Differential and Integral Equations

VI. Nonlinear boundary value problems (Perturbation theory)

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VI. *Nonlinear boundary value problems (perturbation theory)*

1. Preliminaries

In this chapter we shall prove some theorems on the existence of solutions to nonlinear boundary value problems for nonlinear ordinary differential equations of the form

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}) + \varepsilon\, \boldsymbol{g}(t, \boldsymbol{x}, \varepsilon), \qquad \boldsymbol{S}(\boldsymbol{x}) + \varepsilon\, \boldsymbol{R}(\boldsymbol{x}, \varepsilon) = \boldsymbol{0}$$

under the assumption that the existence of a solution to the corresponding shortened boundary value problem

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}), \qquad \boldsymbol{S}(\boldsymbol{x}) = \boldsymbol{0}$$

is guaranteed. ($\boldsymbol{S}$ and $\boldsymbol{R}$ are n-vector valued functionals; $\boldsymbol{x} \in R_n$, $\boldsymbol{f}$: $\mathscr{D} \subset R \times R_n \to R_n$, $\boldsymbol{g}$: $\mathfrak{D} \subset R \times R_n \times R \to R_n$ and $\varepsilon > 0$ is a small parameter.)

The present section provides the survey of the basic theory for the equation

(1,1) $$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}).$$

The proofs may be found in many textbooks on ordinary differential equations (e.g. Coddington, Levinson [1] or Reid [1]).

1.1. Notation. Let $\mathscr{D} \subset R_{p+q}$, $\boldsymbol{u}_0 \in R_p$ and $\boldsymbol{v}_0 \in R_q$. Then

$$\mathscr{D}_{(\boldsymbol{u}_0, .)} = \{\boldsymbol{v} \in R_q;\ (\boldsymbol{u}_0, \boldsymbol{v}) \in \mathscr{D}\} \quad \text{and} \quad \mathscr{D}_{(.,\boldsymbol{v}_0)} = \{\boldsymbol{u} \in R_p;\ (\boldsymbol{u}, \boldsymbol{v}_0) \in \mathscr{D}\}.$$

If $\boldsymbol{f}$ maps $\mathscr{D}$ into R_n, then $\boldsymbol{f}(., \boldsymbol{v}_0)$ and $\boldsymbol{f}(\boldsymbol{u}_0, .)$ denote the mappings given by

$$\boldsymbol{f}(., \boldsymbol{v}_0):\ \boldsymbol{u} \in \mathscr{D}_{(.,\boldsymbol{v}_0)} \to \boldsymbol{f}(\boldsymbol{u}, \boldsymbol{v}_0) \in R_n$$

and

$$\boldsymbol{f}(\boldsymbol{u}_0, .):\ \boldsymbol{v} \in \mathscr{D}_{(\boldsymbol{u}_0, .)} \to \boldsymbol{f}(\boldsymbol{u}_0, \boldsymbol{v}) \in R_n.$$

1.2. Definition. Let $\mathscr{D} \subset R_{n+1}$ be open and let the n-vector valued function $\boldsymbol{f}(t, \boldsymbol{x})$ be defined for $(t, \boldsymbol{x}) \in \mathscr{D}$.

(a) We shall say that $\boldsymbol{f}$ fulfils the *Carathéodory conditions* on $\mathscr{D}$ and write $\boldsymbol{f} \in \mathrm{Car}(\mathscr{D})$ if

(i) for a.e. $t \in R$ such that $\mathscr{D}_{(t,\cdot)} \neq \emptyset$, $\boldsymbol{f}(t, .)$ is continuous;
(ii) given $\boldsymbol{x} \in R_n$ such that $\mathscr{D}_{(\cdot, x)} \neq \emptyset$, $\boldsymbol{f}(., \boldsymbol{x})$ is measurable;
(iii) given $(t_0, \boldsymbol{x}_0) \in \mathscr{D}$, there exist $\delta_1 > 0$, $\delta_2 > 0$ and $m \in L^1[t_0 - \delta_1, t_0 + \delta_1]$ such that $[t_0 - \delta_1, t_0 + \delta_1] \times \mathfrak{B}(\boldsymbol{x}_0, \delta_2; R_n) \subset \mathscr{D}$ and $|\boldsymbol{f}(t, \boldsymbol{x})| \leq m(t)$ for a.e. $t \in [t_0 - \delta_1, t_0 + \delta_1]$ and any $\boldsymbol{x} \in \mathfrak{B}(\boldsymbol{x}_0, \delta_2; R_n)$;

(b) We shall write $\boldsymbol{f} \in \mathrm{Lip}(\mathscr{D})$ if

(iv) given $(t_0, \boldsymbol{x}_0) \in \mathscr{D}$, there exist $\delta_1 > 0$, $\delta_2 > 0$ and $\omega \in L^1[t_0 - \delta_1, t_0 + \delta_1]$ such that $[t_0 - \delta_1, t_0 + \delta_1] \times \mathfrak{B}(\boldsymbol{x}_0, \delta_2; R_n) \subset \mathscr{D}$ and $|\boldsymbol{f}(t, \boldsymbol{x}_1) - \boldsymbol{f}(t, \boldsymbol{x}_2)| \leq \omega(t) |\boldsymbol{x}_1 - \boldsymbol{x}_2|$ for a.e. $t \in [t_0 - \delta_1, t_0 + \delta_1]$ and all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathfrak{B}(\boldsymbol{x}_0, \delta_2; R_n)$.

1.3. Definition. An n-vector function $\boldsymbol{x}(t)$ is said to be a *solution to the equation* (1,1) *on the interval* $\Delta \subset R$ if it is absolutely continuous on Δ and such that $(t, \boldsymbol{x}(t)) \in \mathscr{D}$ for a.e. $t \in \Delta$ and

$$\boldsymbol{x}'(t) = \boldsymbol{f}(t, \boldsymbol{x}(t)) \qquad \text{a.e. on } \Delta .$$

1.4. Theorem (Carathéodory). *Let* $\mathscr{D} \subset R_{n+1}$ *be open and* $\boldsymbol{f} \in \mathrm{Car}(\mathscr{D})$. *Given* $(t_0, \boldsymbol{c}) \in \mathscr{D}$, *there exists* $\delta > 0$ *such that the equation* (1,1) *possesses a solution* $\boldsymbol{x}(t)$ *on* $(t_0 - \delta, t_0 + \delta)$ *such that* $\boldsymbol{x}(t_0) = \boldsymbol{c}$.

1.5. Remark. Obviously, if $\boldsymbol{f} \in C(\mathscr{D})$, then $\boldsymbol{f} \in \mathrm{Car}(\mathscr{D})$ and the equation (3,1) possesses for any $(t_0, \boldsymbol{c}_0) \in \mathscr{D}$ a solution $\boldsymbol{x}(t)$ on a neighbourhood Δ of t_0 such that $\boldsymbol{x}(t_0) = \boldsymbol{c}_0$. Since the function $t \in \Delta \to \boldsymbol{f}(t, \boldsymbol{x}(t)) \in R_n$ is continuous on Δ, it follows immediately that $\boldsymbol{x}'$ is continuous on Δ $(\boldsymbol{x} \in C^1_n(\Delta))$.

1.6. Definition. The equation (1,1) has the *property* $(\mathscr{U})$ (*local uniqueness*) on $\mathscr{D} \in R_{n+1}$, if for any couple of its solutions $\boldsymbol{x}_1(t)$ on Δ_1 and $\boldsymbol{x}_2(t)$ on Δ_2 such that $\boldsymbol{x}_1(t_0) = \boldsymbol{x}_2(t_0)$ for some $t_0 \in \Delta_1 \cap \Delta_2$, $\boldsymbol{x}_1(t) \equiv \boldsymbol{x}_2(t)$ *on* $\Delta_1 \cap \Delta_2$.

1.7. Theorem. *Let* $\mathscr{D} \subset R_{n+1}$ *and* $\boldsymbol{f} \in \mathrm{Lip}(\mathscr{D})$. *Then the equation* (1,1) *has the property* $(\mathscr{U})$ *on* $\mathscr{D}$.

1.8. Definition. The *solution* $\boldsymbol{x}(t)$ of (1,1) on Δ is said to be *maximal* if for any solution $\boldsymbol{x}_1(t)$ of (1,1) on Δ_1 such that $\Delta \subset \Delta_1$ and $\boldsymbol{x}(t) = \boldsymbol{x}_1(t)$ on Δ we have $\Delta = \Delta_1$.

1.9. Lemma. *If the definition domain* $\mathscr{D}$ *of* $\boldsymbol{f}(t, \boldsymbol{x})$ *is open and the solution* $\boldsymbol{x}(t)$ *of* (1,1) *on* Δ *is maximal, then* Δ *is open.*

1.10. Notation. Given $(t_0, \mathbf{c}) \in \mathscr{D}$, $\boldsymbol{\varphi}(\,.\,; t_0, \mathbf{c})$ denotes the corresponding maximal solution of (1,1), $\Delta(t_0, \mathbf{c})$ its definition interval and

$$\Omega = \{(t, t_0, \mathbf{c}) \in R \times R \times R_n;\ (t_0, \mathbf{c}) \in \mathscr{D},\ t \in \Delta(t_0, \mathbf{c})\}\,.$$

1.11. Theorem. *Let $\mathscr{D} \subset R_{n+1}$ be open, $\mathbf{f} \in \mathrm{Car}(\mathscr{D})$ and let the equation (1,1) have the property $(\mathscr{U})$. Then for any $(t_0, \mathbf{c}) \in \mathscr{D}$ there exists a unique maximal solution $\mathbf{x}(t) = \boldsymbol{\varphi}(t; t_0, \mathbf{c})$ of (1,1) on $\Delta = \Delta(t_0, \mathbf{c}) \subset R$ such that $\mathbf{x}(t_0) = \mathbf{c}$. The set Ω (cf. 1.10) is open and the mapping $\boldsymbol{\varphi}$: $(t, t_0, \mathbf{c}) \in \Omega \to \boldsymbol{\varphi}(t; t_0, \mathbf{c}) \in R_n$ is continuous ($\boldsymbol{\varphi} \in C(\Omega)$).*

1.12. Corollary. *Let $\mathscr{D} \subset R_{n+1}$, $\mathbf{f} \in \mathrm{Car}(\mathscr{D})$ and (1,1) have the property $(\mathscr{U})$. Let $(t_0, \mathbf{c}_0) \in \mathscr{D}$, $-\infty < a < b < \infty$ and let $[a, b] \subset \Delta(t_0, \mathbf{c}_0)$. Then there exists $\delta > 0$ such that $|\mathbf{c} - \mathbf{c}_0| \leq \delta$ implies $(t_0, \mathbf{c}) \in \mathscr{D}$ and $\Delta(t_0, \mathbf{c}) \supset [a, b]$, i.e. for any $\mathbf{c} \in \mathfrak{B}(\mathbf{c}_0, \delta; R_n)$ the corresponding maximal solution $\boldsymbol{\varphi}(t, t_0, \mathbf{c})$ of (1,1) is defined on $[a, b]$.*

1.13. Remark. Let us recall that if $\mathbf{f}$: $\mathscr{D} \to R_n$ possesses on $\mathscr{D}$ partial derivatives with respect to the components x_j of $\mathbf{x}$, then $\partial\mathbf{f}/\partial\mathbf{x}$ denotes the Jacobi matrix of $\mathbf{f}$ with respect to $\mathbf{x}$ which is formed by the rows $(\partial\mathbf{f}/\partial x_j)$ $(j = 1, 2, \ldots, n)$. If the $n \times n$-matrix valued function $(t, \mathbf{x}) \in \mathscr{D} \to (\partial\mathbf{f}/\partial\mathbf{x})(t, \mathbf{x}) \in L(R_n)$ fulfils the Carathéodory condition (iii) in 1.2, then making use of the Mean Value Theorem I.7.4 we obtain easily that $\mathbf{f} \in \mathrm{Lip}(\mathscr{D})$.

1.14. Theorem. *Let $\mathscr{D} \subset R_{n+1}$, $\mathbf{f} \in \mathrm{Car}(\mathscr{D})$ and $(\partial\mathbf{f}/\partial\mathbf{x}) \in \mathrm{Car}(\mathscr{D})$. Then the equation (1,1) has the property $(\mathscr{U})$ and hence there exist $\Omega \subset R_{n+2}$ and the continuous mapping $\boldsymbol{\varphi}$: $\Omega \to R_n$ defined in 1.11. Furthermore $(\partial\boldsymbol{\varphi}/\partial\mathbf{c})(t, t_0, \mathbf{c})$ exists and is continuous in $(t, t_0, \mathbf{c})$ on Ω. For any $(t_0, \mathbf{c}) \in \mathscr{D}$ the $n \times n$-matrix valued function $\mathbf{A}(t) = (\partial\mathbf{f}/\partial\mathbf{x})(t, \boldsymbol{\varphi}(t, t_0, \mathbf{c}))$ is L-integrable on each compact subinterval of $\Omega_{(.,t_0,\mathbf{c})} = \Delta(t_0, \mathbf{c})$ and $\mathbf{U}(t) = (\partial\boldsymbol{\varphi}/\partial\mathbf{c})(t, t_0, \mathbf{c})$ is the maximal solution of the linear matrix differential equation $\mathbf{U}' = \mathbf{A}(t)\,\mathbf{U}$ such that $\mathbf{U}(t_0) = \mathbf{I}_n$.*

1.15. Remark. It follows from 1.14 that $(\partial\boldsymbol{\varphi}/\partial\mathbf{c})(t, t_0, \mathbf{c})$ is for any $(t_0, \mathbf{c}) \in \mathscr{D}$ the fundamental matrix solution of the variational equation

$$\mathbf{u}' = \left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(t, \boldsymbol{\varphi}(t, t_0, \mathbf{c}))\right)\mathbf{u}$$

on $\Delta(t_0, \mathbf{c})$. Consequently for any $(t, t_0, \mathbf{c}) \in \Omega$ it possesses an inverse matrix $((\partial\boldsymbol{\varphi}/\partial t)(t, t_0, \mathbf{c}))^{-1}$.

1.16. Theorem. *Let $\mathscr{D} \subset R_{n+1}$, $\mathbf{f} \in \mathrm{Car}(\mathscr{D})$, $(\partial\mathbf{f}/\partial\mathbf{x}) \in \mathrm{Car}(\mathscr{D})$ and $\partial^2\mathbf{f}/(\partial x_i\,\partial x_j) \in \mathrm{Car}(\mathscr{D})$ for any $i, j = 1, 2, \ldots, n$. Then the n-vector valued function $\boldsymbol{\varphi}$ from 1.11* *possesses on Ω all the partial derivatives $\partial^2\boldsymbol{\varphi}/(\partial c_i\,\partial c_j)$ $(i,j = 1,2,\ldots,n)$ and they are continuous in $(t, t_0, \mathbf{c})$ on Ω $(\boldsymbol{\varphi} \in C^2(\Omega))$.*

1.17. Remark. Let $\mathfrak{D} \subset R_1 \times R_n \times R_p$ be open and let the n-vector valued function $\mathbf{h}(t, \mathbf{u}, \mathbf{v})$ map $\mathfrak{D}$ into R_n. The differential equation

$$\mathbf{x}' = \mathbf{h}(t, \mathbf{x}, \mathbf{v}) \tag{1,2}$$

is said to be an equation with a parameter $\mathbf{v} \in R_p$. Let us put

$$\boldsymbol{\xi} = (\mathbf{x}, \mathbf{v}) \quad \text{for} \quad \mathbf{x} \in R_n \quad \text{and} \quad \mathbf{v} \in R_p,$$
$$\tilde{\mathbf{h}}(t, \boldsymbol{\xi}) = \mathbf{h}(t, \mathbf{x}, \mathbf{v}) \quad \text{for} \quad (t, \boldsymbol{\xi}) = (t, (\mathbf{x}, \mathbf{v})) \in \mathfrak{D}$$

and

$$\tilde{\mathbf{f}}(t, \boldsymbol{\xi}) = \begin{pmatrix} \tilde{\mathbf{h}}(t, \boldsymbol{\xi}) \\ \mathbf{0}_p \end{pmatrix} \in R_{n+p} \quad \text{for} \quad (t, \boldsymbol{\xi}) \in \mathfrak{D}.$$

Now, applying the above theorems to the equation

$$\boldsymbol{\xi}' = \tilde{\mathbf{f}}(t, \boldsymbol{\xi}) \qquad \begin{pmatrix} \mathbf{x}' = \mathbf{h}(t, \mathbf{x}, \mathbf{v}) \\ \mathbf{v}' = \mathbf{0} \end{pmatrix}$$

we can easily obtain theorems on the existence, uniqueness, continuous dependence of a solution $\mathbf{x}(t) = \boldsymbol{\varphi}(t;\, t_0, \mathbf{c}, \mathbf{v})$ of (1,2) on the initial data $(t_0, \mathbf{c})$ and on the parameter $\mathbf{v}$ as well as theorems on the differentiability of $\boldsymbol{\varphi}$ with respect to t, $\mathbf{c}$ and $\mathbf{v}$. The formulation of the general statements may be left to the reader. For our purposes only the following lemma is needed.

