II. Integral equations in the space $BV_n[0, 1]$

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1. Some integral operators in the space $BV_n[0, 1]$

In this paragraph we assume that on the twodimensional interval $I = [0, 1] \times [0, 1] \subset R_2$ an $n \times n$-matrix valued function $\mathbf{K}(s, t) = k_{ij}(s, t))$, $i, j = 1, 2, \dots, n$ is given, i.e. $\mathbf{K}: I \to L(R_n)$. Moreover let the twodimensional variation of $\mathbf{K}: I \to L(R_n)$ be finite, i.e. (cf. I.6.1)

$$\text{(1,1)} \qquad \mathrm{v}_I(\mathbf{K}) < \infty .$$

The operator $\int_0^1 \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t)$

Let us assume that $\mathbf{x} \in BV_n[0, 1] = BV_n$ is given, i.e. $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))^*$; $t \in [0, 1]$. If it is assumed that

$$\text{(1,2)} \qquad \mathrm{var}_0^1 \mathbf{K}(0, .) < \infty ,$$

then by I.6.6 we obtain $\mathrm{var}_0^1 \mathbf{K}(s, .) \le \mathrm{v}_I(\mathbf{K}) + \mathrm{var}_0^1 \mathbf{K}(0, .) < +\infty$ for every $s \in [0, 1]$. This yields by I.4.19 the existence of the Perron-Stieltjes integral

$$\text{(1,3)} \qquad \int_0^1 \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) = \mathbf{y}(s)$$

for any $s \in [0, 1]$. The integral (1,3) evidently defines a function $\mathbf{y}: [0, 1] \to R_n$. By I.6.18 we have

$$\text{(1,4)} \qquad \mathrm{var}_0^1 \mathbf{y} \le \sup_{t \in [0,1]} |\mathbf{x}(t)|\, \mathrm{v}_I(\mathbf{K})$$

and consequently $\mathbf{y} \in BV_n$. Hence the integral (1,3) defines an operator acting in the Banach space BV_n. Let us denote this operator by

$$\text{(1,5)} \qquad \mathbf{K}\mathbf{x} = \int_0^1 \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) , \qquad \mathbf{x} \in BV_n .$$

1.1. Theorem. *If* $\boldsymbol{K}\colon I \to L(R_n)$ *satisfies* (1,1) *and* (1,2) *then the operator* $\boldsymbol{K}$ *defined by* (1,5) *is a bounded linear operator on* BV_n ($\boldsymbol{K} \in B(BV_n)$) *and*

$$\|\boldsymbol{K}\|_{B(BV_n)} \leq \operatorname{var}_0^1 \boldsymbol{K}(0, .) + \mathrm{v}_I(\boldsymbol{K}). \tag{1,6}$$

Proof. The linearity of the operator $\boldsymbol{K}$ is evident. Further for any $\boldsymbol{x} \in BV_n$ it is

$$\|\boldsymbol{Kx}\|_{BV_n} = \left|\int_0^1 \mathrm{d}_t[\boldsymbol{K}(0,t)]\,\boldsymbol{x}(t)\right| + \operatorname{var}_0^1\left(\int_0^1 \mathrm{d}_t[\boldsymbol{K}(.,t)]\,\boldsymbol{x}(t)\right)$$
$$\leq \sup_{t\in[0,1]} |\boldsymbol{x}(t)|\,(\operatorname{var}_0^1 \boldsymbol{K}(0, .) + \mathrm{v}_I(\boldsymbol{K})) \leq (\operatorname{var}_0^1 \boldsymbol{K}(0, .) + \mathrm{v}_I(\boldsymbol{K}))\,\|\boldsymbol{x}\|_{BV_n}$$

where (I.6,13) and (I.6,14) from I.6.18 was used. This implies the boundedness of $\boldsymbol{K}$ and the inequality (1,6).

