. [19]

2.2 Scientific building of methods of representation

The main contribution of Karel Pelz in this branch was that he have precised the elements of orthogonal axonometry as an autonomous method of representation. He has started his work with the very modest and noble-minded intention *to correct some of the non precise conclusions in the works devoted to this method* ([5]) and was the first who has turned attention to the axonometric triangle and proved that the method of orthogonal axonometry was by this triangle fully determined.[20] In this method of representation Pelz has solved all metric problems on basic space elements (lines and planes perpendicular to each other, the representation of a circle, the length of a segment, the distance of the origin of the system of coordinates from a plane given by its traces, the revolvement of a plane). Even problems of both central and parallel illumination could not escape to his attention, e. g.: illumination of a sphere; construction of isophotes on the sphere and on the cylindrical and conic surfaces of revolution; the parallel illumination of the semi-sphere surface. Pelz was also the first who – for the construction of the shadow of a sphere (in a parallel illumination) into an auxiliary plane of projection – used a sphere symmetrically conjugate to the given one with respect to the plane of shadows. The axes of the contour of the shadow (generally an ellipse) were lying on diagonals of a rhomb generated by the contour of the projection of two cylindrical surfaces circumscribed to the spheres.[21]

Karel Pelz dealt of course with the fundamental theorem of oblique axonometry as well. It was the Theorem of Pohlke, which was named in this way by mathematicians after its publication in the first volume of Pohlke's textbook on descriptive geometry[22] in 1860, where Pohlke noted, that an elementary proof of the theorem does probably not exist, and therefore it would be delayed to the second volume of the textbook. The first elementary and genially simple complete proof of the theorem of Pohlke was given by young H. Schwarz, a student of Pohlke. Schwarz proved in 1863 the generalized theorem: "An arbitrary quadruple of non-collinear points of the plane of projection can be considered as a parallel projection of the vertices of a tetrahedron, for which the proportions of the lengths of its sides and their mutual angles are known".[23] Pohlke published the proof of Schwarz by mutual consent in the second edition of the first volume of his textbook in 1866.[24] The original proof of Pohlke was never published. In the course of the next half century were many proofs of this theorem published, synthetic

---

[19] Analogous theorems related to the illumination of quadrics are direct consequences of the theorems 2–4.

[20] The property of axonometric triangle that the axonometric projection of the origin of the orthonormal system of coordinates is its orthocenter has been proved by the German mathematician O. Schlömilch (1823–1901) in 1856. See [5].

[21] One of the surfaces is the tangent illumination figure of a given sphere; the second one is symmetrically conjugate to it with respect to the plane of shades.

[22] Karl Pohlke (1810–1876), professor of descriptive geometry at the University of technology in Berlin; *Darstellende Geometrie* (Descriptive geometry, Berlin, 1860, 1866, 1872 – first volume, Berlin 1876 – second volume)

[23] The concidered tetrahedron is similar to an arbitrary chosen tetrahedron.

[24] More details about the history of the proof of this theorem and its importance in the didactics of mathematics can be found in [18].

and analytic. One of the first was the proof of Karel Pelz in 1877 ([9]). The proof of Pelz was important from the historical standpoint; Schwarz recognized in it the proof of Pohlke, which he was acquainted with, but was not able to reproduce (in the time of the publication of Pelz's proof Karl Pohlke was already dead). Next we present at least the draft of the main idea of the proof of Karel Pelz, who proved the fundamental theorem of oblique axonometry in approximately the following wording:

Theorem 5. The vertices of an arbitrary quadrilateral $O'X'Y'Z'$ of the plane of projection can be considered as the parallel projection of the vertices of the orthonormal coordinate tetrahedron *OXYZ*.[25]

Outline of the proof: Let us choose in the plane of projection ε three arbitrary segments $O'X'$, $O'Y'$, $O'Z'$ in order the points X', Y', Z' to have the required character; moreover let the point O be lying in the plane ε of projection ($O = O'$).