1.18. Lemma. *Let $\mathscr{D} \subset R_{n+1}$ and $\mathfrak{D} \subset R_{n+2}$ be open, $\varkappa > 0$, $\mathscr{D} \times [0, \varkappa] \subset \mathfrak{D}$, $\mathbf{f}\colon \mathscr{D} \to R_n$ and $\mathbf{g}\colon \mathfrak{D} \to R_n$. Let us put $\tilde{\mathbf{g}}(t, \mathbf{y}) = \mathbf{g}(t, \mathbf{x}, \varepsilon)$ for $(t, \mathbf{x}, \varepsilon) \in \mathfrak{D}$ and $\mathbf{y} = (\mathbf{x}, \varepsilon)$. Let $\mathbf{f} \in \mathrm{Car}(\mathscr{D})$, $\tilde{\mathbf{g}} \in \mathrm{Car}(\mathfrak{D})$ and let for any $\varepsilon \in [0, \varkappa]$ the equation*

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) + \varepsilon\, \mathbf{g}(t, \mathbf{x}, \varepsilon) \tag{1,3}$$

possess the property $(\mathscr{U})$ on $\mathfrak{D}$. Then

(i) *given $(t_0, \mathbf{c}, \varepsilon) \in \mathscr{D} \times [0, \varkappa]$, there exists a unique maximal solution $\mathbf{x}(t) = \boldsymbol{\psi}(t;\, t_0, \mathbf{c}, \varepsilon)$ of (1,3) on the interval $\Delta = \Delta(t_0, \mathbf{c}, \varepsilon)$ such that $\mathbf{x}(t_0) = \mathbf{c}$;*

(ii) *the set $\Omega = \{(t, t_0, \mathbf{c}, \varepsilon);\ (t_0, \mathbf{c}, \varepsilon) \in \mathscr{D} \times [0, \varkappa],\ t \in \Delta(t_0, \mathbf{c}, \varepsilon)\} \subset R_{n+3}$ is open and the mapping $\boldsymbol{\psi}\colon \Omega \to R_n$ is continuous;*

(iii) *if $-\infty < a < b < \infty$, $(a, \mathbf{c}_0) \in \mathscr{D}$ and $[a, b] \subset \Delta(a, \mathbf{c}_0, 0)$, then there exist $\varrho_0 > 0$ and $\varkappa_0 > 0$, $\varkappa_0 \leq \varkappa$ such that $[a, b] \subset \Delta(a, \mathbf{c}, \varepsilon)$ for any $\mathbf{c} \in \mathfrak{B}(\mathbf{c}_0, \varrho_0;\, R_n)$ and $\varepsilon \in [0, \varkappa_0]$.*

The following theorem provides an example of conditions which assure the existence of a solution to the equation on the given compact interval $[a, b] \subset R$.

1.19. Theorem. *Let* $-\infty < a < b < \infty$, $[a,b] \times R_n \subset \mathscr{D} \subset R_{n+1}$, $\mathscr{D}$ *open and let the n-vector valued function* $\boldsymbol{f}: \mathscr{D} \to R_n$ *fulfil the assumptions*

(i) $\boldsymbol{f}(t, .)$ *is continuous on* R_n *for a.e.* $t \in [a,b]$;

(ii) $\boldsymbol{f}(., \boldsymbol{x})$ *is measurable on* $[a,b]$ *for any* $\boldsymbol{x} \in R_n$;

(iii) *there exist* $\alpha \in R$, $0 \le \alpha \le 1$, *and L-integrable on* $[a,b]$ *scalar functions* $p(t)$ *and* $q(t)$ *such that*

$$|\boldsymbol{f}(t, \boldsymbol{x})| \le p(t) + q(t)\,|\boldsymbol{x}|^\alpha \qquad \text{for any} \quad \boldsymbol{x} \in R_n \quad \text{and a.e.} \quad t \in [a,b].$$

Let the $n \times n$*-matrix valued function* $\boldsymbol{A}: [a,b] \to L(R_n)$ *be L-integrable on* $[a,b]$. *Then for any* $t_0 \in [a,b]$ *and* $\boldsymbol{c} \in R_n$ *there exists a solution* $\boldsymbol{x}(t)$ *of the equation*

$$\boldsymbol{x}' = \boldsymbol{A}(t)\,\boldsymbol{x} + \boldsymbol{f}(t, \boldsymbol{x})$$

on $[a,b]$ *such that* $\boldsymbol{x}(t_0) = \boldsymbol{c}$.

This auxiliary section will be completed by proving the following lemmas which illustrate the assumptions on the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ employed in this chapter.

1.20. Lemma. *Let* $\mathscr{D} \subset R_{n+1}$ *and* $\mathfrak{D} \subset R_{n+2}$ *be open,* $\varkappa > 0$, $[0,1] \times R_n \subset \mathscr{D}$ *and* $\mathscr{D} \times [0,\varkappa] \subset \mathfrak{D}$. *Furthermore, let us assume that the functions* $\boldsymbol{f}: \mathscr{D} \to R_n$ *and* $\boldsymbol{g}: \mathfrak{D} \to R_n$ *are such that* $\boldsymbol{f} \in \mathrm{Car}(\mathscr{D})$ *and* $\tilde{\boldsymbol{g}} \in \mathrm{Car}(\mathfrak{D})$, *where* $\tilde{\boldsymbol{g}}(t, \boldsymbol{y}) = \boldsymbol{g}(t, \boldsymbol{x}, \varepsilon)$ *for* $(t, \boldsymbol{x}, \varepsilon) \in \mathfrak{D}$ *and* $\boldsymbol{y} \in (\boldsymbol{x}, \varepsilon)$. *Let us put*

$$(\boldsymbol{F}(\boldsymbol{x}))(t) = \boldsymbol{f}(t, \boldsymbol{x}(t)) \quad \text{and} \quad (\boldsymbol{G}(\boldsymbol{x}, \varepsilon))(t) = \boldsymbol{g}(t, \boldsymbol{x}(t), \varepsilon)$$

for $\boldsymbol{x} \in C_n$, $\varepsilon \in [0,\varkappa]$ *and* $t \in [0,1]$. *Then* $\boldsymbol{F}(\boldsymbol{x}) \in L^1_n$ *and* $\boldsymbol{G}(\boldsymbol{x}, \varepsilon) \in L^1_n$ *for any* $\boldsymbol{x} \in C_n$ *and* $\varepsilon \in [0,\varkappa]$. *The operators* $\boldsymbol{F}: \boldsymbol{x} \in C_n \to \boldsymbol{F}(\boldsymbol{x}) \in L^1_n$ *and* $\boldsymbol{G}: (\boldsymbol{x}, \varepsilon) \in C_n \times [0,\varkappa] \to \boldsymbol{G}(\boldsymbol{x}, \varepsilon) \in L^1_n$ *are continuous.*

Proof. It is sufficient to show only the assertions concerning $\boldsymbol{G}$.

(a) Let $\varrho > 0$. Since $\tilde{\boldsymbol{g}} \in \mathrm{Car}(\mathfrak{D})$ ($\tilde{\boldsymbol{g}}(t,\boldsymbol{y}) = \boldsymbol{g}(t, \boldsymbol{x}, \varepsilon)$, where $\boldsymbol{y} = (\boldsymbol{x}, \varepsilon)$), applying the Borel Covering Theorem it is easy to find a function $m \in L^1$ such that

$$|\boldsymbol{g}(t, \boldsymbol{x}, \varepsilon)| \le m(t) \qquad \text{for any} \quad \boldsymbol{x} \in \mathfrak{B}(\boldsymbol{0}, \varrho; R_n), \quad \varepsilon \in [0,\varkappa] \quad \text{and a.e.} \quad t \in [0,1]. \tag{1,4}$$

Let the functions $\boldsymbol{x}_k: [0,1] \to R_n$ and the numbers $\varepsilon_k \in [0,\varkappa]$ $(k = 0,1,2,\ldots)$ be such that $\lim_{k\to\infty} \boldsymbol{x}_k(t) = \boldsymbol{x}_0(t)$ on $[0,1]$ and $\lim_{k\to\infty} \varepsilon_k = \varepsilon_0$. Under our assumptions on $\boldsymbol{g}$ this implies that

$$\lim_{k\to\infty} \boldsymbol{g}(t, \boldsymbol{x}_k(t), \varepsilon_k) = \boldsymbol{g}(t, \boldsymbol{x}_0(t), \varepsilon_0) \qquad \text{a.e. on} \ [0,1]. \tag{1,5}$$

If each of the functions $\chi_k(t) = \boldsymbol{g}(t, \boldsymbol{x}_k(t), \varepsilon_k)$ $(k = 0,1,2,\ldots)$ is measurable on $[0,1]$ and $|\boldsymbol{x}_k(t)| \le \varrho$ on $[0,1]$ for any $k = 0,1,2,\ldots$, then by the Lebesgue Dominated

Convergence Theorem

$$\lim_{k\to\infty}\int_0^1 |\mathbf{g}(t,\mathbf{x}_k(t),\varepsilon_k)-\mathbf{g}(t,\mathbf{x}_0(t),\varepsilon_0)|\,\mathrm{d}t=0\,.$$

(b) Let $\mathbf{x}_0\in C_n$ and $\varrho=\|\mathbf{x}_0\|_C+1$. It is well-known that there exist functions $\mathbf{x}_k\colon [0,1]\to R_n$ $(k=1,2,\ldots)$ piecewise constant on $[0,1]$ and such that $|\mathbf{x}_k(t)|\le\varrho$ $(k=1,2,\ldots)$ and $\lim_{k\to\infty}\mathbf{x}_k(t)=\mathbf{x}_0(t)$ on $[0,1]$. In particular, (1,5) with $\varepsilon_k=\varepsilon$ $(k=0,1,\ldots)$ holds and since any function $\gamma_k\colon t\in[0,1]\to\mathbf{g}(t,\mathbf{x}_k(t),\varepsilon)$ $(\varepsilon\in[0,\varkappa]$, $k=1,2,\ldots)$ is obviously measurable, $\gamma_0\colon t\in[0,1]\to\mathbf{g}(t,\mathbf{x}_0(t),\varepsilon)$ is measurable for any $\varepsilon\in[0,\varkappa]$ and hence according to (1,4) $\gamma_0\in L^1_n$.

The continuity of the operator $\mathbf{G}$ follows easily from the first part of the proof.

1.21. Lemma. *Let $\mathscr{D}\subset R_{n+1}$ and $\mathbf{f}\colon\mathscr{D}\to R_n$ satisfy the corresponding assumptions of 1.20. In addition, let $\partial\mathbf{f}/\partial\mathbf{x}\in\mathrm{Car}(\mathscr{D})$. Then $\mathbf{F}$ defined in 1.20 possesses on C_n the Gâteaux derivative $\mathbf{F}'(\mathbf{x})$ continuous in $\mathbf{x}$ on C_n. Given $\mathbf{x},\mathbf{u}\in C_n$,*

$$([\mathbf{F}'(\mathbf{x})]\,\mathbf{u})(t)=\left[\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(t,\mathbf{x}(t))\right]\mathbf{u}(t)\qquad \textit{for a.e.}\quad t\in[0,1]\,.$$

Proof. (a) Let us put for $\mathbf{x}\in C_n$ and $t\in[0,1]$

$$[\mathbf{A}(\mathbf{x})](t)=\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(t,\mathbf{x}(t))\,.$$

By 1.14 the $n\times n$-matrix valued function $\mathbf{A}(\mathbf{x})$ is L-integrable on $[0,1]$ for any $\mathbf{x}\in C_n$. If f_j $(j=1,2,\ldots,n)$ are the components of $\mathbf{f}$, then

$$[\mathbf{A}_j(\mathbf{x})](t)=\frac{\partial f_j}{\partial\mathbf{x}}(t,\mathbf{x}(t))\qquad(j=1,2,\ldots,n)$$

are columns of $[\mathbf{A}(\mathbf{x})](t)$. By 1.20 the mappings

$$\mathbf{x}\in C_n\to\mathbf{A}_j(\mathbf{x})\in L^1_n\qquad(j=1,2,\ldots,n)\tag{1,6}$$

are continuous. Obviously, for any $\mathbf{x}\in C_n$

$$\mathbf{J}(\mathbf{x})\colon\ \mathbf{u}\in C_n\to[\mathbf{A}(\mathbf{x})](t)\,\mathbf{u}(t)\in L^1_n$$

is a linear bounded operator. Moreover,

$$\|\mathbf{J}(\mathbf{x})\|=\sup_{\|\mathbf{u}\|_C\le 1}\|\mathbf{J}(\mathbf{x})\,\mathbf{u}\|_{L^1}\le\|\mathbf{A}(\mathbf{x})\|_{L^1}=\max_{j=1,2,\ldots,n}\|\mathbf{A}_j(\mathbf{x})\|_{L^1}$$

and consequently the operator $\mathbf{x}\in C_n\to\mathbf{J}(\mathbf{x})\in B(C_n,L^1_n)$ is continuous.

(b) By the Mean Value Theorem I.7.4

$$\frac{(\boldsymbol{F}(\boldsymbol{x}_0+\vartheta\boldsymbol{u}))(t)-(\boldsymbol{F}(\boldsymbol{x}_0))(t)}{\vartheta}=\frac{\boldsymbol{f}(t,\boldsymbol{x}_0(t)+\vartheta\,\boldsymbol{u}(t))-\boldsymbol{f}(t,\boldsymbol{x}_0(t))}{\vartheta}$$

$$=\left(\int_0^1\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}(t,\boldsymbol{x}_0(t)+\lambda\vartheta\,\boldsymbol{u}(t))\,\mathrm{d}\lambda\right)\boldsymbol{u}(t)$$

and

$$\left\|\frac{\boldsymbol{F}(\boldsymbol{x}_0+\vartheta\boldsymbol{u})-\boldsymbol{F}(\boldsymbol{x}_0)}{\vartheta}-\boldsymbol{J}(\boldsymbol{x}_0)\,\boldsymbol{u}\right\|_{L^1}$$

$$\leq\int_0^1\left(\int_0^1\left|\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}(t,\boldsymbol{x}_0(t)+\lambda\vartheta\,\boldsymbol{u}(t))-\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}(t,\boldsymbol{x}_0(t))\right|\mathrm{d}\lambda\right)\mathrm{d}t\,\|\boldsymbol{u}\|_C\,.$$

By the Tonelli-Hobson Theorem I.4.36 we may change the order of the integration in the last integral. The continuity of the mappings (1,6) yields

$$\lim_{\vartheta\to0+}\int_0^1\left|\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}(t,\boldsymbol{x}_0(t)+\lambda\vartheta\,\boldsymbol{u}(t))-\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}(t,\boldsymbol{x}_0(t))\right|\mathrm{d}t=0$$

uniformly with respect to $\lambda\in[0,1]$. Consequently,

$$\lim_{\vartheta\to0+}\left\|\frac{\boldsymbol{F}(\boldsymbol{x}_0+\vartheta\boldsymbol{u})-\boldsymbol{F}(\boldsymbol{x}_0)}{\vartheta}-\boldsymbol{J}(\boldsymbol{x})\,\boldsymbol{u}\right\|_{L^1}=0$$

for any $\boldsymbol{x}_0\in C_n$ and $\boldsymbol{u}\in C_n$. This completes the proof.

1.22. Remark. Given $\boldsymbol{x}\in AC_n$ and $\boldsymbol{L}\in B(C_n,L_n^1)$, $\|\boldsymbol{x}\|_C\leq\|\boldsymbol{x}\|_{AC}$, $\boldsymbol{L}\in B(AC_n,L_n^1)$ and

$$\|\boldsymbol{L}\|_{B(AC_n,L_n^1)}=\sup_{\|u\|_{AC}\leq1}\|\boldsymbol{Lu}\|_{L^1}\leq\sup_{\|u\|_C\leq1}\|\boldsymbol{Lu}\|_{L^1}=\|\boldsymbol{L}\|_{B(C_n,L_n^1)}\,.$$

It follows readily that 1.20 and 1.21 remain valid also if in their formulations C_n is replaced everywhere by AC_n.

1.23. Remark. If moreover $\partial^2\boldsymbol{f}/(\partial x_i\,\partial x_j)\in\mathrm{Car}(\mathscr{D})$ $(i,j=1,2,\ldots,n)$, it may be shown that for any $\boldsymbol{x}\in C_n$, $\boldsymbol{F}$ possesses the second order Gâteaux derivative $\boldsymbol{F}''(\boldsymbol{x})$ such that the mapping $\boldsymbol{x}\in C_n\to\boldsymbol{F}''(\boldsymbol{x})\in B(C_n,B(C_n,L_n^1))$ is continuous. Given $\boldsymbol{x},\boldsymbol{u},\boldsymbol{v}\in C_n$, the components of the n-vector $([\boldsymbol{F}''(\boldsymbol{x})\,\boldsymbol{u}]\,\boldsymbol{v})(t)$ are given by

$$\sum_{i=1}^n\left(\sum_{j=1}^n\frac{\partial^2f_k}{\partial x_i\,\partial x_j}(t,\boldsymbol{x}(t))\,u_i(t)\right)v_j(t)\,,\qquad k=1,2,\ldots,n\,.$$

Let $[0,1] \times \{0\} \times R_n \subset \Omega$. Let us put for $\mathbf{y} \in C_n$

$$\begin{aligned}
(1,7) \qquad \boldsymbol{\Phi}(\mathbf{y})(t) &= \varphi(t,0,\mathbf{y}(t)) \quad \text{on } [0,1],\\
\boldsymbol{\Phi}_t(\mathbf{y})(t) &= \frac{\mathrm{d}\varphi}{\mathrm{d}t}(t,0,\mathbf{y}(t)) \quad \text{a.e. on } [0,1],\\
\boldsymbol{\Phi}_c(\mathbf{y})(t) &= \frac{\partial \varphi}{\partial \mathbf{c}}(t,0,\mathbf{y}(t)) \quad \text{on } [0,1].
\end{aligned}$$

It is easy to verify that $\boldsymbol{\Phi}$ and $\mathbf{F}\boldsymbol{\Phi}$ are continuous mappings of C_n into C_n and L^1_n, respectively, and $\boldsymbol{\Phi}_c$ is a continuous mapping of C_n into the space of $n \times n$-matrix valued function which are continuous on $[0,1]$ (cf. 1.14 and 1.20). Since $\|\mathbf{y}\|_{AC} = |\mathbf{y}(0)| + \|\mathbf{y}'\|_{L^1}$ for any $\mathbf{y} \in AC_n$, it follows readily that $\boldsymbol{\Phi}$ is a continuous mapping of AC_n into AC_n. Analogously $\boldsymbol{\Phi}_c$ is a continuous mapping of AC_n into the space of $n \times n$-matrix valued functions absolutely continuous on $[0,1]$, i.e. if $\boldsymbol{\Phi}_c(\mathbf{y})$ denotes also the linear operator $\mathbf{h} \in AC_n \to \boldsymbol{\Phi}_c(\mathbf{y})(t)\,\mathbf{h}(t)$, then $\mathbf{y} \in AC_n \to \boldsymbol{\Phi}_c(\mathbf{y})$ is a continuous mapping of AC_n into $B(AC_n)$. Let us notice that for any $\mathbf{y} \in C_n$

$$(1,8) \qquad \frac{\partial \varphi}{\partial t}(t,0,\mathbf{y}(t)) = \mathbf{f}(t, \varphi(t,0,\mathbf{y}(t))) = \mathbf{F}(\boldsymbol{\Phi}(\mathbf{y}))(t) \qquad \text{a.e. on } [0,1]$$

and by 1.14

$$\begin{aligned}
(1,9) \qquad \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial \mathbf{c}}(t,0,\mathbf{y}(t))\right) &= \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t, \varphi(t,0,\mathbf{y}(t))\right]\frac{\partial \varphi}{\partial \mathbf{c}}(t,0,\mathbf{y}(t))\\
&= ([\mathbf{F}'(\boldsymbol{\Phi}(\mathbf{y}))]\,\boldsymbol{\Phi}_c(\mathbf{y}))(t) \qquad \text{a.e. on } [0,1].
\end{aligned}$$

Moreover, for any $\mathbf{y} \in AC_n$

$$\boldsymbol{\Phi}_t(\mathbf{y})(t) = \mathbf{F}(\boldsymbol{\Phi}(\mathbf{y}))(t) + \boldsymbol{\Phi}_c(\mathbf{y})(t)\,\mathbf{y}'(t)$$

and thus $\boldsymbol{\Phi}_t$ is a continuous operator $AC_n \to L^1_n$.

