PEXIDER'S FUNCTIONAL EQUATION

Ivan Netuka

1. Atmospheric pressure

This section serves as motivation for our subsequent exposition. We will derive an equation describing the dependence of atmospheric pressure on height from the Earth's surface.

The reference point is at height $x_0 = 0$ where the pressure equals p_0. Let us denote by $p(x)$ the value of the pressure at height $x \geq 0$. The following two assumptions are quite natural from the point of view of physics: p is a strictly positive function on $[0, \infty)$ and, for fixed y, the pressure at height $x+y$ is proportional to the pressure at height x. As a consequence there exists a function $q : [0, \infty) \to (0, 1]$ such that

$$p(x+y) = q(y) \cdot p(x), \quad x, y \in [0, \infty). \tag{1}$$

Hence we arrive at *one* equation for *two* unknown functions p and q. In Section 4, we shall prove that there exists $\gamma \geq 0$ such that

$$p(x) = p_0 e^{-\gamma x}, \quad x \in [0, \infty); \tag{2}$$

cf. [KS], [S].

It is worth emphasizing that this result can be derived without any continuity or differentiability assumption imposed on functions p and q.

For further discussion, it is convenient to define $f := \log p$, $h := \log q$. Then Eq. (1) has the form

$$f(x+y) = f(x) + h(y), \quad x, y \in [0, \infty).$$

This is a special case of the equation

$$f(x+y) = g(x) + h(y), \tag{3}$$

which is a functional equation for *three* functions. Eq. (3) is called *Pexider's equation.* It is remarkable that this *one* equation determines the functions f, g, and h.

Eq. (3), and analogous equations

$$\begin{aligned} f(x+y) &= g(x) \cdot h(y), \\ f(x \cdot y) &= g(x) + h(y), \\ f(x \cdot y) &= g(x) \cdot h(y), \end{aligned}$$

were investigated in the paper [P12]. Nowadays, the terms Pexider's equation, and equation of Pexider's type, are quite common in the mathematical literature; see, for instance, [K], [AD]. It is worth mentioning that Pexider's name appears also in recently published mathematical papers. A search of MathSciNet in 2006 reveals that the item *Pexider* appears 152 times, of which 94 occurrences are after 1990. Let us note that J. V. Pexider dealt with functional equations in [P4], [P5], [P6], [P10] and [P12].

2. Solution of Pexider's equation

We are interested in finding all real-valued functions f, g, and h such that

$$f(x+y) = g(x) + h(y), \quad x, y \in \mathbb{R}.$$

We shall show that a general solution can be obtained from knowledge of all solutions of this equation for the special case $f = g = h$.

Let us recall that a function $F : \mathbb{R} \to \mathbb{R}$ is said to be *additive*, if

$$F(x+y) = F(x) + F(y), \quad x, y \in \mathbb{R}. \tag{4}$$

Eq. (4) is called the *Cauchy functional equation* and will be studied in Section 3. Here we note only that it appeared as early as 1791 (A. M. Legendre) and 1809 (C. F. Gauss). It was, however, A. L. Cauchy who described all *continuous* solutions in 1821; for references, see, for instance, [K].

A close connection between Pexider's equation and additive functions is obvious: Given an additive function F and real numbers b and c, the functions

$$f := F + b + c, \quad g := F + b, \quad \text{and} \quad h := F + c$$

clearly satisfy Eq. (3). However, we are interested in *all* solutions of Eq. (3). To this end, let us suppose that functions f, g, and h satisfy Eq. (3). Let us put $b := g(0)$, $c := h(0)$ and define $F : z \mapsto f(z) - b - c$, $z \in \mathbb{R}$. For $y = 0$, Eq. (3) yields $f(x) = g(x) + c$, $x \in \mathbb{R}$. Similarly for $x = 0$ we obtain $f(y) = b + h(y)$. Inserting $g(x) = f(x) - c$, $h(y) = f(y) - b$, $x, y \in \mathbb{R}$, into Eq. (3), we arrive at

$$f(x+y) = f(x) + f(y) - b - c,$$

or

$$f(x+y) - b - c = (f(x) - b - c) + (f(y) - b - c), \quad x, y \in \mathbb{R}.$$

Hence the function F is additive. So we have the following result:

Theorem 1. *Let f, g, and h be real-valued functions such that*

$$f(x+y) = g(x) + h(y), \quad x, y \in \mathbb{R}.$$

Then there exists an additive function F and numbers b and c such that

$$f = F + b + c, \quad g = F + b, \quad \text{and} \quad h = F + c. \tag{5}$$

Conversely, if F is an arbitrary additive function and b and c are arbitrary real numbers, then the functions f, g, and h defined by Eq. (5) satisfy Eq. (3).

In this sense, the solvability of Eq. (3) is reduced to a description of additive functions.

Clearly, for every $a \in \mathbb{R}$, the function $F_a : x \mapsto ax$, $x \in \mathbb{R}$, is additive. These are the expected "natural" solutions of Eq. (4). However, there exist additive functions which are not of the form F_a; see Section 3.

Let us notice that, in proving Theorem 1, only the simplest algebraic properties of real numbers were used. Therefore, it is not surprising that Theorem 1 can be expressed in a purely algebraic form; cf. [AD], [S].

Theorem 2. *Let $(D,+)$ be a groupoid (with addition) and $(R,+)$ be an additive group (not necessarily Abelian). Let f, g, and h be mappings from D to R. Then f, g, and h satisfy the equation*

$$f(x+y) = g(x) + h(y), \quad x, y \in D,$$

if and only if there exists a homomorphism $A : D \to R$ and elements $b, c \in R$ such that

$$f(x) = b + A(x) + c, \quad g(x) = b + A(x), \quad \text{and} \quad h(x) = A(x) + c, \quad x \in D.$$

To say that $A : D \to R$ is a *homomorphism* simply means that $A(x+y) = A(x) + A(y)$, whenever $x, y \in D$.

3. The Cauchy equation

In this section we study solutions of Eq. (4). It is easy to see that every solution of Eq. (4) satisfies $F(r) = F(1) \cdot r$, whenever r is a rational number. This immediately implies that every *continuous* solution of Eq. (4) is necessarily of the form F_a with $a = F(1)$. The continuity assumption can be weakened. Here are the classical results: F is of the form F_a, provided that F is continuous at a point, or even if F is merely bounded from below or above on an interval, see [J], chap. 5, §13. This also follows from Theorem 3 proved below.

In 1905, G. Hamel showed in [H] that there are *discontinuous* solutions of Eq. (4).

A general result of linear algebra says that every vector space possesses a basis. In particular, the vector space $\mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers has a basis (usually called a *Hamel basis*). Given a Hamel basis B, it is useful to notice that an arbitrary function $f : B \to \mathbb{R}$ can be uniquely extended to a solution F of Eq. (4). Choosing two different elements $b_1, b_2 \in B$ and two numbers y_1, y_2 in such a way that $y_1 b_2 \neq y_2 b_1$, no solution F of Eq. (4) with $F(b_j) = y_j$, $j = 1, 2$, can be of the form F_a, so F is necessarily discontinuous.

G. Hamel proved that the graph of an arbitrary discontinuous solution of Eq. (4) is dense in $\mathbb{R}^2$, so such an additive function behaves rather "wildly".

Recall that the graph of a function F is the set

$$G(F) = \{(x, y) \in \mathbb{R}^2 : y = F(x)\}.$$

A set $Z \subset \mathbb{R}^2$ is said to be dense, if for every $z \in \mathbb{R}^2$ there exist $z_n \in Z$ converging to z.