a) At first let us consider the ellipse k' in the plane ε given by two conjugate half-diameters, that are coincident with an arbitrary pair of given segments (for example $O'X'$, $O'Y'$). The ellipse k' can be considered as an oblique projection of the circle k with its diameter in an arbitrary chosen diameter of the ellipse k' ($k \subset \alpha$, $\alpha \neq \varepsilon$), i.e. $A = A'$, $B = B'$. The angle of the planes α, ε can be chosen arbitrarily; it corresponds to an arbitrary choice of the orthogonal projection of one endpoint C of the diameter CD ($CD \perp AB$) of the circle k into the plane ε.[26] (Fig. 3) The system of oblique projection g: $k \mapsto k'$ is given with the line CC', where $C'D'$ is the diameter of ellipse k' conjugate to the diameter $A'B'$. (These oblique projections are two; the second one projects into point C' the point $D \in k$.) Let us construct point E: $OE \cong OA$ ($\cong OC$) and $OE \perp \alpha$ (the construction of point E_1 is evident from the theorem on the orthogonal projection of a circle[27]), and its oblique projection E' into the plane of projection ε ($\Delta E'E_1(E) \sim \Delta\, C'C_1(C)$). Then it is sufficient to construct the points X, Y in a way, that $X = (g/\alpha)^{-1}(X')$, $Y = (g/\alpha)^{-1}(Y')$. The figure *OXYE* is an orthonormal tetrahedron, the vertices of which are in the projection g projected into the points O', X', Y', E'. The point E' is in a general case different from point Z', that's why it is needed further on to answer the following two questions: – what is the set of all points E' for all possible positions of plane α by its rotation around the line AB; – what is the set of all points E' for all possibilities of the choice of the diameter $A'B'$?

b) By the rotation of the plane α about the line AB the point C moves on a circular path of the circle $k^* \subset \lambda$, while the projector CC' is passing through the generators of a circular conical surface $\boldsymbol{K}^*$ with the directrix k^* and the vertex C'. At the same time the projector of the point E is passing through a set of parallel lines with the mentioned projectors. (The point E is a point of the circle k^*, for which the oriented angle $\angle COE$ is congruent to the positive right angle; the orientation of the angle (by the motion of the point C) remains invariant. Pelz has proved (analytically) that the set of traces of the projectors of all positions of the point E was a hyperbola $h \subset \varepsilon$ coaxial with the ellipse k'; the (linear) excentricities of

[25] The theorem is introduced in a modern – from the viewpoint of terminology correct – version. *OXYZ* is a tetrahedron, in which all edges *OX*, *OY*, *OZ* are congruent and perpendicular one to each other (= *orthonormal coordinate tetrahedron*).

[26] Normal projections into the plane ε will have a right subscript "1". The angle of planes α and ε can be determined by the revolvement of the plane λ ($CD \subset \lambda$, $\lambda \perp \varepsilon$): $\angle\alpha\varepsilon \cong \angle\,(C)OC_1$; $(O) = O_1 = O$.

[27] From it follows: $E \in \lambda$, $OE \cong OC$, $OE \perp OC \Rightarrow |O_1E_1| = e$, where e is the linear excentricity of the ellipse k_1. The construction can be made by help of the revolvement of plane λ *to the plane of projection* ε; E, C, are the end points of perpendicular half-diameters of the circle $k^* = \lambda \cap \boldsymbol{G}$, where $\boldsymbol{G}$ is a sphere with the center O and diameter OA.

both conic sections were equal (the real axis of the hyperbola h coincided with the minor axis of the ellipse k'). The normal projection of the projector of the point E ($\overleftrightarrow{E_1E'}$) is a tangent to the hyperbola h at the point E' and the asymptotes of the hyperbola h are parallel to the lines $C'K$ and $C'L$ (the points K, L being points of the circle k^* in the plane ε.) [28] Karl Pohlke has probably proved (synthetically) that the set of points of the projectors of every point $E \in k^*$ (corresponding to the point C that pass through points of the circle k^*) is a quadratic surface. Every two rectilinear generators of this surface are the skew lines; from the classification of quadrics it follows that this surface is a one-sheet hyperboloid. The traces of its generators lie on the intersection of this hyperboloid with the plane of projection (the Pelz's hyperbola h).