Let $\mathbf{y}, \mathbf{h} \in AC_n$ and $\vartheta \in (0,1)$. Then

$$\begin{aligned}
(1,10) \qquad & \left\| \frac{\boldsymbol{\Phi}(\mathbf{y} + \vartheta \mathbf{h}) - \boldsymbol{\Phi}(\mathbf{y})}{\vartheta} - \boldsymbol{\Phi}_c(\mathbf{y})\,\mathbf{h} \right\|_{AC}\\
&\le \left| \frac{\varphi(0,0,\mathbf{y}(0) + \vartheta\,\mathbf{h}(0)) - \varphi(0,0,\mathbf{y}(0))}{\vartheta} - \frac{\partial \varphi}{\partial \mathbf{c}}(0,0,\mathbf{y}(0))\,\mathbf{h}(0) \right|\\
&+ \int_0^1 \left| \frac{\frac{\partial \varphi}{\partial t}(t,0,\mathbf{y}(t) + \vartheta\,\mathbf{h}(t)) - \frac{\partial \varphi}{\partial t}(t,0,\mathbf{y}(t))}{\vartheta} - \frac{\partial^2 \varphi}{\partial t\,\partial \mathbf{c}}(t,0,\mathbf{y}(t))\,\mathbf{h}(t) \right| \mathrm{d}t
\end{aligned}$$

$$+\int_0^1 \left| \frac{\dfrac{\partial \varphi}{\partial \mathbf{c}}(t,0,\mathbf{y}(t)+\vartheta\,\mathbf{h}(t))\,\vartheta\,\mathbf{h}'(t)}{\vartheta} - \frac{\partial \varphi}{\partial \mathbf{c}}(t,0,\mathbf{y}(t))\,\mathbf{h}'(t)\right| \mathrm{d}t$$

$$+\int_0^1 \left| \frac{\left(\dfrac{\partial \varphi}{\partial \mathbf{c}}(t,0,\mathbf{y}(t)+\vartheta\,\mathbf{h}(t)) - \dfrac{\partial \varphi}{\partial \mathbf{c}}(t,0,\mathbf{y}(t))\right)\mathbf{y}'(t)}{\vartheta} - \frac{\partial^2 \varphi}{\partial \mathbf{c}^2}(t,0,\mathbf{y}(t))\,\mathbf{y}'(t)\,\mathbf{h}(t)\right| \mathrm{d}t\,.$$

Obviously, the first and the third terms on the right-hand side of (1,10) tend to 0 as $\vartheta \to 0+$. Furthermore, by (1,8), (1,9) and the Mean Value Theorem the second one becomes

$$\int_0^1 \left| \frac{\mathbf{f}(t,\ \varphi(t,0,\mathbf{y}(t)+\vartheta\,\mathbf{h}(t))) - \mathbf{f}(t,\ \varphi(t,0,\mathbf{y}(t)))}{\vartheta} - \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t,\ \varphi(t,0,\mathbf{y}(t)))\right]\left(\frac{\partial \varphi}{\partial \mathbf{c}}(t,0,\mathbf{y}(t))\right)\mathbf{h}(t)\right| \mathrm{d}t$$

$$\le \int_0^1 \left(\int_0^1 \left\| \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t,\ \varphi(t,0,\mathbf{y}(t)+\vartheta\lambda\,\mathbf{h}(t)))\right]\left(\frac{\partial \varphi}{\partial \mathbf{c}}(t,0,\mathbf{y}(t)+\vartheta\lambda\,\mathbf{h}(t))\right) - \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t,\ \varphi(t,0,\mathbf{y}(t)))\right]\left(\frac{\partial \varphi}{\partial \mathbf{c}}(t,0,\mathbf{y}(t))\right)\right\| |\mathbf{h}(t)|\,\mathrm{d}\lambda\right)\mathrm{d}t\,.$$

It is easy to verify that this last expression tends to 0 as $\vartheta \to 0+$. (Obviously, $\mathbf{F}'\boldsymbol{\Phi}_c$ is a continuous operator $C_n \to B(C_n, L^1_n)$.) Analogously, the Mean Value Theorem yields that also the fourth term of the right-hand side of (1,10) tends to 0 as $\vartheta \to 0+$.

1.24. Lemma. *Under the assumptions of* 1.16, *the operator* $\boldsymbol{\Phi}$ *given by* (1,7) *is a continuous mapping of* AC_n *into* AC_n *which is Gâteaux differentiable at any* $\mathbf{x} \in AC_n$. *Given* $\mathbf{y}, \mathbf{h} \in AC_n$,

$$([\boldsymbol{\Phi}'(\mathbf{y})]\,\mathbf{h})(t) = \left[\frac{\partial \varphi}{\partial \mathbf{c}}(t,0,\mathbf{y}(t))\right]\mathbf{h}(t)\,.$$

The mapping $\mathbf{y} \in AC_n \to \boldsymbol{\Phi}'(\mathbf{y}) \in B(AC_n)$ *is continuous.*

1.25. Definition. Let $\mathfrak{D} \subset R_{n+2}$ be open, $\varkappa > 0$, $[0,1] \times R_n \times [0,\varkappa] \subset \mathfrak{D}$ and $\mathbf{g}\colon \mathfrak{D} \to R_n$. Let $\varepsilon_0 \in [0,\varkappa]$ and let for given $t \in [0,1]$ and $\mathbf{x}_0 \in R_n$ there exist $\delta_0 = \delta_0(t,\mathbf{x}_0) > 0$, $\varrho_0 = \varrho_0(t,\mathbf{x}_0) > 0$, $\varkappa_0 = \varkappa_0(t,\mathbf{x}_0) > 0$ and $\omega \in L^1(t-\delta_0, t+\delta_0)$ such that $|\tau - t| < \delta_0$, $|\mathbf{x}_1 - \mathbf{x}_0| < \varrho_0$, $|\mathbf{x}_2 - \mathbf{x}_0| < \varrho_0$, $\varepsilon \ge 0$ and $|\varepsilon - \varepsilon_0| < \varkappa_0$ implies $(\tau,\mathbf{x}_1,\varepsilon) \in \mathfrak{D}$, $(\tau,\mathbf{x}_2,\varepsilon) \in \mathfrak{D}$ and

$$|\mathbf{g}(\tau,\mathbf{x}_2,\varepsilon) - \mathbf{g}(\tau,\mathbf{x}_1,\varepsilon)| \le \omega(\tau)\,|\mathbf{x}_2 - \mathbf{x}_1|\,.$$

Then $\mathbf{g}$ is said to be *locally lipschitzian in* $\mathbf{x}$ *near* $\varepsilon = \varepsilon_0$ and we shall write $\mathbf{g} \in \mathrm{Lip}\,(\mathfrak{D}, \varepsilon_0)$.

1.26. Lemma. *Let* $\mathfrak{D} \subset R_{n+2}$ *and* $\boldsymbol{g}$: $\mathfrak{D} \to R_n$ *satisfy the corresponding assumptions of* 1.20. *In addition, let* $\boldsymbol{g} \in \operatorname{Lip}(\mathfrak{D}, \varepsilon_0)$. *Then* $\boldsymbol{G}$ *defined in* 1.20 *is locally lipschitzian in* $\boldsymbol{x}$ *near* $\varepsilon = \varepsilon_0$.

Proof follows from Definition 1.25 applying the Borel Covering Theorem.

1.27. Remark. In order that the operators $\boldsymbol{F}$ and $\boldsymbol{G}$ might possess the properties from 1.20 – 1.26 locally, it is sufficient to require that the assumptions of the corresponding lemmas are fulfilled only locally.

2. Nonlinear boundary value problems for functional-differential equations

Let $\varkappa > 0$ and let $\boldsymbol{F}$: $C_n \to L^1_n$, $\boldsymbol{G}$: $AC_n \times [0, \varkappa] \to L^1_n$, $\boldsymbol{S}$: $C_n \to R_n$ and $\boldsymbol{R}$: $AC_n \times [0, \varkappa] \to R_n$ be continuous operators. To a given $\varepsilon \in [0, \varkappa]$ we want to find a solution $\boldsymbol{x}$ of the functional-differential equation

$$\boldsymbol{x}' = \boldsymbol{F}(\boldsymbol{x}) + \varepsilon\, \boldsymbol{G}(\boldsymbol{x}, \varepsilon) \tag{2,1}$$

on the interval $[0, 1]$ which verifies the side condition

$$\boldsymbol{S}(\boldsymbol{x}) + \varepsilon\, \boldsymbol{R}(\boldsymbol{x}, \varepsilon) = \boldsymbol{0}\,. \tag{2,2}$$

This boundary value problem will be referred to as BVP $(\mathscr{P}_\varepsilon)$. The limit problem for $\varepsilon = 0$

$$\boldsymbol{x}' = \boldsymbol{F}(\boldsymbol{x})\,, \tag{2,3}$$

$$\boldsymbol{S}(\boldsymbol{x}) = \boldsymbol{0} \tag{2,4}$$

is denoted by $(\mathscr{P}_0)$.

2.1. Definition. Let $\varepsilon \in [0, \varkappa]$. An n-vector valued function $\boldsymbol{x}$ *is a solution to* (2,1) *on* $[0, 1]$ if $\boldsymbol{x} \in AC_n$ and

$$\boldsymbol{x}'(t) = (\boldsymbol{F}(\boldsymbol{x}))(t) + \varepsilon(\boldsymbol{G}(\boldsymbol{x}, \varepsilon))(t) \qquad \text{a.e. on } [0, 1]\,.$$

2.2. Remark. Let $\boldsymbol{x}_0 \in C_n$, $\omega \in L^1$, $\varrho > 0$ and

$$|(\boldsymbol{F}(\boldsymbol{x}_2))(t) - (\boldsymbol{F}(\boldsymbol{x}_1))(t)| \le \omega(t)\, \|\boldsymbol{x}_2 - \boldsymbol{x}_1\|_C \tag{2,5}$$

for any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathfrak{B}(\boldsymbol{x}_0, \varrho;\, C_n)$ and a.e. $t \in [0, 1]$. Then

$$\left|\frac{\boldsymbol{F}(\boldsymbol{x}_0 + \vartheta \boldsymbol{u})(t) - \boldsymbol{F}(\boldsymbol{x}_0)(t)}{\vartheta}\right| \le \omega(t)\, \|\boldsymbol{u}\|_C$$

for any $\vartheta > 0$, $\boldsymbol{u} \in C_n$ and a.e. $t \in [0, 1]$. If $\boldsymbol{F}$ possesses the Gâteaux derivative $\boldsymbol{F}'(\boldsymbol{x}_0)$ at $\boldsymbol{x}_0$, then

$$\lim_{\vartheta \to 0+} \left| \frac{\boldsymbol{F}(\boldsymbol{x}_0 + \vartheta \boldsymbol{u})(t) - \boldsymbol{F}(\boldsymbol{x}_0)(t)}{\vartheta} - ([\boldsymbol{F}'(\boldsymbol{x}_0)] \boldsymbol{u})(t) \right| = 0 \qquad \text{a.e. on } [0, 1].$$

It follows easily that

$$|([\boldsymbol{F}'(\boldsymbol{x}_0)] \boldsymbol{u})(t)| \leq \omega(t) \|\boldsymbol{u}\|_C \qquad \text{for any} \quad \boldsymbol{u} \in C_n \quad \text{and a.e.} \quad t \in [0, 1].$$

In particular, there exists a function $\boldsymbol{P}$: $[0, 1] \times [0, 1] \to L(R_n)$ such that $\boldsymbol{P}(., s)$ is measurable on $[0, 1]$ for any $s \in [0, 1]$, $\varrho(t) = |\boldsymbol{P}(t, 0)| + \operatorname{var}_0^1 \boldsymbol{P}(t, .) < \infty$ for a.e. $t \in [0, 1]$, $\varrho \in L^1$ ($\boldsymbol{P}$ is an $L^1[BV]$-kernel) and

$$([\boldsymbol{F}'(\boldsymbol{x}_0)] \boldsymbol{u})(t) = \int_0^1 \mathrm{d}_s[\boldsymbol{P}(t, s)] \boldsymbol{u}(s) \qquad \text{for any} \quad \boldsymbol{u} \in C_n \quad \text{and a.e.} \quad t \in [0, 1]$$

(cf. Kantorovič, Pinsker, Vulich [1]).

2.3. Theorem. *Let $\boldsymbol{x}_0 \in AC_n$ be a solution to BVP $(\mathscr{P}_0)$, where $\boldsymbol{F}$: $C_n \to L_n^1$ and $\boldsymbol{S}$: $C_n \to R_n$ are continuous operators. Furthermore, let us assume that (2,5) holds and $\boldsymbol{F}, \boldsymbol{S} \in C^1(\mathfrak{B}(\boldsymbol{x}_0, \varrho; C_n))$ for some $\varrho > 0$. If the linear BVP for $\boldsymbol{u} \in AC_n$*

$$\boldsymbol{u}' = [\boldsymbol{F}'(\boldsymbol{x}_0)] \boldsymbol{u}, \tag{2,6}$$

$$[\boldsymbol{S}'(\boldsymbol{x}_0)] \boldsymbol{u} = \boldsymbol{0} \tag{2,7}$$

possesses only the trivial solution, then there exists $\varrho_0 > 0$ such that there is no other solution $\boldsymbol{x}$ of BVP $(\mathscr{P}_0)$ such that $\|\boldsymbol{x} - \boldsymbol{x}_0\|_{AC} \leq \varrho_0$.

Proof. Let us put

$$\mathscr{F}: \ \boldsymbol{x} \in AC_n \to \begin{pmatrix} \boldsymbol{x}' - \boldsymbol{F}(\boldsymbol{x}) \\ \boldsymbol{S}(\boldsymbol{x}) \end{pmatrix} \in L_n^1 \times R_n. \tag{2,8}$$

By the assumption $\mathscr{F}(\boldsymbol{x}_0) = \boldsymbol{0}$ and $\mathscr{F} \in C^1(\mathfrak{B}(\boldsymbol{x}_0, \varrho; AC_n))$,

$$\mathscr{F}'(\boldsymbol{x}): \ \boldsymbol{u} \in AC_n \to \begin{pmatrix} \boldsymbol{u}' - \boldsymbol{F}'(\boldsymbol{x}) \boldsymbol{u} \\ \boldsymbol{S}'(\boldsymbol{x}) \boldsymbol{u} \end{pmatrix} \in L_n^1 \times R_n \tag{2,9}$$

for any $\boldsymbol{x} \in \mathfrak{B}(\boldsymbol{x}_0, \varrho; AC_n)$.

Let $\mathscr{F}(\boldsymbol{x}) = \boldsymbol{0}$ for some $\boldsymbol{x} \in \mathfrak{B}(\boldsymbol{x}_0, \varrho; AC_n)$, $\boldsymbol{x} \neq \boldsymbol{x}_0$. By the Mean Value Theorem I.7.4 we have

$$\boldsymbol{0} = \mathscr{F}(\boldsymbol{x}) - \mathscr{F}(\boldsymbol{x}_0) = \int_0^1 [\mathscr{F}'(\boldsymbol{x}_0 + \vartheta(\boldsymbol{x} - \boldsymbol{x}_0))] (\boldsymbol{x} - \boldsymbol{x}_0) \, \mathrm{d}\vartheta.$$

By 2.2 and V.3.12 $\mathscr{F}'(\boldsymbol{x}_0)$ possesses a bounded inverse

$$\boldsymbol{\Gamma} = [\mathscr{F}'(\boldsymbol{x}_0)]^{-1}: \ L_n^1 \times R_n \to AC_n.$$

Hence

$$\boldsymbol{x} - \boldsymbol{x}_0 = \int_0^1 \Gamma[\mathscr{F}'(\boldsymbol{x}_0) - \mathscr{F}'(\boldsymbol{x}_0 + \vartheta(\boldsymbol{x} - \boldsymbol{x}_0))]\,(\boldsymbol{x} - \boldsymbol{x}_0)\,\mathrm{d}\vartheta$$

and

$$(2{,}10)\quad \|\boldsymbol{x} - \boldsymbol{x}_0\|_{AC} \le \|\Gamma\|\,(\sup_{\vartheta\in[0,1]} \|\mathscr{F}'(\boldsymbol{x}_0) - \mathscr{F}'(\boldsymbol{x}_0 + \vartheta(\boldsymbol{x} - \boldsymbol{x}_0))\|)\,\|\boldsymbol{x} - \boldsymbol{x}_0\|_{AC}\,.$$

Since the mapping

$$\boldsymbol{x} \in \mathfrak{B}(\boldsymbol{x}_0, \varrho;\, AC_n) \to \mathscr{F}'(\boldsymbol{x}) \in B(AC_n, L_n^1)$$

is continuous, there is $\varrho_0 > 0$ such that $\varrho_0 \le \varrho$ and

$$\|\mathscr{F}'(\boldsymbol{x}_0) - \mathscr{F}'(\boldsymbol{x}_0 + \vartheta(\boldsymbol{x} - \boldsymbol{x}_0))\| \le \|\Gamma\|^{-1}$$

for any $\boldsymbol{x} \in \mathfrak{B}(\boldsymbol{x}_0, \varrho_0;\, AC_n)$ and $\vartheta \in [0,1]$. Consequently for $\boldsymbol{x} \in \mathfrak{B}(\boldsymbol{x}_0, \varrho_0;\, AC_n)$, $\boldsymbol{x} \neq \boldsymbol{x}_0$ (2,10) becomes a contradiction $\|\boldsymbol{x} - \boldsymbol{x}_0\|_{AC} < \|\boldsymbol{x} - \boldsymbol{x}_0\|_{AC}$. This proves that $\boldsymbol{x} = \boldsymbol{x}_0$ if $\mathscr{F}(\boldsymbol{x}) = \boldsymbol{0}$ and $\boldsymbol{x} \in \mathfrak{B}(\boldsymbol{x}_0, \varrho_0;\, AC_n)$.