1.2. Lemma. *If* $\boldsymbol{K}\colon I \to L(R_n)$ *and* $\tilde{\boldsymbol{K}}\colon I \to L(R_n)$ *satisfy* (1,1) *and* (1,2), *then*

$$\int_0^1 \mathrm{d}_t[\boldsymbol{K}(s,t)]\,\boldsymbol{x}(t) = \int_0^1 \mathrm{d}_t[\tilde{\boldsymbol{K}}(s,t)]\,\boldsymbol{x}(t) \tag{1,7}$$

for every $\boldsymbol{x} \in BV_n$ *and* $s \in [0, 1]$ *if and only if the difference*

$$\boldsymbol{W}(s,t) = \boldsymbol{K}(s,t) - \tilde{\boldsymbol{K}}(s,t)$$

satisfies

$$\boldsymbol{W}(s,t+) = \boldsymbol{W}(s,t-) = \boldsymbol{W}(s,1-) = \boldsymbol{W}(s,0+) = \boldsymbol{W}(s,1) = \boldsymbol{W}(s,0) \tag{1,8}$$

for every $s \in [0, 1]$ *and* $t \in (0, 1)$.

Proof. The assumptions on $\boldsymbol{K}, \tilde{\boldsymbol{K}}$ guarantee that for $\boldsymbol{W}\colon I \to L(R_n)$ we have $\mathrm{v}_I(\boldsymbol{W}) < \infty$ and $\operatorname{var}_0^1 \boldsymbol{W}(0, .) < \infty$. Hence by I.6.6 also $\operatorname{var}_0^1 \boldsymbol{W}(s, .) < \infty$ for every $s \in [0, 1]$. The equality (1,7) can be written in the form $\int_0^1 \mathrm{d}_t[\boldsymbol{W}(s,t)]\,\boldsymbol{x}(t) = 0$. The assertion of our lemma follows now immediately from I.6.5 since (1,8) is equivalent to the fact that for every $s \in [0, 1]$ the elements of the matrix $\boldsymbol{W}(s, .)$ belong to $S[0, 1]$.

1.3. Corollary. *If* $\boldsymbol{K}, \tilde{\boldsymbol{K}}\colon I \to L(R_n)$ *satisfies* (1,1) *and* (1,2) *where for the difference* $\boldsymbol{W}(s,t) = \boldsymbol{K}(s,t) - \tilde{\boldsymbol{K}}(s,t)$ *the chain of equalities* (1,8) *holds for any* $s \in [0, 1]$ *and* $t \in (0, 1)$, *then the operator* $\tilde{\boldsymbol{K}} \in B(BV_n)$ *defined by the relation*

$$\tilde{\boldsymbol{K}}\boldsymbol{x} = \int_0^1 \mathrm{d}_t[\tilde{\boldsymbol{K}}(s,t)]\,\boldsymbol{x}(t), \qquad \boldsymbol{x} \in BV_n$$

is identical with the operator $\boldsymbol{K} \in B(BV_n)$ *defined by* (1,5).

If we define for any $s \in [0, 1]$

$$\hat{\boldsymbol{K}}(s,t) = \boldsymbol{K}(s,t+) - \boldsymbol{K}(s,0) \quad \text{for} \quad t \in (0,1), \tag{1,9}$$
$$\hat{\boldsymbol{K}}(s,0) = 0, \qquad \hat{\boldsymbol{K}}(s,1) = \boldsymbol{K}(s,1) - \boldsymbol{K}(s,0),$$

then $v_I(\hat{K}) < \infty$, $\text{var}_0^1 \hat{K}(0, .) < \infty$ *and the difference* $\hat{W}(s, t) = K(s, t) - \hat{K}(s, t)$ *satisfies* (1,8) *for any* $s \in [0, 1]$ *and* $t \in (0, 1)$. *Hence the operator*

$$\hat{K}x = \int_0^1 d_t[\hat{K}(s, t)]\, x(t), \qquad x \in BV_n$$

is the same as the operator $K \in B(BV_n)$ *defined by* (1,5), *i.e.* $K = \hat{K}$.