Theorem 3. *Let F be a solution of Eq. (4) and suppose that $F = F_a$ for no $a \in \mathbb{R}$. Then the graph $G(F)$ of F is dense in $\mathbb{R}^2$.*

Standard proofs may be found, for instance, in [AD] and [B]. Here we offer a proof based on an idea from [R]; cf. also [S].

For $x \in \mathbb{R}$, we denote, as usual, by $[x]$ the largest integer not exceeding x and we put $\{x\} := x - [x]$.

To prove Theorem 3, let F be a solution of Eq. (4) such that $F = F_a$ for no $a \in \mathbb{R}$. Then there exists $s \in \mathbb{R}$ such that $F(s) \neq F(1) \cdot s$ (of course, the number s must be irrational). For a natural number n,

$$\begin{aligned} F(\{ns\}) &= F(ns) - F([ns]) = nF(s) - [ns]F(1) = \\ &= nF(s) - nsF(1) + \{ns\}F(1) = n(F(s) - sF(1)) + \{ns\}F(1). \end{aligned}$$

Fix $z_0 = (x_0, y_0) \in \mathbb{R}^2$ and choose sequences (r_n) and (s_n) of rational numbers in such a way that

$$r_n \to x_0, \quad s_n \to (y_0 - x_0F(1))/(F(s) - sF(1)), \quad n \to \infty.$$

Defining $x_n := r_n + s_n \cdot \{ns\}/n$, we have $x_n \to x_0$ and

$$\begin{aligned} F(x_n) &= r_nF(1) + (s_n/n)F(\{ns\}) = \\ &= r_nF(1) + s_n(F(s) - sF(1)) + (s_n/n)\{ns\}F(1). \end{aligned}$$

We see that

$$F(x_n) \to x_0F(1) + y_0 - x_0F(1) = y_0, \quad n \to \infty.$$

Consequently, for $z_n = (x_n, F(x_n))$, we have $z_n \to z_0$, $n \to \infty$, which shows that the graph $G(F)$ of F is dense in $\mathbb{R}^2$.

As already noted, one-sided boundedness of an additive function F on an interval implies $F = F_a$ for a suitable $a \in \mathbb{R}$. This assertion can be substantially strengthened as follows:

Theorem 4. *Let $F : \mathbb{R} \to \mathbb{R}$ be an additive function, $M \subset \mathbb{R}$, and let F be lower or upper bounded on M. If M has positive Lebesgue measure, then there exists $a \in \mathbb{R}$ such that $F = F_a$.*

Corollary 5. *Every measurable additive function F is of the form F_a.*

To see this, define $M_k := F^{-1}((-\infty, k))$ for each integer k. Then M_k is measurable and $\bigcup_k M_k = \mathbb{R}$. Hence at least one of the sets M_k has positive Lebesgue measure and F is upper bounded there.

The idea of the proof of Theorem 4 goes back to A. Ostrowski [O] and is based on the following remarkable result of H. Steinhaus; see [K], p. 69:

If $M \subset \mathbb{R}$ has positive Lebesgue measure, then the set

$$M + M := \{x + y : x, y \in M\}$$

contains an open interval.

To prove Theorem 4 for, say, an additive function F which is upper bounded on M, fix $m \in \mathbb{R}$ such that $F \leq m$ on M. If $z \in M + M$, then $z = x + y$ for some $x, y \in M$, and hence

$$F(z) = F(x + y) = F(x) + F(y) \leq 2m.$$

We see that F is upper bounded on an open interval, so the graph $G(F)$ of F is not dense in $\mathbb{R}^2$. Now we apply Theorem 3 (or [J]; chap. 3, §13).

The results summarized above lead to the following informal statement: a solution of the Cauchy functional equation either is a linear function of the form F_a, or behaves in a "pathological" way. Further justification for the term "pathological" may be found in [K], chap. 12.