2.4. Definition. Let $\boldsymbol{x}_0 \in AC_n$ be a solution of BVP $(\mathscr{P}_0)$ and let the operators $\boldsymbol{F}$ and $\boldsymbol{S}$ fulfil the assumptions of 2.3. The problem of determining a solution $\boldsymbol{u} \in AC_n$ of (2,6) which verifies the side condition (2,7) is called the *variational boundary value problem corresponding to* $\boldsymbol{x}_0$ and is denoted by $(\mathscr{V}_0(\boldsymbol{x}_0))$.

2.5. Remark. BVP

$$x' = x + 1\,, \qquad \boldsymbol{S}(x) = (x(0))^2 + (x(1) + 1 - \exp(1))^2 = 0$$

indicates that in general the converse statement to 2.3 is not true. In fact, the solutions to $x' = x + 1$ are of the form $x(t) = c\exp(t) - 1$, where $c \in R$. The only solution to

$$(2{,}11)\qquad \boldsymbol{S}(x) = (c-1)^2 + (c-1)^2(\exp(1))^2 = 0$$

is $c = 1$. Hence $x_0(t) = \exp(t) - 1$ is the only solution to (2,11). The corresponding variational BVP is given by

$$(2{,}12)\qquad u' = u\,, \qquad [x_0(0)]\,u(0) + [x_0(1) + 1 - \exp(1)]\,u(1) = 0\,.$$

Since $x_0(0) = 0$, $x_0(1) = \exp(1) - 1$, $u(t) = d\exp(t)$ is a solution to (2,12) for any $d \in R$.

2.6. Definition. A solution $\boldsymbol{x}_0$ of BVP $(\mathscr{P}_0)$ is said to be *isolated* if there is $\varrho_0 > 0$ such that there is no solution $\boldsymbol{x}$ to $(\mathscr{P}_0)$ such that $\boldsymbol{x} \neq \boldsymbol{x}_0$ and $\boldsymbol{x} \in \mathfrak{B}(\boldsymbol{x}_0, \varrho_0;\, AC_n)$. It is *regular* if the corresponding variational BVP $(\mathscr{V}(\boldsymbol{x}_0))$ is defined and possesses only the trivial solution.

2.7. Theorem. *Let* $\boldsymbol{x}_0 \in AC_n$ *be a solution to BVP* $(\mathscr{P}_0)$ *where* $\boldsymbol{F}\colon C_n \to L_n^1$ *and* $\boldsymbol{S}\colon C_n \to R_n$ *are continuous operators such that* (2,5) *holds and* $\boldsymbol{F}, \boldsymbol{S} \in C^1(\mathfrak{B}(\boldsymbol{x}_0, \varrho;\, C_n))$

for some $\varrho > 0$. *Furthermore,* $\varkappa > 0$ *and* $\mathbf{G}$: $AC_n \times [0, \varkappa] \to L_n^1$ *and* $\mathbf{R}$: $AC_n \times [0, \varkappa] \to R_n$ *are continuous operators which are locally lipschitzian in* $\mathbf{x}$ *near* $\varepsilon = 0$.

If $\mathbf{x}_0$ *is a regular solution of* $(\mathscr{P}_0)$, *then there exist* $\varepsilon_0 > 0$ *and* $\varrho_0 > 0$ *such that for any* $\varepsilon \in [0, \varepsilon_0]$ *BVP* $(\mathscr{P}_\varepsilon)$ *possesses a unique solution* $\mathbf{x}(\varepsilon)$ *in* $\mathfrak{B}(\mathbf{x}_0, \varrho_0; AC_n)$. *The mapping* $\varepsilon \in [0, \varepsilon_0] \to \mathbf{x}(\varepsilon) \in AC_n$ $(\mathbf{x}(0) = \mathbf{x}_0)$ *is continuous.*

Proof follows by applying I.7.8 to the operator equation

$$\mathscr{F}(\mathbf{x}) + \varepsilon \mathscr{G}(\mathbf{x}, \varepsilon) = \mathbf{0},$$

where $\mathscr{F}$: $AC_n \to L_n^1 \times R_n$ is given by (2,8) and

$$\mathscr{G}\colon (\mathbf{x}, \varepsilon) \in AC_n \times [0, \varkappa] \to \begin{pmatrix} \mathbf{G}(\mathbf{x}, \varepsilon) \\ \mathbf{R}(\mathbf{x}, \varepsilon) \end{pmatrix} \in L_n^1 \times R_n .$$

(Under our assumptions there exists a bounded inverse of $\mathscr{F}'(\mathbf{x}_0)$, cf. the proof of 2.3.)

2.8. Remark. The conclusion of Theorem 2.7 may be reformulated as follows.

If $\mathbf{x}_0$ is a regular solution of $(\mathscr{P}_0)$, then there exists for any $\varepsilon > 0$ sufficiently small a unique solution $\mathbf{x}(\varepsilon)$ of BVP $(\mathscr{P}_\varepsilon)$ which is continuous in ε and tends to $\mathbf{x}_0$ as $\varepsilon \to 0$.

Theorem 2.7 assures the existence of an isolated solution to BVP $(\mathscr{P}_\varepsilon)$ which is close to the regular solution $\mathbf{x}_0$ of the limit problem $(\mathscr{P}_0)$. If also the perturbations $\mathbf{G}$ and $\mathbf{R}$ are differentiable with respect to $\mathbf{x}$, then we can prove that for any $\varepsilon > 0$ sufficiently small this solution is regular, too.

2.9. Theorem. *Let the assumptions of* 2.7 *hold. In addition, let us assume that* $\mathbf{G}$ *and* $\mathbf{R}$ *possess the Gâteaux derivatives* $\mathbf{G}'(\mathbf{x}, \varepsilon)$ *and* $\mathbf{R}'(\mathbf{x}, \varepsilon)$ *with respect to* $\mathbf{x}$ *for any* $(\mathbf{x}, \varepsilon) \in \mathfrak{B}(\mathbf{x}_0, \varrho; AC_n) \times [0, \varkappa]$ *continuous in* $(\mathbf{x}, \varepsilon)$ *on* $\mathfrak{B}(\mathbf{x}_0, \varrho; AC_n) \times [0, \varkappa]$.

Then there exists ε_1, $0 < \varepsilon_1 \leq \varepsilon_0$ *such that for any* $\varepsilon \in [0, \varepsilon_1]$ *the corresponding solution* $\mathbf{x}(\varepsilon)$ *of BVP* $(\mathscr{P}_\varepsilon)$ *is regular.*

Proof. Given $\varepsilon \in [0, \varepsilon_0]$, the variational BVP $(\mathscr{V}_\varepsilon(\mathbf{x}(\varepsilon)))$ corresponding to the solution $\mathbf{x}(\varepsilon)$ of BVP $(\mathscr{P}_\varepsilon)$ is given by

$$\mathbf{u}' = [\mathbf{F}'(\mathbf{x}(\varepsilon)) + \varepsilon\, \mathbf{G}'(\mathbf{x}(\varepsilon), \varepsilon)]\, \mathbf{u},$$
$$[\mathbf{S}'(\mathbf{x}(\varepsilon)) + \varepsilon\, \mathbf{R}'(\mathbf{x}(\varepsilon), \varepsilon)]\, \mathbf{u} = \mathbf{0}.$$

Let $\mathbf{u}$ be its solution, i.e.

$$\mathscr{J}(\varepsilon)\, \mathbf{u} = [\mathscr{F}'(\mathbf{x}(\varepsilon)) + \varepsilon\, \mathscr{G}'(\mathbf{x}(\varepsilon), \varepsilon)]\, \mathbf{u} = \mathbf{0}.$$

Let $\boldsymbol{\Gamma} = [\mathscr{F}'(\mathbf{x}_0)]^{-1}$. Then $\mathbf{u} = \boldsymbol{\Gamma}[\mathscr{J}(0) - \mathscr{J}(\varepsilon)]\, \mathbf{u}$ and

$$\|\mathbf{u}\|_{AC} \leq \|\boldsymbol{\Gamma}\|\ \|\mathscr{J}(0) - \mathscr{J}(\varepsilon)\|\ \|\mathbf{u}\|_{AC} .$$

Since the operators $\varepsilon \in [0, \varepsilon_0] \to \mathbf{x}(\varepsilon) \in AC_n$ and $(\mathbf{x}, \varepsilon) \in \mathfrak{B}(\mathbf{x}_0, \varrho; AC_n) \times [0, \varkappa] \to \mathscr{F}'(\mathbf{x}) + \varepsilon\, \mathscr{G}'(\mathbf{x}, \varepsilon) \in B(AC_n, L^1_n \times R_n)$ are continuous, their composition $\varepsilon \in [0, \varepsilon_0] \to \mathscr{J}(\varepsilon) \in B(AC_n, L^1_n \times R_n)$ is also continuous.

Choosing ε_1, $0 < \varepsilon_1 \leq \varepsilon_0$ in such a way that $\varepsilon \in [0, \varepsilon_1]$ implies $\|\mathscr{J}(0) - \mathscr{J}(\varepsilon)\| \leq \|\boldsymbol{\Gamma}\|^{-1}$ we derive a contradiction $\|\mathbf{u}\|_{AC} < \|\mathbf{u}\|_{AC}$ whenever $\mathbf{u} \neq \mathbf{0}$.

2.10. Remark. The case when $\mathbf{x}_0$ is a regular solution of BVP $(\mathscr{P}_0)$ has appeared to be simple. It is said to be noncritical. The case when $\mathbf{x}_0$ is not a regular solution of $(\mathscr{P}_0)$ is more complicated and said to be critical.

2.11. The critical case. Let $\mathbf{x}_0 \in AC_n$ be a solution to BVP $(\mathscr{P}_0)$, where $\mathbf{F}$: $C_n \to L^1_n$ and $\mathbf{S}$: $C_n \to R_n$ are continuous operators such that (2,5) holds and $\mathbf{F}, \mathbf{S} \in C^2(\mathfrak{B}(\mathbf{x}_0, \varrho; C_n)$ for some $\varrho > 0$. Furthermore, $\varkappa > 0$ and $\mathbf{G}$: $AC_n \times [0, \varkappa] \to L^1_n$ and $\mathbf{R}$: $AC_n \times [0, \varkappa] \to R_n$ are continuous operators such that $\mathbf{G}, \mathbf{R} \in C^{1,1}(\mathfrak{B}(\mathbf{x}_0, \varrho; AC_n) \times [0, \varkappa])$. In general, we do not assume that $\mathbf{x}_0$ is a regular solution of BVP $(\mathscr{P}_0)$. Let us try to find a solution to $(\mathscr{P}_\varepsilon)$ in the form

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \varepsilon\, \boldsymbol{\chi}(t). \tag{2,13}$$

Inserting (2,13) into (2,1) we obtain

$$\mathbf{x}_0' + \varepsilon \boldsymbol{\chi}' = \mathbf{F}(\mathbf{x}_0) + (\mathbf{F}(\mathbf{x}_0 + \varepsilon\boldsymbol{\chi}) - \mathbf{F}(\mathbf{x}_0)) + \varepsilon\, \mathbf{G}(\mathbf{x}_0 + \varepsilon\boldsymbol{\chi}, \varepsilon),$$

i.e.

$$\begin{aligned}\boldsymbol{\chi}' &= \int_0^1 [\mathbf{F}'(\mathbf{x}_0) + (\mathbf{F}'(\mathbf{x}_0 + \varepsilon\vartheta\boldsymbol{\chi}) - \mathbf{F}'(\mathbf{x}_0))]\, \boldsymbol{\chi}\, \mathrm{d}\vartheta \\ &\quad + \mathbf{G}(\mathbf{x}_0, 0) + (\mathbf{G}(\mathbf{x}_0 + \varepsilon\boldsymbol{\chi}, \varepsilon) - \mathbf{G}(\mathbf{x}_0, 0)) \\ &= [\mathbf{F}'(\mathbf{x}_0)]\, \boldsymbol{\chi} + \mathbf{G}(\mathbf{x}_0, 0) + \varepsilon\, \mathbf{H}(\boldsymbol{\chi}, \varepsilon),\end{aligned}$$

where

$$\begin{aligned}\mathbf{H}(\boldsymbol{\chi}, \varepsilon) &= \left(\int_0^1 \left(\int_0^\vartheta \mathbf{F}''(\mathbf{x}_0 + \varepsilon\vartheta_1\vartheta\boldsymbol{\chi})\, \mathrm{d}\vartheta_1\right) \vartheta\, \mathrm{d}\vartheta\boldsymbol{\chi}\right) \boldsymbol{\chi} \\ &\quad + \left(\int_0^1 \mathbf{G}'_x(\mathbf{x}_0 + \varepsilon\vartheta\boldsymbol{\chi}, \vartheta\varepsilon)\, \mathrm{d}\vartheta\right) \boldsymbol{\chi} + \int_0^1 \mathbf{G}'_\varepsilon(\mathbf{x}_0 + \varepsilon\vartheta\boldsymbol{\chi}, \vartheta\varepsilon)\, \mathrm{d}\vartheta.\end{aligned}$$

Thus (2,13) is a solution to (2,1) on $[0, 1]$ if and only if

$$\boldsymbol{\chi}' = [\mathbf{F}'(\mathbf{x}_0)]\, \boldsymbol{\chi} + \mathbf{G}(\mathbf{x}_0, 0) + \varepsilon\, \mathbf{H}(\boldsymbol{\chi}, \varepsilon). \tag{2,14}$$

Analogously, (2,13) verifies (2,2) if and only if

$$[\mathbf{S}'(\mathbf{x}_0)]\, \boldsymbol{\chi} + \mathbf{R}(\mathbf{x}_0, 0) + \varepsilon\, \mathbf{Q}(\boldsymbol{\chi}, \varepsilon) = \mathbf{0}, \tag{2,15}$$

where

$$\mathbf{Q}(\chi, \varepsilon) = \left(\int_0^1 \left(\int_0^\vartheta \mathbf{S}''(\mathbf{x}_0 + \varepsilon\vartheta_1\vartheta\chi)\,\mathrm{d}\vartheta_1\right)\vartheta\,\mathrm{d}\vartheta\chi\right)\chi$$

$$+ \left(\int_0^1 \mathbf{R}'_x(\mathbf{x}_0 + \varepsilon\vartheta\chi, \vartheta\varepsilon)\,\mathrm{d}\vartheta\right)\chi + \int_0^1 \mathbf{R}'_\varepsilon(\mathbf{x}_0 + \varepsilon\vartheta\chi, \vartheta\varepsilon)\,\mathrm{d}\vartheta\,.$$

It follows that the given BVP $(\mathscr{P}_\varepsilon)$ possesses a solution of the form (2,13) for any $\varepsilon > 0$ sufficiently small if and only if the weakly nonlinear problem (2,14), (2,15) possesses a solution for any $\varepsilon > 0$ sufficiently small. In particular, a necessary condition for the existence of a solution of the form (2,13) to BVP $(\mathscr{P}_\varepsilon)$ is that the linear nonhomogeneous problem

$$\chi' = [\mathbf{F}'(\mathbf{x}_0)]\,\chi + \mathbf{G}(\mathbf{x}_0, 0)\,, \qquad [\mathbf{S}'(\mathbf{x}_0)]\,\chi = \mathbf{R}(\mathbf{x}_0, 0)$$

has a solution. Applying the procedure from I.7.10 to BVP (2,14), (2,15) we should obtain furthermore that BVP $(\mathscr{P}_\varepsilon)$ may possess a solution of the form (2,13) for any $\varepsilon > 0$ sufficiently small only if there exists a solution γ_0 of a certain (determining) equation $\mathbf{T}_0(\gamma) = \mathbf{0}$ for a finite dimensional vector γ and if, moreover, $\mathbf{F}$ and $\mathbf{S} \in C^3$, $\mathbf{G}$ and $\mathbf{R} \in C^{2,1}$ and $\det\left((\partial \mathbf{T}_0/\partial\gamma)(\gamma_0)\right) \neq 0$, then such a solution exists (cf. I.7.11).

The critical case will be treated in more detail in the following paragraph concerning ordinary differential equations with arbitrary side conditions.

2.12. Remark. If $\mathbf{P}\colon [0,1] \times [0,1] \to L(R_n)$ is an $L^1[BV]$-kernel, $\mathbf{f} \in L^1_n$, $\mathbf{S} \in B(AC_n, R_m)$,

$$\mathbf{F}\colon\ \mathbf{x} \in AC_n \to \int_0^1 \mathrm{d}_s[\mathbf{P}(t,s)]\,\mathbf{x}(s) + \mathbf{f}(t)\,,$$

$\mathbf{G}\colon AC_n \times [0,\varkappa] \to L^1_n$, $\mathbf{R}\colon AC_n \times [0,\varkappa] \to R_m$, then the weakly nonlinear BVP (cf. V.2.4)

$$\mathscr{D}\mathbf{x} = \begin{pmatrix} \mathbf{D}\mathbf{x} - \mathbf{P}\mathbf{x} \\ \mathbf{S}\mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} = \varepsilon \begin{pmatrix} \mathbf{G}(\mathbf{x}, \varepsilon) \\ \mathbf{R}(\mathbf{x}, \varepsilon) \end{pmatrix} \tag{2,16}$$

becomes a special case of BVP $(\mathscr{P}_\varepsilon)$ studied in this section. In particular, if $\mathbf{R}$ and $\mathbf{G}$ are sufficiently smooth and the limit problem $(\mathscr{P}_0)$ possesses a unique solution for any $\begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \in L^p_n \times R_m$, then by 2.9 BVP (2,16) possesses a unique solution for $\varepsilon > 0$ sufficiently small.