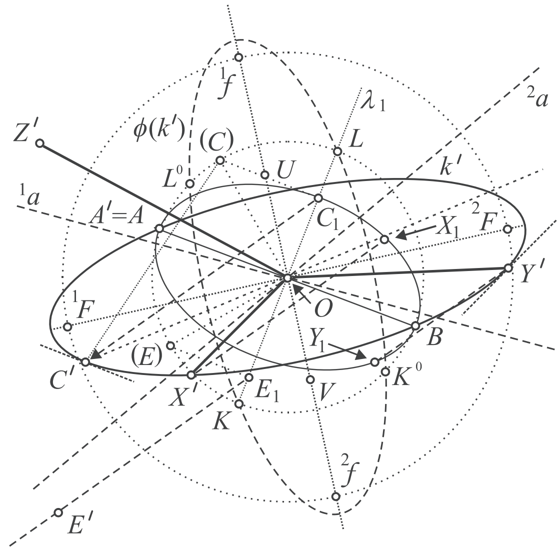

Fig. 3

c) When choosing another diameter ${}^1A{}^1B$ of the ellipse k' we get a system of one-sheet hyperboloids that intersect the plane ε in a system of confocal hyperbolas. Evidently through every point Z' in the plane ε (of the required characteristic) passes exactly one hyperbola ${}^ih$ ($i \in I$) that is determined by its foci ${}^1f$, ${}^2f$ and by the point Z'.[29] To every hyperbola ${}^ih$ corresponds exactly one sphere ${}^i\boldsymbol{G}$; its diameter ${}^iK{}^iL \subset \varepsilon$ is the revolved position of the diameter ${}^iA{}^iB$ of the ellipse k' by a right angle. From it follows the construction of the points ${}^iK$, ${}^iL$, and of the diameter ${}^iA{}^iB$.[30] Two conic sections $\varphi(k')$ and ${}^ih$ have generally the endpoints of two diameters common: ${}^iK{}^iL$, ${}^iK^o{}^iL^o$. Consequently there exist two planes ${}^i\lambda$, ${}^i\lambda^o$ each perpendicular to the plane ε and coinciding with exactly one of these diameters. Let us consider the plane ${}^i\lambda$. The normal projection of the point $Z \in {}^i\lambda$ lies on the tangent of the

[28] KL is a diameter of the hyperbola h in the plane λ with vertices U, V. (Fig. 3)
[29] The construction of the major axis and asymptotes of hyperbola ${}^1h$ follows from its focal characteristics.
[30] Evidently this is valid: ${}^iK, {}^iL \in \varphi(k') \cap {}^ih$, where $({}^iK\,{}^iLO) = -1$ and φ is the rotation in the plane of projection with the center O by a right angle.

hyperbola ${}^{i}h$ at the point Z'. The construction of the normal projection ${}^{i}C_1$ of the point ${}^{i}C \in {}^{i}k^*$ (${}^{i}k^* = {}^{i}\boldsymbol{G} \cap {}^{i}\lambda$), as well as of the point ${}^{i}C' \in k'$ is evident from *a*), *b*). The parallel projection ${}^{i}g$ (${}^{i}g$: ${}^{i}C \mapsto {}^{i}C'$) has the required characteristic: it projects the vertices *O*, *X*, *Y*, *Z* of the orthonormal coordinate tetrahedron to the vertices O', X', Y', Z' of the given quadrilateral of the plane of projection.[31]

In conclusion it can be said, that Karel Pelz was able in a masterly way to use the synthetic methods of projective and descriptive geometry to deduce and fill in known and to trace new constructions and methods. He considered every problem from the next viewpoint: the problem solving has to be *compendious* (with respect to the number of basic operations) and *exact*. He put the most emphasis on transparent clearness and simplicity. It was reflected as well in his own lectures, which enjoyed great popularity. It really can be regretted, that Pelz have not left behind a systematic publication on descriptive geometry. At least his axonometry got into a book compilation in the work of his former assistant and later successor in Graz *Rudolf Schüssler "Orthogonale Axonometrie. Ein Lehrbuch zum Selbststudium"* (Orthogonal axonometry. A textbook for self-study, Berlin 1905). The author wanted to repay to a great descriptive geometer his favor for the use of descriptive geometry and its tuition at universities. In the work not only the ideas of Pelz are methodically explained for those, who are already acquainted with the basics of this discipline, but they are also applied on a wide scale of interesting problems theoretical and practical too.

It is a great pleasure for us to express our deep gratitude to Professor Ján Čižmár for his help both in the study of documents and the thorough and competent recension of this paper.

References

[1] Folta J.: *Česká geometrická škola* (*Historická analýza*). Nakladatelství ČSAV, Academia, Praha, no. 9, 1982.

[2] Kadeřávek F., Klíma J., Kounovský J.: *Deskriptivní geometrie*. Díl druhý, Jednota česko-slovenských matematiků a fysiků, Praha, 1932.

[3] Kadeřávek F., Klíma J., Kounovský J.: *Deskriptivní geometrie.* Díl I., Jednota česko--slovenských matematiků a fysiků, Praha, 1932.

[4] Klapka J.: *Deskriptivní geometrie*. Vědecko-technické nakladatelství, Praha, 1951.

[5] Loria G.: *Storia della geometria descrittiva*. Ulrico Hoepli, Milano, 1921.