4. Back to atmospheric pressure

We shall start with the following simple result:

Theorem 6. *Suppose that* $H : [0, \infty) \to [0, \infty)$ *satisfies*

$$H(x + y) = H(x) + H(y), \quad x, y \in [0, \infty).$$

Then there exists $a \geq 0$ *such that* $H(x) = ax$, $x \in [0, \infty)$.

The proof is easy: If $x_1, x_1', x_2, x_2' \in [0, \infty)$ are such that $x_1 - x_1' = x_2 - x_2'$, then

$$H(x_1) + H(x_2') = H(x_1 + x_2') = H(x_2 + x_1') = H(x_2) + H(x_1'),$$

and hence

$$H(x_1) - H(x_1') = H(x_2) - H(x_2').$$

It follows that the function

$$F(x) = H(x_1) - H(x_1')$$

is well defined on $\mathbb{R}$ for x of the form $x = x_1 - x_1'$, where $x_1, x_1' \in [0, \infty)$. It is easy to see that F is additive, $F = H$ on $[0, \infty)$ and $F \geq 0$ on $[0, \infty)$, which gives the result.

In Section 1, we were interested in functions $f : [0, \infty) \to \mathbb{R}$ and $h : [0, \infty) \to (-\infty, 0]$ satisfying

$$f(x + y) = f(x) + h(y), \quad x, y \in [0, \infty).$$

Now we shall apply Theorem 2 for $D = [0, \infty)$ and $R = \mathbb{R}$. We know that there exist a function $A : [0, \infty) \to \mathbb{R}$ satisfying $A(x+y) = A(x)+A(y)$, $x, y \in [0, \infty)$ and numbers $b, c \in \mathbb{R}$ such that

$$f(x) = b + A(x) + c, \quad f(x) = b + A(x), \quad \text{and} \quad h(x) = A(x) + c,$$

whenever $x \in [0, \infty)$. This yields $b = f(0)$ and $c = 0$, since obviously $A(0) = 0$. Now we apply Theorem 6 for $H := -A$ $(= -h)$. We conclude that there exists $\gamma \geq 0$ such that $A(x) = -\gamma x$, $x \in [0, \infty)$. Recalling from Section 1 that $f = \log p$, $h = \log q$, we arrive at

$$q(x) = e^{-\gamma x}, \quad p(x) = e^{f(x)} = e^{b} \cdot e^{-\gamma x}, \quad x \in [0, \infty).$$

Now $e^b = p(0) = p_0$. So we have verified that the dependence of the atmospheric pressure on height is given by Eq. (2).

References

[AD] J. Aczél, J. Dhombres, *Functional equations in several variables*, Cambridge Univ. Press, Cambridge, 1989.

[B] R. P. Boas, *A primer of real functions*, MAA, Washington, 1981.

[H] G. Hamel, *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung $f(x+y) = f(x) + f(y)$*, Math. Ann. 60 (1905), 459–462.

[J] V. Jarník, *Differential calculus II* (in Czech), NČSAV, Praha, 1956.

[KS] P. Kahlig, J. Schwaiger, *Logarithmic wind profile and Pexider's functional equation*, Beitr. Phys. Atmosph. 64 (1991), 153–155.

[K] M. Kuczma, *An introduction to the theory of functional equations and inequalities*, Państwowe wydawnictwo naukowe, Kraków, 1985.

[O] A. Ostrowski, *Über die Funktionalgleichung der Exponentialfunktion und verwandte Funktionalgleichungen*, Jber. Deutsch. Math.-Verein. 38 (1929), 54–62.

[R] L. Reich, *Über Approximation durch Werte additiver Funktionen*, Sitz. Ber. Öster. Akad. d. Wiss. 201 (1992), 169–181.

[S] J. Schwaiger, *Funktionalgleichungen: Grundlegende Gleichungen mit Anwendungen – ein kleiner Überblick*, Jahrbuch Überblicke Mathematik 1994 (Eds. S. D. Chatterji et al.), Braunschweig, Vieweg, 1994.