Since according to V.1.8, V.2.5 and V.2.8 $\mathscr{D}\colon AC_n \to L^1_n \times R_m$ verifies (I.7,5), the procedure from I.7.10 may be applied to BVP (2,16). Let us mention that in the special case when $\mathbf{P}$ is an $L^2[BV]$-kernel, $\mathbf{f} \in L^2_n$ and $R(\mathbf{G}) \subset L^2_n$ the transformation of BVP $(\mathscr{P}_0)$ to an algebraic equation exhibited in section V.4 may also be used (cf. Tvrdý, Vejvoda [1]).

3. Nonlinear boundary value problems for ordinary differential equations

In this section we shall treat special cases of the problems $(\mathscr{P}_\varepsilon)$ from the previous section, namely the problems of the form (Π_ε)

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) + \varepsilon\, \mathbf{g}(t, \mathbf{x}, \varepsilon), \tag{3,1}$$

$$\mathbf{S}(\mathbf{x}) + \varepsilon\, \mathbf{R}(\mathbf{x}, \varepsilon) = \mathbf{0} \tag{3,2}$$

and (Π_0)

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \tag{3,3}$$

$$\mathbf{S}(\mathbf{x}) = \mathbf{0}. \tag{3,4}$$

Our aim is again to obtain conditions for the existence of a solution to the perturbed problem (Π_ε) under the assumption that the limit problem (Π_0) possesses a solution. In doing this only such solutions of BVP (Π_ε) are sought which tend to some solution of BVP (Π_0) as $\varepsilon \to 0+$.

The following assumptions are pertinent.

3.1. Assumptions.

(i) $\mathscr{D} \subset R_{n+1}$ *and* $\mathfrak{D} \subset R_{n+2}$ *are open,* $\varkappa > 0$ *and* $[0,1] \times R_n \subset \mathscr{D}$, $\mathscr{D} \times [0, \varkappa] \subset \mathfrak{D}$;

(ii) $\mathbf{f}\colon \mathscr{D} \to R_n$, $\mathbf{f} \in \mathrm{Car}(\mathscr{D})$, $\partial \mathbf{f}/\partial \mathbf{x}$ *exists on* $\mathscr{D}$ *and* $\partial \mathbf{f}/\partial \mathbf{x} \in \mathrm{Car}(\mathscr{D})$ (*cf.* 1.2);

(iii) $\mathbf{g}\colon \mathfrak{D} \to R_n$, $\mathbf{g} \in \mathrm{Lip}(\mathscr{D}; 0)$ (*i.e.* $\mathbf{g}$ *is locally lipschitzian in* $\mathbf{x}$ *near* $\varepsilon = 0$, *cf.* 1.25) *and if* $\tilde{\mathbf{g}}(t, \mathbf{y}) = \mathbf{g}(t, \mathbf{x}, \varepsilon)$ *for* $(t, \mathbf{x}, \varepsilon) \in \mathfrak{D}$ *and* $\mathbf{y} = (\mathbf{x}, \varepsilon)$, *then* $\tilde{\mathbf{g}} \in \mathrm{Car}(\mathfrak{D})$;

(iv) $\mathbf{S}$ *is a continuous mapping of* AC_n *into* R_n, $\mathbf{S} \in C^1(AC_n)$, $\mathbf{R}$ *is a continuous mapping of* $AC_n \times [0, \varkappa]$ *into* R_n *which is locally lipschitzian in* $\mathbf{x}$ *near* $\varepsilon = 0$ (*cf.* I.7.1).

3.2. Remark. Under the assumptions 3.1 for any $(\mathbf{c}, \varepsilon) \in R_n \times [0, \varkappa]$ there exists a unique maximal solution $\mathbf{x}(t) = \boldsymbol{\psi}(t; 0, \mathbf{c}, \varepsilon)$ of (3,1) on $\Delta = \Delta(\mathbf{c}, \varepsilon)$ such that $0 \in \Delta$ and $\mathbf{x}(0) = \mathbf{c}$ (cf. 1.4, 1.7, 1.11 and 1.13). The set

$$\tilde{\Omega}_{(\cdot, 0, \cdot, \cdot)} = \Omega_0 = \{(t, 0, \mathbf{c}, \varepsilon);\ (\mathbf{c}, \varepsilon) \in R_n \times [0, \varkappa],\ t \in \Delta(\mathbf{c}, \varepsilon)\}$$

is open and the mapping

$$\boldsymbol{\xi}\colon (t, \mathbf{c}, \varepsilon) \in \Omega_0 \to \boldsymbol{\psi}(t; 0, \mathbf{c}, \varepsilon) \in R_n$$

is continuous.

3.3. Notation. In the sequel $\boldsymbol{\xi}(t; \mathbf{c}, \varepsilon) = \boldsymbol{\psi}(t; 0, \mathbf{c}, \varepsilon)$ for $(t, \mathbf{c}, \varepsilon) \in \Omega_0$. In particular, $\boldsymbol{\eta}(t; \mathbf{c}) = \boldsymbol{\psi}(t; 0, \mathbf{c}, 0) = \boldsymbol{\varphi}(t; 0, \mathbf{c})$ for $\mathbf{c} \in R_n$ and $t \in \Delta(\mathbf{c}, 0)$.

3.4. Remark. Given $\mathbf{x} \in AC_n$, the corresponding variational BVP $(\mathscr{V}_0(\mathbf{x}))$ to (Π_0) is given by the linear ordinary differential equation

$$\mathbf{u}' - \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t, \mathbf{x}(t))\right]\mathbf{u} = \mathbf{0} \tag{3,5}$$

and by the side condition

$$[\mathbf{S}'(\mathbf{x})]\,\mathbf{u} = \mathbf{0}. \tag{3,6}$$

According to 1.14, given a solution $\mathbf{x}(t) = \boldsymbol{\eta}(t; \mathbf{c})$ to (3,3) on $[0, 1]$, the $n \times n$-matrix valued function

$$\mathbf{A}(t) = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t, \boldsymbol{\eta}(t; \mathbf{c}))\right]$$

is L-integrable on $[0, 1]$. Moreover, the $n \times n$-matrix valued function

$$\mathbf{U}(t) = \left[\frac{\partial \boldsymbol{\eta}}{\partial \mathbf{c}}(t; \mathbf{c})\right]$$

is the fundamental matrix solution to (3,5) on $[0, 1]$ such that $\mathbf{U}(0) = \mathbf{I}_n$.

3.5. Remark. Let us notice (cf. 1.20 – 1.27) that under our assumptions 3.1 the operators $\mathbf{F}$: $AC_n \to L^1_n$ and $\mathbf{G}$: $AC_n \times [0, \varkappa] \to L^1_n$ defined as in 1.21 fulfil all the corresponding assumptions of theorems 2.3 and 2.7 (with AC_n in place of C_n). Moreover, if $\mathbf{x}(t) = \boldsymbol{\eta}(t; \mathbf{c})$ and the variational BVP $(\mathscr{V}_0(\mathbf{x}))$ given now by (3,5), (3,6) has only the trivial solution, then according to V.3.12 the linear operator

$$\mathscr{F}'(\mathbf{x}):\ \mathbf{u} \in AC_n \to \begin{pmatrix} \mathbf{u}' - [(\partial \mathbf{f}/\partial \mathbf{x})(t, \mathbf{x}(t))]\,\mathbf{u} \\ [\mathbf{S}'(\mathbf{x})]\,\mathbf{u} \end{pmatrix} \in L^1_n \times R_n$$

possesses a bounded inverse. Thus applying the same argument as in the proofs of Theorems 2.3 and 2.7 we can prove the following assertion.

3.6. Theorem. *Let 3,1 hold. Let $\mathbf{x}_0$ be a solution to BVP (Π_0) and let the corresponding variational BVP $(\mathscr{V}_0(\mathbf{x}_0))$ possess only the trivial solution. Then $\mathbf{x}_0$ is an isolated solution of (Π_0) and for $\varepsilon > 0$ sufficiently small BVP (Π_ε) has a solution $\mathbf{x}(\varepsilon)$ which is continuous in ε and tends to $\mathbf{x}_0$ as $\varepsilon \to 0+$.*

To obtain some results also for the critical case we shall strengthen our hypotheses.

3.7. Assumptions. *For any $i, j = 1, 2, \dots, n$ $\mathbf{f}$ possesses on $\mathscr{D}$ the partial derivatives $\partial^2 \mathbf{f}/(\partial x_i\, \partial x_j)$ with respect to the components x_j of $\mathbf{x}$ and $\partial^2 \mathbf{f}/(\partial x_i\, \partial x_j) \in \mathrm{Car}(\mathscr{D})$ $(i, j = 1, 2, \dots, n)$. Furthermore, $\partial \mathbf{g}/\partial \mathbf{x}$ exists on $\mathfrak{D}$ and if $\mathbf{h}(t, \mathbf{y}) = (\partial \mathbf{g}/\partial \mathbf{x})(t, \mathbf{x}, \varepsilon)$ for $(t, \mathbf{x}, \varepsilon) \in \mathfrak{D}$ and $\mathbf{y} = (\mathbf{x}, \varepsilon)$, then $\mathbf{h} \in \mathrm{Car}(\mathfrak{D})$.*

$\mathbf{S} \in C^2(AC_n)$ and $\mathbf{R} \in C^{1,0}(AC_n \times [0, \varkappa])$ (i.e. given $(\mathbf{x}, \varepsilon) \in AC_n \times [0, \varkappa]$, $\mathbf{R}'_x(\mathbf{x}, \varepsilon)$ exists and the mapping $(\mathbf{x}, \varepsilon) \in AC_n \times [0, \varkappa] \to \mathbf{R}'_x(\mathbf{x}, \varepsilon) \in B(AC_n, R_n)$ is continuous).

The following lemma provides the principal tool for proving theorems on the existence of solutions to BVP (Π_ε) in the critical case. It establishes the *variation-of-constants method for nonlinear equations.*

3.8. Lemma. *Let* 3.1 *and* 3.7 *hold. Let the equation* (3,3) *possess a solution* $\mathbf{x}_0(t) = \boldsymbol{\eta}(t; \mathbf{c}_0)$ *on* $[0,1]$. *Then there exist* $\varrho_0 > 0$ *and* $\varkappa_0 > 0$ *such that for any* $(\mathbf{c}, \varepsilon) \in \mathfrak{B}(\mathbf{c}_0, \varrho_0; R_n) \times [0, \varkappa_0]$ *the equation* (3,1) *possesses a unique solution* $\mathbf{x}(t)$ *on* $[0,1]$ *such that* $\mathbf{x}(0) = \mathbf{c}$. *This solution is given by*

$$\mathbf{x}(t) = \boldsymbol{\xi}(t; \mathbf{c}, \varepsilon) = \boldsymbol{\eta}(t; \boldsymbol{\beta}(t; \mathbf{c}, \varepsilon)) \qquad on\ [0,1], \tag{3,7}$$

where for any $(\mathbf{c}, \varepsilon) \in \mathfrak{B}(\mathbf{c}_0, \varrho_0; R_n) \times [0, \varkappa_0]$ $\mathbf{b}(t) = \boldsymbol{\beta}(t, \mathbf{c}, \varepsilon)$ *is a unique solution to*

$$\mathbf{b}' = \varepsilon \left[\frac{\partial \boldsymbol{\eta}}{\partial \mathbf{c}}(t; \mathbf{b}) \right]^{-1} \mathbf{g}(t, \boldsymbol{\eta}(t; \mathbf{b}), \varepsilon) \tag{3,8}$$

on $[0,1]$ *such that* $\mathbf{b}(0) = \mathbf{c}$. *The mapping* $(t, \mathbf{c}, \varepsilon) \in \mathfrak{B} = [0,1] \times \mathfrak{B}(\mathbf{c}_0, \varrho_0; R_n) \times [0, \varkappa_0] \to \boldsymbol{\beta}(t; \mathbf{c}, \varepsilon) \in R_n$ *is continuous and possesses the Jacobi matrix* $(\partial \boldsymbol{\beta}/\partial \mathbf{c})(t; \mathbf{c}, \varepsilon)$ *continuous in* $(t, \mathbf{c}, \varepsilon)$ *on* $\mathfrak{B}$.

Proof. (a) According to 1.12 there exist an open subset $\Omega \in R_{n+1}$ and $\delta > 0$ such that $\boldsymbol{\eta}(t; \mathbf{c})$ is defined for any $(t, \mathbf{c}) \in \Omega$ and $[0,1] \times \mathfrak{B}(\mathbf{c}_0, \delta; R_n) \subset \Omega$. Furthermore, in virtue of 1.16 the Jacobi matrix $\mathbf{U}(t, \mathbf{c}) = (\partial \boldsymbol{\eta}/\partial \mathbf{c})(t; \mathbf{c})$ and its partial derivatives $\partial \mathbf{U}(t, \mathbf{c})/\partial c_j$ $(j = 1, 2, \ldots, n)$ with respect to the components c_j of $\mathbf{c}$ exist and are continuous on Ω. Since by 1.15 $\mathbf{U}^{-1}(t, \mathbf{c})$ exists on Ω and for any $j = 1, 2, \ldots, n$ and $(t, \mathbf{c}) \in \Omega$

$$\mathbf{0} = \frac{\partial}{\partial c_j}(\mathbf{U}(t, \mathbf{c})\, \mathbf{U}^{-1}(t, \mathbf{c})) = \left(\frac{\partial}{\partial c_j} \mathbf{U}(t, \mathbf{c}) \right) \mathbf{U}^{-1}(t, \mathbf{c}) + \mathbf{U}(t, \mathbf{c}) \left(\frac{\partial}{\partial c_j} \mathbf{U}^{-1}(t, \mathbf{c}) \right),$$

$\mathbf{U}^{-1}(t, \mathbf{c})$ possesses on Ω all the partial derivatives

$$\frac{\partial}{\partial c_j} \mathbf{U}^{-1}(t, \mathbf{c}) = -\mathbf{U}^{-1}(t, \mathbf{c}) \left(\frac{\partial}{\partial c_j} \mathbf{U}(t, \mathbf{c}) \right) \mathbf{U}^{-1}(t, \mathbf{c}) \qquad (j = 1, 2, \ldots, n).$$

It is easy to see now that the right-hand side

$$\mathbf{h}(t, \mathbf{b}, \varepsilon) = \varepsilon\, \mathbf{U}^{-1}(t, \mathbf{b})\, \mathbf{g}(t, \boldsymbol{\eta}(t; \mathbf{b}), \varepsilon) \tag{3,9}$$

of (3,8) possesses the Jacobi matrix $(\partial \mathbf{h}/\partial \mathbf{b})(t, \mathbf{b}, \varepsilon)$ on some open subset $\tilde{\Omega}$ of R_{n+2} such that $\Omega \times [0, \varkappa] \subset \tilde{\Omega}$ and if we put $\boldsymbol{\chi}(t, \boldsymbol{\mu}) = (\partial \mathbf{h}/\partial \mathbf{b})(t, \mathbf{b}, \varepsilon)$ for $\boldsymbol{\mu} = (\mathbf{b}, \varepsilon)$ and $(t, \boldsymbol{\mu}) \in \tilde{\Omega}$, then $\boldsymbol{\chi} \in \mathrm{Car}(\tilde{\Omega})$. By 1.14 (cf. also 1.17) this implies that for any $(\mathbf{c}, \varepsilon) \in R_n \times [0, \varkappa]$ sufficiently close to $(\mathbf{c}_0, 0)$ the equation (3,8) possesses a unique solution $\mathbf{b}(t) = \boldsymbol{\beta}(t; \mathbf{c}, \varepsilon)$ on $[0,1]$ such that $\mathbf{b}(0) = \mathbf{c}$. Moreover, since for $\varepsilon = 0$, $\mathbf{b}(t) \equiv \mathbf{c}_0$ is a solution to (3,8) on $[0,1]$, there exist $\varrho_0 > 0$ and $\varkappa_0 > 0$ such that $\boldsymbol{\beta}(t; \mathbf{c}, \varepsilon)$ is defined and possesses the required properties on $\mathfrak{B} = [0,1] \times \mathfrak{B}(\mathbf{c}_0, \varrho_0; R_n) \times [0, \varkappa_0]$ and in addition $|\boldsymbol{\beta}(t, \mathbf{c}, \varepsilon)| \le \delta$ for any $(t, \mathbf{c}, \varepsilon) \in \mathfrak{B}$.

(b) Let $(\mathbf{c}, \varepsilon) \in \mathfrak{B}(\mathbf{c}_0, \varrho_0; R_n) \times [0, \varkappa]$. By the first part of the proof (3,7) is defined on $[0, 1]$ and according to the definitions of $\boldsymbol{\eta}(t, \mathbf{c})$ and $\boldsymbol{\beta}(t; \mathbf{c}, \varepsilon)$

$$\begin{aligned} \mathbf{x}'(t) &= \frac{\partial \boldsymbol{\eta}}{\partial t}(t; \boldsymbol{\beta}(t; \mathbf{c}, \varepsilon)) + \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{c}}(t; \boldsymbol{\beta}(t; \mathbf{c}, \varepsilon)) \frac{\partial \boldsymbol{\beta}}{\partial t}(t; \mathbf{c}, \varepsilon) \\ &= \mathbf{f}(t, \boldsymbol{\eta}(t; \boldsymbol{\beta}(t; \mathbf{c}, \varepsilon))) + \varepsilon\, \mathbf{g}(t; \boldsymbol{\beta}(t; \mathbf{c}, \varepsilon)), \varepsilon) \qquad \text{for a.e.} \quad t \in [0, 1], \end{aligned}$$

while $\mathbf{x}(0) = \boldsymbol{\eta}(0; \boldsymbol{\beta}(0; \mathbf{c}, \varepsilon)) = \boldsymbol{\eta}(0; \mathbf{c}) = \mathbf{c}$. Since (3,1) possesses obviously the property $(\mathscr{U})$, it means that

$$\mathbf{x}(t) = \boldsymbol{\eta}(t; \boldsymbol{\beta}(t; \mathbf{c}, \varepsilon)) = \boldsymbol{\xi}(t; \mathbf{c}, \varepsilon) \qquad \text{on } [0, 1].$$

3.9. Notation. $\mathscr{N}$ denotes the naturally ordered set $\{1, 2, \ldots, n\}$. If $\mathscr{I}$ is a naturally ordered subset of $\mathscr{N}$, then $\mathscr{N} \setminus \mathscr{I}$ denotes the naturally ordered complement of $\mathscr{I}$ with respect to $\mathscr{N}$. The number of elements of a set $\mathscr{I} \subset \mathscr{N}$ is denoted by $\nu(\mathscr{I})$. Let $\mathbf{C} = (c_{i,j})_{i,j=1,2,\ldots,n} \in L(R_n)$ and let $\mathscr{I} = \{i_1, i_2, \ldots, i_p\}$ and $\mathscr{J} = \{j_1, j_2, \ldots, j_q\}$ be naturally ordered subsets of $\mathscr{N}$, then $\mathbf{C}_{\mathscr{I},\mathscr{J}}$ denotes the $p \times q$-matrix $(d_{k,l})_{k=1,2,\ldots,p;\, l=1,2,\ldots,q}$, where $d_{k,l} = c_{i_k, j_l}$ for $k = 1, 2, \ldots, p$ and $l = 1, 2, \ldots, q$. In particular, if $\mathbf{b} \in R_n$ $(\mathbf{b} = (b_1, b_2, \ldots, b_n)^*)$, then $\mathbf{b}_{\mathscr{I}}$ denotes the p-vector $(d_1, d_2, \ldots, d_p)^*$, where $d_k = b_{i_k}$ for $k = 1, 2, \ldots, p$. (Analogously for matrix or vector valued functions and operators.)