Proof. The first part of this corollary simply follows from 1.2. For the second part it is necessary to show that $\hat{K}$: $I \to L(R_n)$ from (1,9) satisfies (1,1) and (1,2).

Assume that $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ is an arbitrary subdivision of $[0, 1]$ and $J_{ij} = [\alpha_{i-1}, \alpha_i] \times [\alpha_{j-1}, \alpha_j]$, $i, j = 1, \ldots, k$ is the corresponding net-type subdivision of I (see I.6.3). We have for any given $\delta > 0$

$$\sum_{i=1}^{k} |K(\alpha_i, \alpha_1 + \delta) - K(\alpha_i, \alpha_0) - K(\alpha_{i-1}, \alpha_1 + \delta) + K(\alpha_{i-1}, \alpha_0)|$$
$$+ \sum_{j=2}^{k} \sum_{i=1}^{k} |K(\alpha_i, \alpha_j + \delta) - K(\alpha_i, \alpha_{j-1} + \delta) - K(\alpha_{i-1}, \alpha_j + \delta) + K(\alpha_{i-1}, \alpha_{j-1} + \delta)| \le v_I(K)$$

where we assume that $K(s, t) = K(s, 1)$ if $t > 1$. Since for K: $I \to L(R_n)$ (1,1) and (1,2) hold, the limit $\lim_{\delta \to 0+} K(s, t + \delta) = K(s, t+)$ exists for every $s \in [0, 1]$, $t \in [0, 1]$. Passing to the limit $\delta \to 0+$ in the above inequality we obtain for $\hat{K}$ the inequality

$$\sum_{j=1}^{k} \sum_{i=1}^{k} |m_{\hat{K}}(J_{ij})|$$
$$\sum_{j=1}^{k} \sum_{i=1}^{k} |\hat{K}(\alpha_i, \alpha_j) - \hat{K}(\alpha_i, \alpha_{j-1}) - \hat{K}(\alpha_{i-1}, \alpha_j) + \hat{K}(\alpha_{i-1}, \alpha_{j-1})| \le v_I(K)$$

which holds for every net-type subdivision J_{ij} of I. Hence (see I.6.3) we obtain $v_I(\hat{K}) \le v_I(K) < \infty$. Since $\text{var}_0^1 K(0, .) < \infty$ and $\hat{K}(0, t) = K(0, t+) - K(0, 0)$ differs from $K(0, t) - K(0, 0)$ only on an at most countable set of points in $[0, 1]$, the variation $\text{var}_0^1 \hat{K}(0, .)$ is finite. For $\hat{W}(s, t) = K(s, t) - \hat{K}(s, t)$ we have evidently

$$\hat{W}(s, t-) = K(s, t-) - \hat{K}(s, t-) = K(s, t-) - \lim_{\tau \to t-} K(s, \tau+) + K(s, 0) = K(s, 0)$$

if $s \in [0, 1]$, $t \in (0, 1)$. Similarly also $\hat{W}(s, t+) = \hat{W}(s, 1-) = \hat{W}(s, 1) = \hat{W}(s, 0+) = \hat{W}(s, 0) = K(s, 0)$ holds and the assertion of the second part of the corollary is valid.

1.4. Remark. The corollary 1.3 states that we can assume without any loss of generality that the kernel K: $I \to L(R_n)$, which defines by (1,5) the operator $K \in B(BV_n)$, satisfies

(1,10) $$K(s, t+) = K(s, t) \quad \text{for any} \quad s \in [0, 1], \quad t \in (0, 1)$$

and

(1,11) $$K(s, 0) = 0 \quad \text{for any} \quad s \in [0, 1].$$

It is clear that if in (1,9) the right-hand limit $\mathbf{K}(s, t+)$ is replaced by the left-hand limit $\mathbf{K}(s, t-)$, then 1.3 holds too. This justifies the possibility of replacing the condition (1,10) by

$$(1,10') \qquad \mathbf{K}(s, t-) = \mathbf{K}(s, t) \qquad \text{for any} \quad s \in [0, 1], \quad t \in (0, 1).$$

Hence without any restriction it can be assumed that the kernel $\mathbf{K}: I \to L(R_n)$ defining the operator $\mathbf{K} \in B(BV_n)$ by (1,5) satisfies (1,10′) and (1,11), $\mathbf{K}$ remaining unchanged also in this case.