[6] Pelz K.: *Über die Bestimmung der Achsen von Zentralprojekzionen des Kreises.* Věstník královské české společnosti nauk v Praze, 1872.[32]

[7] Pelz K.: *Die Axenbestimmung der Kegelflächen zweiten Grades.* Věstník královské české společnosti nauk v Praze, 1874.

[8] Pelz K.: *Construction der Axen einer Ellipse aus zwei conjugirten Diametern.* Výročná správa c. k. reálky v Těšíně, Těšín, 1876.

---

[31] On the Fig. 3 there is constructed (within the analysis of the problem) the orthonormal coordinate tetrahedron *OXYE*; the quadrangle *O′X′Y′E′* is its parallel projection into the plane ε. As the hyperbola ${}^{i}h$ – passing through the point *Z′* – is by this point (and its foci) fully determined as well as its intersection points with the ellipse $\varphi(k')$ (endpoints of the diameters ${}^{i}K\ {}^{i}L$, ${}^{i}K^o\ {}^{i}L^o$), the construction of the orthonormal tetrahedron *OXYZ* is the trivial consequence of a), b).

[32] The journal is presented in the bibliography also under the title: Sitzungsberichte der königl. Böhmischen Gesellschaft der Wissenschaften, Prag

[9] Pelz K.: *Über einen neuen Beweis des Fundamentalsatzes von Pohlke.* Sitzungsb. der kaiser. Akademie der Wissenschaften in Wien, LXXVI, 1877, 123–138.

[10] Pelz K.: *Beiträge zur Bestimmung der Selbstschatten- und Schlagschattengrenzen der Flächen zweiten Grades bei Centralbeleuchtung.* Sitzungsb. der kaiser. Akademie der Wissen. in Wien, 1878.

[11] Pelz K.: *Über die Focalkurven des Quetelet.* Sitzungsb. der kaiser. Akademie der Wissenschaften in Wien, LXXXII, 1880, 1207–1219.

[12] Pelz K.: *Zur Joachimsthal'schen Lösung des Normalenproblems.* Věstník královské české společnosti nauk v Praze, 1898.

[13] Pelz K.: *Deskriptivní geometrie dle přednášek v r. 1906–7.* ČVUT, Praha, 1907, 479 pages, 515 pictures. – Litografované prednášky. (Identified in accordance with the quotation in the book Jarolímek V. – Procházka B.: *Deskriptivní geometrie pro vysoké školy technické*, Praha, 1909).

[14] Procházka B.: *Vybrané statě z deskriptivní geometrie*. Česká matica technická, Praha, 1912.

[15] Sklenáriková Z.: *Emil Müller – vrcholný predstaviteľ viedenskej geometrickej školy.* ***G*** – Slovenský časopis pre geometriu a grafiku, vol. 1, n. 2, STU Bratislava 2004.

[16] Sklenáriková Z.: *Z dejín deskriptívnej geometrie v Rakúsko-Uhorsku* (First part of a double article). ***G*** – slovenský časopis pre geometriu a grafiku, vol. 3, n. 5, STU Bratislava 2006.

[17] Sklenáriková Z.: *Z dejín deskriptívnej geometrie v Rakúsko-Uhorsku. Niektorí najvýznamnejší predstavitelia* (Second part of a double article). ***G*** – slovenský časopis pre geometriu a grafiku, vol. 3, n. 6, STU Bratislava 2006.

[18] Sklenáriková Z., Pémová M.: *The Pohlke-Schwarz Theorem and its Relevancy in the Didactics of Mathematics.* Quaderni di Ricerca in Didattica, n. 17, 2007; G.R.I.M. (http://math.unipa.it/~grim/quaderno17.htm).

[19] Sklenáriková Z.: *100 rokov od smrti Karla Pelza.* In: ***G*** – slovenský časopis pre geometriu a grafiku, vol. 5, n. 9, STU Bratislava 2008.

[20] Sobotka J.: *O životě a činnosti Karla Pelze.* Časopis pro pěstování mathematiky a fysiky 39(1910), 433– 460.

Adress

RNDr. Zita Sklenáriková, PhD.
Department of algebra, geometry and didactics of mathematics
Faculty of mathematics, physics and informatics, Comenius University
Mlynská dolina
842 48 Bratislava
Slovakia
e-mail: *zita.sklenarikova@fmph.uniba.sk*

PaedDr. Marta Pémová, PhD.
Department of algebra, geometry and didactics of mathematics
Faculty of mathematics, physics and informatics, Comenius University
Mlynská dolina
842 48 Bratislava
Slovakia
e-mail: *marta.pemova@fmph.uniba.sk*

Marko Razpet