3.10. Remark. Let $\mathbf{x}_0(t) = \boldsymbol{\eta}(t; \mathbf{c}_0)$ be a solution to the limit problem (Π_0) and let the corresponding variational BVP $(\mathscr{V}_0(\mathbf{x}_0))$ possess exactly k linearly independent solutions on $[0, 1]$ $(\dim N(\mathscr{F}'(\mathbf{x}_0)) = k)$. This means that $\operatorname{rank}(\Delta(\mathbf{c}_0)) = n - k$, where

$$\Delta(\mathbf{c}_0) = [\mathbf{S}'(\mathbf{x}_0)] \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{c}}(\,.\,; \mathbf{c}_0)$$

denotes the $n \times n$-matrix formed by the columns $[\mathbf{S}'(\mathbf{x}_0)]\, \mathbf{u}_j$ $(\mathbf{u}_j(t) = (\partial \boldsymbol{\eta}/\partial \mathbf{c}_j)(t; \mathbf{c}_0)$ on $[0, 1]$; $j = 1, 2, \ldots, n)$. Hence there exist naturally ordered subsets $\mathscr{I}$, $\mathscr{J}$ of $\mathscr{N} = \{1, 2, \ldots, n\}$ with k elements such that

$$\det(\Delta(\mathbf{c}_0))_{\mathscr{N} \setminus \mathscr{I}, \mathscr{N} \setminus \mathscr{J}} \neq 0.$$

Let us denote $(\mathbf{c}_0)_{\mathscr{J}} = \gamma_0$ and $(\mathbf{c}_0)_{\mathscr{N} \setminus \mathscr{J}} = \boldsymbol{\delta}_0$. Since for any $\mathbf{c} \in R_n$ sufficiently close to $\mathbf{c}_0$ the value of the Jacobi matrix of the function $\mathbf{d} \in R_n \to \mathbf{S}(\boldsymbol{\eta}(\,.\,; \mathbf{d})) \in R_n$ is given by $[\mathbf{S}'(\boldsymbol{\eta}(\,.\,; \mathbf{c}))]\, (\partial \boldsymbol{\eta}/\partial \mathbf{c})(\,.\,; \mathbf{c})$, the Implicit Function Theorem yields that there exist $\sigma > 0$ and a function $\mathbf{p}_0\colon \mathfrak{B}(\gamma_0, \sigma; R_k) = \Gamma \to R_{n-k}$ such that $\mathbf{p}_0(\gamma_0) = \boldsymbol{\delta}_0$, $(\partial \mathbf{p}_0/\partial \gamma)(\gamma)$ exists and is continuous on Γ $(\mathbf{p}_0 \in C^1(\Gamma))$ and if the function $\mathbf{q}_0\colon \Gamma \to R_n$ is defined by $(\mathbf{q}_0(\gamma))_{\mathscr{J}} = \gamma$ and $(\mathbf{q}_0(\gamma))_{\mathscr{N} \setminus \mathscr{J}} = \mathbf{p}_0(\gamma)$, then

$$\mathbf{S}_{\mathscr{N} \setminus \mathscr{I}}(\boldsymbol{\eta}(\,.\,; \mathbf{q}_0(\gamma))) = \mathbf{0} \qquad \text{for any} \quad \gamma \in \Gamma.$$

If also $\mathbf{S}_{\mathscr{I}}(\boldsymbol{\eta}(\,.\,; \mathbf{q}_0(\gamma))) = \mathbf{0}$ for any $\gamma \in \Gamma$, then $\mathbf{x}(t) = \boldsymbol{\eta}(t; \mathbf{q}_0(\gamma))$ is a solution to (Π_0) for any $\gamma \in \Gamma$.

3.11. Theorem. *Let the assumptions* 3.1 *and* 3.7 *hold. In addition, let us assume*

(i) *there exist an integer* k, $0 < k < n$, *a naturally ordered subset* $\mathscr{I}$ *of* $\mathscr{N}$ *with* k *elements* $(\nu(\mathscr{I}) = k)$, *an open set* $\Gamma \subset R_k$ *and a function* $\mathbf{p}_0\colon \Gamma \to R_{n-k}$ *such that* $(\partial \mathbf{p}_0/\partial \gamma)(\gamma)$ *exists and is continuous on* Γ *and if* $\mathbf{q}_0\colon \Gamma \to R_n$ *is defined by* $(\mathbf{q}_0(\gamma))_{\mathscr{I}} = \gamma$ *and* $(\mathbf{q}_0(\gamma))_{\mathscr{N}\setminus\mathscr{I}} = \mathbf{p}_0(\gamma)$, *then the function* $t \in [0,1] \to \boldsymbol{\eta}(t; \mathbf{q}_0(\gamma)) \in R_n$ *is a solution to BVP* (Π_0) *for any* $\gamma \in \Gamma$;

(ii) $\operatorname{rank}([\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)))]\,(\partial\boldsymbol{\eta}/\partial\mathbf{c})\,(.\,; \mathbf{q}_0(\gamma)) = n - k$ *for any* $\gamma \in \Gamma$.

Let $\mathscr{I}$ *be a naturally ordered subset of* $\mathscr{N}$ *with* k *elements such that*

$$\operatorname{rank}\left([\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)))]\,\frac{\partial\boldsymbol{\eta}}{\partial\mathbf{c}}(.\,; \mathbf{q}_0(\gamma))\right)_{\mathscr{N}\setminus\mathscr{I},\mathscr{N}} = n - k \tag{3,10}$$

and let $\boldsymbol{\Theta}\colon \Gamma \to L(R_{n-k}, R_k)$ *be a matrix valued function such that*

$$\begin{aligned}&\left([\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)))]\,\frac{\partial\boldsymbol{\eta}}{\partial\mathbf{c}}(.\,; \mathbf{q}_0(\gamma))\right)_{\mathscr{I},\mathscr{N}} \\ &\quad = \boldsymbol{\Theta}(\gamma)\left(\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)))\,\frac{\partial\boldsymbol{\eta}}{\partial\mathbf{c}}(.\,; \mathbf{q}_0(\gamma))\right)_{\mathscr{N}\setminus\mathscr{I},\mathscr{N}} \\ &\qquad \text{for any} \quad \gamma \in \Gamma\,.\end{aligned} \tag{3,11}$$

Then the mapping

$$\begin{aligned}\mathbf{T}_0\colon \gamma \in \Gamma \subset R_k \to &\left([\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)))]\,\frac{\partial\boldsymbol{\eta}}{\partial\mathbf{c}}(.\,; \mathbf{q}_0(\gamma))\,\zeta_\gamma + \mathbf{R}(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)), 0)\right)_{\mathscr{I}} \\ &- \boldsymbol{\Theta}(\gamma)\left([\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)))]\,\frac{\partial\boldsymbol{\eta}}{\partial\mathbf{c}}(.\,; \mathbf{q}_0(\gamma))\,\zeta_\gamma + \mathbf{R}(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)), 0)\right)_{\mathscr{N}\setminus\mathscr{I}} \in R_k\end{aligned} \tag{3,12}$$

where

$$\zeta_\gamma(t) = \int_0^t \left[\frac{\partial\boldsymbol{\eta}}{\partial\mathbf{c}}(\tau; \mathbf{q}_0(\gamma))\right]^{-1} \mathbf{g}(\tau, \boldsymbol{\eta}(\tau; \mathbf{q}_0(\gamma)), 0)\,\mathrm{d}\tau \qquad \text{on } [0,1]\,,$$

possesses the Jacobi matrix $(\partial\mathbf{T}_0/\partial\gamma)(\gamma)$ *on* Γ.

If, moreover, the equation

$$\mathbf{T}_0(\gamma) = \mathbf{0} \tag{3,13}$$

possesses a solution $\gamma_0 \in \Gamma$ *such that*

$$\det\left(\frac{\partial\mathbf{T}_0}{\partial\gamma}(\gamma_0)\right) \neq 0\,, \tag{3,14}$$

then there exists for any $\varepsilon > 0$ *sufficiently small a unique solution* $\mathbf{x}_\varepsilon(t) = \boldsymbol{\xi}(t; \mathbf{c}(\varepsilon), \varepsilon)$ *of BVP* (Π_ε) *which is continuous in* ε *and tends to* $\boldsymbol{\eta}(t; \mathbf{q}_0(\gamma_0))$ *uniformly on* $[0,1]$ *as* $\varepsilon \to 0+$.

Proof. (a) Let us put

$$\Delta_0(\gamma) = [\mathbf{S}'(\boldsymbol{\eta}(.\,;\mathbf{q}_0(\gamma)))]\frac{\partial\boldsymbol{\eta}}{\partial\mathbf{c}}(.\,;\mathbf{q}_0(\gamma)) \qquad \text{for} \quad \gamma\in\Gamma\,.$$

We shall show that

(3,15) $$\det\left((\Delta_0(\gamma))_{\mathcal{N}\setminus\mathcal{I},\mathcal{N}\setminus\mathcal{I}}\right) \neq 0 \qquad \text{for any} \quad \gamma\in\Gamma\,.$$

In fact, if there were $\det(\Delta_0(\gamma_1))_{\mathcal{N}\setminus\mathcal{I},\mathcal{N}\setminus\mathcal{I}} = 0$, then $h\in\mathcal{I}$ and $\boldsymbol{\mu}\in R_{n-k-1}$ should exist such that

(3,16) $$(\Delta_0(\gamma_1))_{h,\mathcal{N}\setminus\mathcal{I}} = \boldsymbol{\mu}^*(\Delta_0(\gamma_1))_{\mathcal{H},\mathcal{N}\setminus\mathcal{I}}$$

where $\mathcal{H} = (\mathcal{N}\setminus\mathcal{I})\setminus\{h\}$. On the other hand, according to our assumptions and the definition of $\boldsymbol{\eta}(t,\mathbf{c})$

(3,17) $$\mathbf{S}(\boldsymbol{\eta}(.\,;\mathbf{q}_0(\gamma)) = \mathbf{0} \qquad \text{for any} \quad \gamma\in\Gamma\,.$$

Differentiating the identity (3,17) with respect to γ, we obtain

$$\Delta_0(\gamma)\frac{\partial\mathbf{q}_0}{\partial\gamma}(\gamma) = (\Delta_0(\gamma))_{\mathcal{N},\mathcal{N}\setminus\mathcal{I}}\frac{\partial\mathbf{p}_0}{\partial\gamma}(\gamma) + (\Delta_0(\gamma))_{\mathcal{N},\mathcal{I}} = \mathbf{0}$$

for any $\gamma\in\Gamma$. By (3,16)

$$(\Delta_0(\gamma_1))_{h,\mathcal{I}} = -(\Delta_0(\gamma_1))_{h,\mathcal{N}\setminus\mathcal{I}}\frac{\partial\mathbf{p}_0}{\partial\gamma}(\gamma_1)$$

$$= -\boldsymbol{\mu}^*(\Delta_0(\gamma_1))_{\mathcal{H},\mathcal{N}\setminus\mathcal{I}}\frac{\partial\mathbf{p}_0}{\partial\gamma}(\gamma_1) = \boldsymbol{\mu}^*(\Delta_0(\gamma_1))_{\mathcal{H},\mathcal{I}}$$

i.e.

$$(\Delta_0(\gamma_1)_{h,\mathcal{N}} = \boldsymbol{\mu}^*(\Delta_0(\gamma_1))_{\mathcal{H},\mathcal{N}}$$

and $\operatorname{rank}(\Delta_0(\gamma_1))_{\mathcal{N}\setminus\mathcal{I},\mathcal{N}} \le n-k-1$. This being a contradiction to (3,10), (3,15) has to hold.

(b) Since (3,10) is assumed, for any $\gamma\in\Gamma$ there exist a $k\times(n-k)$-matrix $\Theta(\gamma)$ such that (3,11) holds on Γ, i.e.

$$(\Delta_0(\gamma))_{\mathcal{I},\mathcal{N}} = \Theta(\gamma)\,(\Delta_0(\gamma))_{\mathcal{N}\setminus\mathcal{I},\mathcal{N}} \qquad \text{on} \quad \Gamma\,.$$

In particular,

$$(\Delta_0(\gamma))_{\mathcal{I},\mathcal{N}\setminus\mathcal{I}} = \Theta(\gamma)\,(\Delta_0(\gamma))_{\mathcal{N}\setminus\mathcal{I},\mathcal{N}\setminus\mathcal{I}}$$

and

(3,18) $$\Theta(\gamma) = (\Delta_0(\gamma))_{\mathcal{I},\mathcal{N}\setminus\mathcal{I}}(\Delta_0(\gamma))^{-1}_{\mathcal{N}\setminus\mathcal{I},\mathcal{N}\setminus\mathcal{I}} \qquad \text{on} \quad \Gamma\,.$$

It is easy to verify that under our assumptions all the partial derivatives $(\partial\Delta_0/\partial\gamma_j)(\gamma)$ $(j = 1,2,\ldots,k)$ exist and are continuous on Γ. Clearly, for any $j = 1,2,\ldots,n$

$$\frac{\partial}{\partial\gamma_j}(\Delta_0(\gamma))^{-1} = -(\Delta_0(\gamma))^{-1}\left(\frac{\partial}{\partial\gamma_j}(\Delta_0(\gamma))\right)(\Delta_0(\gamma))^{-1} \qquad \text{on} \quad \Gamma$$

and in virtue of (3,18) also the $k \times (n-k)$-matrix function $\boldsymbol{\Theta}(\gamma)$ possesses all the partial derivatives $(\partial\boldsymbol{\Theta}/\partial\gamma_j)(\gamma)$ $(j = 1, 2, \ldots, n)$ on Γ and they are continuous on Γ. This implies that the function $\boldsymbol{T}_0\colon \Gamma \subset R_k \to R_k$ defined by (3,12) possesses the Jacobi matrix $(\partial\boldsymbol{T}_0/\partial\gamma)(\gamma)$ on Γ and it is continuous on Γ.

(c) According to the definition of $\boldsymbol{\xi}(t, \boldsymbol{c}, \varepsilon)$ an n-vector valued function $\boldsymbol{x}(t)$ is a solution to BVP (Π_ε) if and only if $\boldsymbol{x}(t) = \boldsymbol{\xi}(t; \boldsymbol{c}, \varepsilon)$ on $[0, 1]$ and $\boldsymbol{c} \in R_n$ fulfils the equation

$$(3,19) \qquad \boldsymbol{W}(\boldsymbol{c}, \varepsilon) \equiv \boldsymbol{S}(\boldsymbol{\xi}(.\,; \boldsymbol{c}, \varepsilon)) + \varepsilon\,\boldsymbol{R}(\boldsymbol{\xi}(.\,; \boldsymbol{c}, \varepsilon), \varepsilon) = \boldsymbol{0}\,.$$

The mappings $\boldsymbol{W}\colon R_n \times [0, \varkappa] \to R_n$ and $\partial\boldsymbol{W}/\partial\boldsymbol{c}\colon R_n \times [0, \varkappa] \to L(R_n)$ are clearly continuous.

Let $\gamma_0 \in \Gamma$ be such that $\boldsymbol{T}_0(\gamma_0) = \boldsymbol{0}$. Then $\boldsymbol{W}(\boldsymbol{q}_0(\gamma_0), 0) = \boldsymbol{0}$. Furthermore, since

$$\frac{\partial\boldsymbol{W}}{\partial\boldsymbol{c}}(\boldsymbol{q}_0(\gamma), 0) = \Delta_0(\gamma) \qquad \text{on } \Gamma\,,$$

(3,15) means

$$\det\left(\frac{\partial\boldsymbol{W}}{\partial\boldsymbol{c}}(\boldsymbol{q}_0(\gamma), 0)\right)_{\mathcal{N}\setminus\mathcal{J},\,\mathcal{N}\setminus\mathcal{J}} \neq 0 \qquad \text{on } \Gamma\,.$$

It follows that there are $\varrho_1 > 0$ and $\varkappa_1 > 0$ such that

$$\det\left(\frac{\partial\boldsymbol{W}}{\partial\boldsymbol{c}}(\boldsymbol{c}, \varepsilon)\right)_{\mathcal{N}\setminus\mathcal{J},\,\mathcal{N}\setminus\mathcal{J}} \neq 0$$

for all $(\boldsymbol{c}, \varepsilon) \in \mathfrak{B}_1 = \mathfrak{B}(\boldsymbol{c}_0, \varrho_1; R_n) \times [0, \varkappa_1]$. By the Implicit Function Theorem there exist $\varrho_2 > 0$, $\varkappa_2 > 0$, $\varkappa_2 \le \varkappa_1$, and a unique function $\boldsymbol{p}\colon \mathfrak{B}_2 = \mathfrak{B}(\gamma_0, \varrho_2; R_k) \times [0, \varkappa_2] \to R_{n-k}$, $\boldsymbol{p} \in C^{1,0}(\mathfrak{B}_2)$ such that if $(\boldsymbol{q}(\gamma, \varepsilon))_{\mathcal{J}} = \gamma$ and $(\boldsymbol{q}(\gamma, \varepsilon))_{\mathcal{N}\setminus\mathcal{J}} = \boldsymbol{p}(\gamma, \varepsilon)$, then $\boldsymbol{q}(\gamma, \varepsilon) \in \mathfrak{B}(\boldsymbol{c}_0, \varrho_1; R_n)$ and

$$(3,20) \qquad \boldsymbol{W}_{\mathcal{N}\setminus\mathcal{J}}(\boldsymbol{q}(\gamma, \varepsilon), \varepsilon) = \boldsymbol{0} \qquad \text{for any } \ (\gamma, \varepsilon) \in \mathfrak{B}_2$$

and $\boldsymbol{q}(\gamma, 0) = \boldsymbol{q}_0(\gamma)$ on $\mathfrak{B}(\gamma_0, \varrho_2; R_k)$.

(d) By 3.8 for any $t \in [0, 1]$ and $(\boldsymbol{c}, \varepsilon)$ sufficiently close to $(\boldsymbol{c}_0, 0)$ the function $\boldsymbol{\xi}(t; \boldsymbol{c}, \varepsilon) = \boldsymbol{\eta}(t; \boldsymbol{\beta}(t; \boldsymbol{c}, \varepsilon))$, where $\boldsymbol{b}(t) = \boldsymbol{\beta}(t; \boldsymbol{c}, \varepsilon)$, is the solution of (3,8) on $[0, 1]$ such that $\boldsymbol{b}(0) = \boldsymbol{c}$. We may assume that this is true for $(\boldsymbol{c}, \varepsilon) \in \mathfrak{B}_1$. Let us put

$$\boldsymbol{\zeta}(t; \boldsymbol{c}, \varepsilon) = \int_0^t \left[\frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{c}}(\tau; \boldsymbol{c}, \varepsilon))\right]^{-1} \boldsymbol{g}(\tau, \boldsymbol{\eta}(\tau; \boldsymbol{\beta}(\tau; \boldsymbol{c}, \varepsilon)), \varepsilon)\,\mathrm{d}\tau$$

for $(t; \boldsymbol{c}, \varepsilon) \in [0, 1] \times \mathfrak{B}_1$. Then

$$\boldsymbol{\beta}(t; \boldsymbol{c}, \varepsilon) = \boldsymbol{c} + \varepsilon\,\boldsymbol{\zeta}(t; \boldsymbol{c}, \varepsilon) \qquad \text{on } [0, 1] \times \mathfrak{B}_1\,.$$

and (cf. (3,12))

$$\lim_{\varepsilon \to 0+} \boldsymbol{\zeta}(t; \boldsymbol{q}(\gamma, \varepsilon), \varepsilon) = \boldsymbol{\zeta}(t; \boldsymbol{q}_0(\gamma), 0) = \boldsymbol{\zeta}_\gamma(t) \qquad \text{for any } \ t \in [0, 1] \ \text{ and } \ \gamma \in \Gamma\,.$$

By (3,17) and I.7.4 we have for any $(\gamma, \varepsilon) \in \mathfrak{B}_2$, $\varepsilon > 0$,

$$(3,21)\qquad \frac{1}{\varepsilon} \mathbf{W}(\mathbf{q}(\gamma, \varepsilon), \varepsilon)$$

$$= \frac{1}{\varepsilon} \left[\mathbf{S}(\boldsymbol{\eta}(.\,; \mathbf{q}(\gamma, \varepsilon) + \varepsilon\zeta)) - \mathbf{S}(\boldsymbol{\eta}(.\,; \mathbf{q}(\gamma, \varepsilon))) \right]$$

$$+ \frac{1}{\varepsilon} \left[\mathbf{S}(\boldsymbol{\eta}(.\,; \mathbf{q}(\gamma, \varepsilon))) - \mathbf{S}(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma))) \right] + \mathbf{R}(\boldsymbol{\eta}(.\,; \mathbf{q}(\gamma, \varepsilon) + \varepsilon\zeta), \varepsilon)$$

$$= \int_0^1 \left[\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}(\gamma, \varepsilon) + \varepsilon\vartheta\zeta)) \right] \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{c}} (.\,; \mathbf{q}(\gamma, \varepsilon) + \varepsilon\vartheta\zeta) \, \mathrm{d}\vartheta\zeta$$

$$+ \left(\int_0^1 \left[\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma) + \vartheta[\mathbf{q}(\gamma, \varepsilon) - \mathbf{q}_0(\gamma)])) \right] \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{c}} (.\,; \mathbf{q}_0(\gamma) + \vartheta[\mathbf{q}(\gamma, \varepsilon) - \mathbf{q}_0(\gamma)]) \, \mathrm{d}\vartheta \right)$$

$$\cdot \frac{\mathbf{q}(\gamma, \varepsilon) - \mathbf{q}_0(\gamma)}{\varepsilon} + \mathbf{R}(\boldsymbol{\eta}(.\,; \mathbf{q}(\gamma, \varepsilon) + \varepsilon\zeta), \varepsilon)$$

$$= \left[\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}(\gamma, \varepsilon))) \right] \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{c}} (.\,; \mathbf{q}(\gamma, \varepsilon)) \, \zeta + \mathbf{R}(\boldsymbol{\eta}(.\,; \mathbf{q}(\gamma, \varepsilon)), \varepsilon)$$

$$+ \left(\int_0^1 \left\{ \left[\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}(\gamma, \varepsilon) + \varepsilon\vartheta\zeta)) \right] \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{c}} (.\,; \mathbf{q}(\gamma, \varepsilon) + \varepsilon\vartheta\zeta) \right. \right.$$

$$\left. \left. - \left[\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}(\gamma, \varepsilon))) \right] \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{c}} (.\,; \mathbf{q}(\gamma, \varepsilon)) \right\} \mathrm{d}\vartheta \right) \zeta + (\Delta(\gamma, \varepsilon))_{\mathcal{N}, \mathcal{N} \setminus \mathcal{J}} \frac{\mathbf{p}(\gamma, \varepsilon) - \mathbf{p}_0(\gamma)}{\varepsilon}$$

$$+ \mathbf{R}(\boldsymbol{\eta}(.\,; \mathbf{q}(\gamma, \varepsilon) + \varepsilon\zeta, \varepsilon) - \mathbf{R}(\boldsymbol{\eta}(.\,; \mathbf{q}(\gamma, \varepsilon)), \varepsilon),$$

where

$$\Delta(\gamma, \varepsilon) = \int_0^1 \left[\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma) + \vartheta[\mathbf{q}(\gamma, \varepsilon) - \mathbf{q}_0(\gamma)])) \right] \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{c}} (.\,; \mathbf{q}_0(\gamma) + \vartheta[\mathbf{q}(\gamma, \varepsilon) - \mathbf{q}_0(\gamma)]) \, \mathrm{d}\vartheta.$$

Since for any $\gamma \in \mathfrak{B}(\gamma_0, \varrho_2; R_k)$

$$\lim_{\varepsilon \to 0+} \Delta(\gamma, \varepsilon) = \left[\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma))) \right] \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{c}} (.\,; \mathbf{q}_0(\gamma)) = \Delta_0(\gamma),$$

(3,10) implies that also

$$(3,22)\qquad \det (\Delta(\gamma, \varepsilon))_{\mathcal{N} \setminus \mathcal{J}, \mathcal{N} \setminus \mathcal{J}} \neq 0$$

for all $\varepsilon > 0$ sufficiently small. Without any loss of generality we may assume that (3,22) holds for all $(\gamma, \varepsilon) \in \mathfrak{B}_2$.

By (3,20)–(3,22) we have for any $\gamma \in \mathfrak{B}(\gamma_0, \varrho_2; R_k)$

$$(3,23)\qquad \lim_{\varepsilon \to 0+} \frac{\mathbf{p}(\gamma, \varepsilon) - \mathbf{p}_0(\gamma)}{\varepsilon}$$

$$= (\Delta_0(\gamma))^{-1}_{\mathcal{N} \setminus \mathcal{J}, \mathcal{N} \setminus \mathcal{J}} \left[(\Delta_0(\gamma))_{\mathcal{N} \setminus \mathcal{J}, \mathcal{N}} \, \zeta_\gamma + R_{\mathcal{N} \setminus \mathcal{J}}(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)), 0) \right].$$

Differentiating (3,21) with respect to γ and making use of (3,22) we may analogously prove that also

$$(3{,}24)\qquad \lim_{\varepsilon\to 0+} \frac{\dfrac{\partial \boldsymbol{p}}{\partial \gamma}(\gamma,\varepsilon) - \dfrac{\partial \boldsymbol{p}_0}{\partial \gamma}(\gamma)}{\varepsilon}$$

exists.

According to (3,20) for $\varepsilon > 0$ the equation (3,19) is near $\boldsymbol{c} = \boldsymbol{q}_0(\gamma_0)$ equivalent to

$$(3{,}25)\qquad \boldsymbol{T}(\gamma,\varepsilon) = \frac{1}{\varepsilon}[\boldsymbol{W}_{\mathscr{I}}(\boldsymbol{q}(\gamma,\varepsilon),\varepsilon) - \boldsymbol{\Theta}(\gamma)\,\boldsymbol{W}_{\mathscr{N}\setminus\mathscr{I}}(\boldsymbol{q}(\gamma,\varepsilon),\varepsilon)] = \boldsymbol{0}\,.$$

Moreover, if for any $\varepsilon > 0$ sufficiently small $\gamma_\varepsilon \in \Gamma$ is the solution to (3,25) which tend to γ_0 as $\varepsilon \to 0+$, then

$$\boldsymbol{x}_\varepsilon(t) = \boldsymbol{\xi}(t;\, \boldsymbol{q}(\gamma_\varepsilon,\varepsilon),\varepsilon)$$

are solutions of BVP (Π_ε) such that

$$\lim_{\varepsilon\to 0+} \|\boldsymbol{x}_\varepsilon - \boldsymbol{\eta}(\,.\,;\, \boldsymbol{q}_0(\gamma_0))\|_C = 0\,.$$

Let $\boldsymbol{r}(\gamma,\varepsilon)$ denote the n-vector

$$\boldsymbol{r}(\gamma,\varepsilon) = (\boldsymbol{\Delta}(\gamma,\varepsilon))_{\mathscr{N},\mathscr{N}\setminus\mathscr{I}}\,\frac{\boldsymbol{p}(\gamma,\varepsilon) - \boldsymbol{p}_0(\gamma)}{\varepsilon}\,.$$

In virtue of (3,11) and (3,23) for any $\gamma \in \mathfrak{B}(\gamma_0, \varrho_2;\, R_k)$

$$\lim_{\varepsilon\to 0+} [\boldsymbol{r}_{\mathscr{I}}(\gamma,\varepsilon) - \boldsymbol{\Theta}(\gamma)\,\boldsymbol{r}_{\mathscr{N}\setminus\mathscr{I}}(\gamma,\varepsilon)] = \boldsymbol{0}\,.$$

Furthermore, (3,11) implies

$$\left(\frac{\partial \boldsymbol{\Delta}_0}{\partial \gamma}(\gamma)\right)_{\mathscr{I},\mathscr{N}} - \boldsymbol{\Theta}(\gamma)\left(\frac{\partial \boldsymbol{\Delta}_0}{\partial \gamma}(\gamma)\right)_{\mathscr{N}\setminus\mathscr{I},\mathscr{N}} - \frac{\partial \boldsymbol{\Theta}}{\partial \gamma}(\gamma)\,(\boldsymbol{\Delta}_0(\gamma))_{\mathscr{N}\setminus\mathscr{I},\mathscr{N}} \equiv \boldsymbol{0} \qquad \text{on } I$$

and hence

$$\lim_{\varepsilon\to 0+} \frac{\partial}{\partial \gamma}[\boldsymbol{r}_{\mathscr{I}}(\gamma,\varepsilon) - \boldsymbol{\Theta}(\gamma)\,\boldsymbol{r}_{\mathscr{N}\setminus\mathscr{I}}(\gamma,\varepsilon)] = \lim_{\varepsilon\to 0+}\Bigg[\left(\frac{\partial \boldsymbol{\Delta}}{\partial \gamma}(\gamma,\varepsilon)\right)_{\mathscr{I},\mathscr{N}\setminus\mathscr{I}}$$

$$- \boldsymbol{\Theta}(\gamma)\left(\frac{\partial \boldsymbol{\Delta}}{\partial \gamma}(\gamma,\varepsilon)\right)_{\mathscr{N}\setminus\mathscr{I},\mathscr{N}\setminus\mathscr{I}} - \frac{\partial \boldsymbol{\Theta}}{\partial \gamma}(\gamma)\,(\boldsymbol{\Delta}(\gamma,\varepsilon))_{\mathscr{N}\setminus\mathscr{I},\mathscr{N}\setminus\mathscr{I}}\Bigg]\frac{\boldsymbol{p}(\gamma,\varepsilon) - \boldsymbol{p}_0(\gamma)}{\varepsilon}$$

$$+ [(\boldsymbol{\Delta}(\gamma,\varepsilon))_{\mathscr{I},\mathscr{N}\setminus\mathscr{I}} - \boldsymbol{\Theta}(\gamma)\,(\boldsymbol{\Delta}(\gamma,\varepsilon))_{\mathscr{N}\setminus\mathscr{I},\mathscr{N}\setminus\mathscr{I}}]\,\frac{\dfrac{\partial \boldsymbol{p}}{\partial \gamma}(\gamma,\varepsilon) - \dfrac{\partial \boldsymbol{p}_0}{\partial \gamma}(\gamma)}{\varepsilon} = \boldsymbol{0}$$

Thus if we put for $\gamma \in \mathfrak{B}(\gamma_0, \varrho_2;\, R_k)$ $\boldsymbol{T}(\gamma, 0) = \boldsymbol{T}_0(\gamma)$, then $\boldsymbol{T}\colon \mathfrak{B}_2 \to R_k$ becomes a continuous operator which possesses the Jacobi matrix $(\partial \boldsymbol{T}/\partial \gamma)(\gamma,\varepsilon)$ for any

$(\gamma, \varepsilon) \in \mathfrak{B}_2$, while the mapping

$$(\gamma, \varepsilon) \in \mathfrak{B}_2 \to \frac{\partial \boldsymbol{T}}{\partial \gamma}(\gamma, \varepsilon) \in L(R_k)$$

is continuous.

Applying the Implicit Function Theorem to (3,25) we complete the proof of the theorem.

Now, let $\gamma_0 \in \Gamma$ and let us assume that given $\varepsilon > 0$ sufficiently small (e.g. $\varepsilon \in (0, \varkappa_0]$), there exists a solution $\boldsymbol{x}_\varepsilon(t) = \boldsymbol{\xi}(t; \boldsymbol{c}_\varepsilon, \varepsilon)$ of BVP (Π_ε) such that $\boldsymbol{x}_\varepsilon(t)$ tends uniformly on $[0, 1]$ to the solution $\boldsymbol{x}_0(t) = \boldsymbol{\eta}(t; \boldsymbol{q}_0(\gamma_0))$ of the limit problem (Π_0) as $\varepsilon \to 0+$. Then, in particular, $\boldsymbol{c}_\varepsilon = \boldsymbol{x}_\varepsilon(0)$ tends to $\boldsymbol{q}_0(\gamma_0)$ and $\gamma_\varepsilon = (\boldsymbol{c}_\varepsilon)_{\mathscr{S}}$ tends to γ_0 as $\varepsilon \to 0+$. Hence $|\gamma_\varepsilon - \gamma_0| < \varrho_2$ for any $\varepsilon > 0$ sufficiently small and analogously as in the proof of Theorem 3.11 we may show that

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \left[\boldsymbol{W}_{\mathscr{S}}(\boldsymbol{q}(\gamma_\varepsilon, \varepsilon)) - \boldsymbol{\Theta}(\gamma_\varepsilon)\, \boldsymbol{W}_{\mathscr{N} \setminus \mathscr{S}}(\boldsymbol{q}(\gamma_\varepsilon, \varepsilon)) \right] = \boldsymbol{T}_0(\gamma_0).$$

Since by the assumption $\boldsymbol{W}(\boldsymbol{q}(\gamma_\varepsilon, \varepsilon)) = \boldsymbol{0}$ for all $\varepsilon \in (0, \varkappa_0]$, this completes the proof of the following theorem.

3.12. Theorem. *Let in addition to* 3.1 *and* 3.7 (i) *and* (ii) *from* 3.11 *hold. Then there exists* $\varepsilon_0 > 0$ *such that given* $\varepsilon \in (0, \varepsilon_0]$, *BVP* (Π_ε) *possesses a solution* $\boldsymbol{x}_\varepsilon(t)$ *tending uniformly on* $[0, 1]$ *to some solution* $\boldsymbol{x}_0(t) = \boldsymbol{\eta}(t; \boldsymbol{q}_0(\gamma))$ *of BVP* (Π_0) *as* $\varepsilon \to 0+$ *only if the equation* (3,13) *has a solution* $\gamma_0 \in \Gamma$.

The next theorem supplements the theorems 3.11 and 3.12.

3.13. Theorem. *Let* 3.1 *and* 3.7 *hold and let* $\Gamma \subset R_n$ *be such an open subset that* $\boldsymbol{x}_\gamma(t) = \boldsymbol{\eta}(t; \gamma)$ *is a solution to BVP* (Π_0) *for any* $\gamma \in \Gamma$.

Let $\gamma_0 \in \Gamma$. *Then a necessary condition for the existence of an* $\varepsilon_0 > 0$ *such that for a given* $\varepsilon \in (0, \varepsilon_0]$ *there exists a solution* $\boldsymbol{x}_\varepsilon(t)$ *of BVP* (Π_ε) *and* $\boldsymbol{x}_\varepsilon(t)$ *tends uniformly on* $[0, 1]$ *to* $\boldsymbol{x}_{\gamma_0}(t)$ *is that* γ_0 *is a solution to*

$$\boldsymbol{T}_0(\gamma) = [\boldsymbol{S}'(\boldsymbol{\eta}(\cdot\,; \gamma))] \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{c}}(\cdot\,; \gamma)\, \zeta_\gamma = \boldsymbol{0}, \tag{3,26}$$

where

$$\zeta_\gamma(t) = \int_0^t \left[\frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{c}}(\tau; \gamma) \right]^{-1} \boldsymbol{g}(\tau, \boldsymbol{\eta}(\tau; \gamma), 0)\, \mathrm{d}\tau.$$

If, moreover, $\det((\partial \boldsymbol{T}_0/\partial \gamma)(\gamma_0)) \neq 0$, *then such an* $\varepsilon_0 > 0$ *exists.*

Proof follows readily by an appropriate modification of the proofs of 3.11 and 3.12.

3.14. Remark. Let us notice that the condition (3,10) of 3.11 holds if and only if any variational problem $(\mathscr{V}_0(\boldsymbol{\eta}(\cdot\,; \boldsymbol{q}_0(\gamma))))$ possesses exactly k linearly independent solutions (cf. IV.2.7). In the next lemma we shall show that the determining equation (3,13) may also be expressed by means of the variational problem.

3.15. Lemma. *Let in addition to* 3.1 *and* 3.7 (i) *and* (ii) *from* 3.11 *hold. Given* $\gamma \in \Gamma$, $\mathbf{T}_0(\gamma) = \mathbf{0}$ *if and only if the nonhomogeneous variational BVP*

$$(3,27) \qquad \mathbf{u}' - \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t, \boldsymbol{\eta}(t; \mathbf{q}_0(\gamma)))\right]\mathbf{u} = \mathbf{g}(t, \boldsymbol{\eta}(t; \mathbf{q}_0(\gamma)), 0),$$

$$(3,28) \qquad [\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)))]\,\mathbf{u} = -\mathbf{R}(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)), 0)$$

possesses a solution.

Proof. Let Ξ be an $n \times n$-matrix such that for a given $\mathbf{r} \in R_n$

$$\Xi \mathbf{r} = \begin{pmatrix} \mathbf{r}_{\mathcal{N}\setminus\mathcal{I}} \\ \mathbf{r}_{\mathcal{I}} \end{pmatrix}.$$

Then the assumption (3,10) means that there exists a $k \times (n - k)$-matrix valued function $\boldsymbol{\Theta}(\gamma)$ defined on Γ and such that

$$(3,29) \qquad \Lambda(\gamma)\,[\mathbf{S}'(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)))]\,\frac{\partial \boldsymbol{\eta}}{\partial \mathbf{c}}(.\,; \mathbf{q}_0(\gamma)) = \mathbf{0} \qquad \text{for any} \quad \gamma \in \Gamma,$$

where

$$(3,30) \qquad \Lambda(\gamma) = -[-\boldsymbol{\Theta}(\gamma), \mathbf{I}_k]\,\Xi.$$

Analogously as in IV.2.2, we may show that to a given $\gamma \in \Gamma$ there exists an $n \times n$-matrix valued function $\mathbf{F}(t, \gamma)$ defined on $[0, 1] \times \Gamma$ and such that

$$\mathbf{T}_0(\gamma) = \Lambda(\gamma)\left(\int_0^1 \mathbf{F}(t, \gamma)\,\mathbf{g}(t, \boldsymbol{\eta}(t; \mathbf{q}_0(\gamma)), 0)\,\mathrm{d}t + \mathbf{R}(\boldsymbol{\eta}(.\,; \mathbf{q}_0(\gamma)), 0)\right) \qquad \text{for any} \quad \gamma \in \Gamma$$

and the couple $(\boldsymbol{\delta}^* \Lambda(\gamma)\,\mathbf{F}(t, \gamma), \boldsymbol{\delta}^* \Lambda(\gamma))$ verifies for any $\boldsymbol{\delta} \in R_n$ and $\gamma \in \Gamma$ the adjoint BVP to BVP (3,27), (3,28). Obviously $\operatorname{rank} \Lambda(\gamma) = k$ for any $\gamma \in \Gamma$. Thus, given $\gamma \in \Gamma$, the rows of $\Lambda(\gamma)\,\mathbf{F}(t, \gamma)$, $\Lambda(\gamma)$ form a basis in the space of all solutions of the adjoint BVP to BVP (3,27), (3,28) (cf. V.2.9). Hence by V.2.6 and V.2.12 our assertion follows.

3.16. Remark. Let us assume that BVP (Π_ε) has the *property* $(\mathcal{T})$ (*translation*): $\boldsymbol{\xi}(t; \mathbf{c}, \varepsilon)$ being a solution to BVP (Π_ε), $\boldsymbol{\xi}(t + \delta; \mathbf{c}, \varepsilon)$ is also a solution to BVP (Π_ε) for any $\delta \in R$ such that $\boldsymbol{\xi}(t + \delta; \mathbf{c}, \varepsilon)$ is defined on $[0, 1]$.

Then, if BVP (Π_ε) has a nonconstant solution $\boldsymbol{\xi}(t; \mathbf{c}, \varepsilon)$, it has at least a one-parametric family of solutions $\boldsymbol{\xi}(t; \boldsymbol{\xi}(\delta; \mathbf{c}, \varepsilon), \varepsilon)$ for all $\delta \in R$ such that $|\delta|$ is sufficiently small. Consequently, Theorem 3.11 cannot be used for proving the existence of a solution $\mathbf{x}_\varepsilon(t)$ of BVP (Π) which tends to some solution $\mathbf{x}_0(t)$ of the shortened

BVP (Π_0) as $\varepsilon \to 0+$. This is clear from the fact that this theorem ensures the existence of an isolated solution. In some cases one component of the initial vector $\mathbf{c} = \mathbf{c}(\varepsilon)$ of the sought solution $\boldsymbol{\xi}(t; \mathbf{c}, \varepsilon)$ may be chosen arbitrary (in a certain range) and another parameter has to be taken as a new unknown instead. Theorems on the existence of solutions to such problems can be then formulated and proved analogously as Theorem 3.11 (cf. Vejvoda [2]–[4]).

The most important problems with the property $(\mathscr{T})$ are those of determining a periodic solution to the autonomous differential equation $\mathbf{x}' = \mathbf{f}(\mathbf{x}) + \varepsilon\, \mathbf{g}(\mathbf{x}, \varepsilon)$. Solving such problems, the period $T = T(\varepsilon)$ of the sought solution is usually chosen as a new unknown. In general, two principal cases have to be distinguished. Either the limit BVP (Π_0) associated to the given BVP (Π_ε) has a k-parametric family of T-periodic solutions $\boldsymbol{\eta}(t; \mathbf{c}(\gamma))$, $\gamma \in \Gamma$, with T independent of γ or their periods depend on γ. The former case occurs e.g. if the equation $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ may be rewritten as the equation $\mathbf{z}' = i\mathbf{z} + \mathbf{z}^2$ for a complex valued function $\mathbf{z}$. (All the solutions of this equation with the initial value sufficiently close to the origin are 2π-periodic, cf. Vejvoda [1], Lemma 5.1.) An example of the latter case is treated in the following section.

4. Froud-Žukovskij pendulum

Let us consider the second order autonomous differential equation of the *Froud-Žukovskij pendulum*

$$x'' + \sin x = \varepsilon\, g(x, x'), \tag{4,1}$$

where g is a sufficiently smooth scalar function and $\varepsilon > 0$ is a small parameter. Given $\varepsilon > 0$, we are looking for a real number $T > 0$ and for a solution $x(t)$ to (4,1) on R such that

$$x(T) = x(0) \quad \text{and} \quad x'(T) = x'(0). \tag{4,2}$$

The limit equation (for $\varepsilon = 0$)

$$y'' + \sin y = 0 \tag{4,3}$$

is known as being equation of the mathematical pendulum. All the solutions $y(t)$ to (4.3) with sufficiently small initial values $y(0)$, $y'(0)$ are defined on the whole real axis R and may be expressed in the form

$$y(t) = \eta(t + h;\, k),$$

where

$$\eta(t; k) = 2 \arcsin(k \operatorname{sn}(t; k)), \qquad h \in R \quad \text{and} \quad k \in (0, 1). \tag{4,4}$$

(cf. Kamke [1], 6.17). Moreover, for any $h \in R$ and $k \in (0, 1)$ the function $y(t)$

$= \eta(t + h;\, k)$ fulfils the periodic boundary conditions (4,2) with $T = 4\,K(k)$, where

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\vartheta}{1 - k^2 \sin^2 \vartheta}.$$

In (4,4) $\operatorname{sn}(t;\, k)$ denotes the value of the Jacobi elliptic sine function with the modulus k at the point t. For the definition and basic properties of the Jacobi elliptic functions sn, cn, dn and of the elliptic integrals $K(k)$, $E(k)$ see e.g. Whittaker-Watson [1], Chapter 22. If no misunderstanding may arise, we write sn, cn, dn instead of $\operatorname{sn}(t;\, k)$, $\operatorname{cn}(t;\, k)$ and $\operatorname{dn}(t;\, k)$, respectively.

Solutions of the perturbed equation (4,1) will be sought in the form

$$x(t) = \xi(t;\, h, k, \varepsilon) = \eta(t + \alpha;\, \beta)\,, \tag{4,5}$$

where $\alpha = \alpha(t) = \alpha(t;\, h, k, \varepsilon)$ and $\beta = \beta(t) = \beta(t;\, h, k, \varepsilon)$ are properly chosen scalar functions such that $\alpha(0) = h$ and $\beta(0) = k$ (cf. 3.8). Differentiating (4,5) with respect to t, we obtain

$$x'(t) = \frac{\partial \eta}{\partial t}(t + \alpha(t);\, \beta(t))\,(1 + \alpha'(t)) + \frac{\partial \eta}{\partial k}(t + \alpha(t);\, \beta(t))\,\beta'(t)\,.$$

Hence, if

$$\frac{\partial \eta}{\partial t}(t + \alpha(t);\, \beta(t))\,\alpha'(t) + \frac{\partial \eta}{\partial k}(t + \alpha(t);\, \beta(t))\,\beta'(t) = 0\,, \tag{4,6}$$

$$\frac{\partial^2 \eta}{\partial t^2}(t + \alpha(t);\, \beta(t))\,\alpha'(t) + \frac{\partial^2 \eta}{\partial k\, \partial t}(t + \alpha(t);\, \beta(t))\,\alpha'(t) = \varepsilon\, g(\eta(t + \alpha(t);\, \beta(t)))\,,$$

then

$$x'(t) = \frac{\partial \eta}{\partial t}(t + \alpha(t);\, \beta(t))$$

and

$$x''(t) - \sin(x(t)) = \varepsilon\, g(\eta(t + \alpha(t);\, \beta(t)))\,.$$

Since

$$\frac{\partial\, \mathrm{sn}}{\partial k} = -k^2 \,.\, \mathrm{cn} \,.\, \mathrm{dn} \,.\, J \quad \text{and} \quad \frac{\partial\, \mathrm{cn}}{\partial k} = k^2 \,.\, \mathrm{sn} \,.\, \mathrm{dn} \,.\, J\,,$$

where

$$J = J(t, k) = \int_0^t \frac{\mathrm{sn}^2(\tau;\, k)}{\mathrm{dn}^2(\tau;\, k)}\, \mathrm{d}\tau\,,$$

we have

$$H(t, k) = \begin{pmatrix} 2k \,.\, \mathrm{cn}\,, & 2\dfrac{\mathrm{sn}}{\mathrm{dn}} - 2k^2 \,.\, \mathrm{cn} \,.\, J \\ -2k \,.\, \mathrm{cn} \,.\, \mathrm{dn}\,, & 2\,\mathrm{cn} + 2k^2 \,.\, \mathrm{sn} \,.\, \mathrm{dn} \,.\, J \end{pmatrix}$$

for

$$H(t,k) = \begin{pmatrix} \dfrac{\partial \eta}{\partial t}(t;k), & \dfrac{\partial \eta}{\partial k}(t;k) \\ \dfrac{\partial^2 \eta}{\partial t^2}(t;k), & \dfrac{\partial^2 \eta}{\partial k\, \partial t}(t;k) \end{pmatrix}.$$

Consequently

$$\det H(t + \alpha(t); \beta(t)) = 4\beta(t).$$

Provided $\beta(t) \neq 0$, the system (4,6) may be written as follows

$$(4,7) \qquad \alpha' = \varepsilon . \frac{1}{2}\left[\frac{\operatorname{sn}(t+\alpha;\beta)}{\operatorname{dn}(t+\alpha;\beta)} - \operatorname{cn}(t+\alpha;\beta)\right] g(\eta(t+\alpha;\beta)),$$
$$\beta' = \varepsilon . \tfrac{1}{2}\operatorname{cn}(t+\alpha;\beta)\, g(\eta(t+\alpha;\beta)).$$

Since for $\varepsilon = 0$ the couple $(\alpha(t), \beta(t)) \equiv (h, k)$ is the unique solution of the system (4,7) on R such that $\alpha(0) = h$, $\beta(0) = k$, Lemma 1.18 implies that for any $T > 0$ there exists $\varepsilon_T > 0$ such that for any $\varepsilon \in (0, \varepsilon_T)$ and $h \in R$, $k \in (0, 1)$ the system (4,7) possesses a unique solution $(\alpha(t), \beta(t)) = (\alpha(t; h, k, \varepsilon), \beta(t; h, k, \varepsilon))$ on $[0, T]$, continuous on $[0, T] \times R \times (0, 1) \times (0, \varepsilon_T)$ and such that $\alpha(0) = h$, $\beta(0) = k$, while $\beta(t) \in (0, 1)$ for any $t \in [0, T]$. Let us put $\alpha(t; h, k, 0) = h$ and $\beta(t; h, k, 0) = k$.

Given a solution $x(t)$ to BVP (4,1), (4,2) and $h \in R$, the function $z(t) = x(t + h)$ is also a solution to this problem. Hence without any loss of generality we may put

$$(4,8) \qquad h = 0.$$

Let $T > 0$ and $k \in (0, 1)$ be for a while fixed. Let $\alpha(t) = \alpha(t; 0, k, \varepsilon)$, $\beta(t) = \beta(t; 0, k, \varepsilon)$ be the corresponding solution of (4,7) on $[0, T]$ $(\varepsilon \in (0, \varepsilon_T))$. Then (4,5) becomes

$$(4,9) \quad x(t) = 2 \arcsin(\beta(t) \operatorname{sn}(t + \alpha(t); \beta(t))) \quad \text{for} \quad t \in [0, T] \quad \text{and} \quad \varepsilon \in (0, \varepsilon_T)$$

and $x(T) = x(0)$ if and only if $\beta(T) \operatorname{sn}(T + \alpha(T); \beta(T)) = 0$ or equivalently $(\beta(T) \neq 0)$

$$(4,10) \qquad T + \alpha(T; 0, k, \varepsilon) - 4K(\beta(T; 0, k, \varepsilon)) = 0.$$

According to (4,6) and (4,9)

$$x'(t) = 2\,\beta(t) \operatorname{cn}(t + \alpha(t); \beta(t))$$

and $x'(T) = x'(0)$ if and only if

$$\beta(T) \operatorname{cn}(T + \alpha(T); \beta(T)) = k \operatorname{cn}(0; k) = k$$

or in virtue of (4,10)

$$(4,11) \qquad \beta(T) = \beta(T) \operatorname{cn}(4K(\beta(T)); \beta(T)) = k.$$

By (4,9)

$$\beta(t) = k + \varepsilon . \tfrac{1}{2}\varkappa(t, k, \varepsilon) \qquad \text{for} \quad t \in [0, T] \quad \text{and} \quad \varepsilon \in (0, \varepsilon_T),$$

where
$$\varkappa(t,k,\varepsilon)=\int_0^t \operatorname{cn}(\tau+\alpha(\tau);\ \beta(\tau))\, g(\eta(\tau+\alpha(\tau);\ \beta(\tau)))\,d\tau .$$
This together with (4,11) implies that $x'(T)=x'(0)$ if and only if

(4,12) $$\varkappa(T,k,\varepsilon)=0 .$$

If $\varepsilon\to 0+$, then the equation (4,10) becomes $T-4\,K(k)=0$ and the system (4,10), (4,12) reduces to the equation

(4,13) $$B(k)=0 ,$$

where
$$B(k)=\int_0^{4\,K(k)} \operatorname{cn}(t;\ k)\, g(\eta(t;\ k))\,dt .$$

This means that a necessary condition for the existence of a solution to the given BVP (4,1), (4,2) for any $\varepsilon>0$ sufficiently small is the existence of a solution $k\in(0,1)$ of the equation (4,13).

Taking into account the properties of the Jacobi elliptic functions it can be shown that if e.g.
$$g(x,x')=x'-3(x')^3 ,$$
then the equation (4,13) possesses a solution $k_0\in(0,1)$ such that $(\partial B/\partial k)(k_0)\neq 0$. By the Implicit Function Theorem there exists $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0]$ the system (4,10), (4,12) possesses a unique solution $T=T_\varepsilon>0$ and $k=k_\varepsilon\in(0,1)$ such that $T_\varepsilon\to 4\,K(k_0)$ and $k_\varepsilon\to k_0$ as $\varepsilon\to 0+$. Given $\varepsilon\in[0,\varepsilon_0]$, $\alpha(t)=\alpha(t;\ 0,k_\varepsilon,\varepsilon)$ and $\beta(t)=\beta(t;\ 0,k_\varepsilon,\varepsilon)$ verify the system (4,7) on $[0,T_\varepsilon]$ and hence $x_\varepsilon(t)=\eta(t+\alpha(t);\ \beta(t))$ is a unique T_ε-periodic solution of the equation
$$x''+\sin x=\varepsilon(x'-3(x')^3)$$
such that
$$x_\varepsilon(t)\to x_0(t)=\eta(t;\ k_0)\qquad \text{as}\quad \varepsilon\to 0+ .$$

Notes

Chapter VI is a generalization of the work by Vejvoda ([4]). The main tools are the Implicit Function Theorem (Newton's method) and the nonlinear variation of constants formula VI.3.8 due to Vejvoda ([4]). Theorems VI.2.3, VI.2.7 and VI.2.9 are contained also in Urabe [2], [3].

The method of a small parameter (perturbation theory) originated from the celestial mechanics (Poincaré [1]). Periodic solutions of nonlinear differential equations were dealt with e.g. by Malkin ([1], [2]), Coddington, Levinson ([1]), Hale [1], Loud ([1], [2]) and others. Further related references concerning the application of the Newton method to perturbed nonlinear BVP are e.g. Antosiewicz [1], [2], Bernfeld, Lakshmikantham [1], Candless [1], Locker [1], Kwapisz [1], Tvrdý, Vejvoda [1], Vejvoda [2], [3] and Urabe [1].