Moreover, any operator $\mathbf{K} \in B(BV_n)$ given by (1,5) with $\mathbf{K}: I \to L(R_n)$ satisfying (1,1) and (1,2) can be represented by a kernel $\mathbf{K}: I \to L(R_n)$ satisfying the additional assumptions (1,10), (1,11) (or (1,10′), (1,11)). Using the notations from I.5 the additional assumptions (1,10), (1,11) ((1,10′), (1,11)) state that the elements $k_{ij}(s, t)$ of $\mathbf{K}: I \to L(R_n)$ as functions of the second variable t belong to the class NBV (NBV^-).

1.5. Theorem. *If $\mathbf{K}: I \to L(R_n)$ satisfies (1,1) and (1,2), then the operator $\mathbf{K} \in B(BV_n)$ defined by (1,5) is compact, i.e. $\mathbf{K} \in K(BV_n)$.*

Proof. For proving $\mathbf{K} \in K(BV_n)$ we use I.3.16. Let $\{\mathbf{x}_k\}$, $\mathbf{x}_k \in BV_n$, $k = 1, 2, \ldots$ be an arbitrary sequence with

$$\|\mathbf{x}_k\|_{BV_n} = |\mathbf{x}_k(0)| + \operatorname{var}_0^1 \mathbf{x}_k \le C = \text{const.}, \qquad k = 1, 2, \ldots.$$

By Helly's Choice Theorem (cf. I.1.4) there exists a function $\tilde{\mathbf{x}} \in BV_n$ and a subsequence $\mathbf{x}_{k_l}$, $l = 1, 2, \ldots$ of $\{\mathbf{x}_k\}$ such that $\lim_{l \to \infty} \mathbf{x}_{k_l}(t) = \tilde{\mathbf{x}}(t)$ for any $t \in [0, 1]$.

Let us put

$$\mathbf{z}_l(t) = \mathbf{x}_{k_l}(t) - \tilde{\mathbf{x}}(t), \qquad t \in [0, 1], \quad l = 1, 2, \ldots.$$

Then $\|\mathbf{z}_l\|_{BV_n} \le C + \|\tilde{\mathbf{x}}\|_{BV_n} < \infty$, $\mathbf{z}_l \in BV_n$, $l = 1, 2, \ldots$ and

$$(1,12) \qquad \lim_{l \to \infty} \mathbf{z}_l(t) = \mathbf{0} \qquad \text{for any} \quad t \in [0, 1].$$

Using I.6.18 (see (I.6,14)) we have

$$(1,13) \qquad \operatorname{var}_0^1 \left(\int_0^1 \mathrm{d}_t[\mathbf{K}(., t)] \, (\mathbf{x}_{k_l}(t) - \tilde{\mathbf{x}}(t)) \right) = \operatorname{var}_0^1 \left(\int_0^1 \mathrm{d}_t[\mathbf{K}(., t)] \, \mathbf{z}_l(t) \right) \le \int_0^1 |\mathbf{z}_l(t)| \, \mathrm{d}\omega_2(t)$$

where $\omega_2: [0, 1] \to R$ is nondecreasing, $\omega_2(0) = 0$, $\omega_2(1) = \mathrm{v}_I(\mathbf{K})$, (see I.6.7). For every $t \in [0, 1]$ and $l = 1, 2, \ldots$ we have evidently $0 \le |\mathbf{z}_l(t)| \le \|\mathbf{z}_l\|_{BV_n} \le C + \|\tilde{\mathbf{x}}\|_{BV_n}$ and the real valued function $|\mathbf{z}_l(t)|: [0, 1] \to R$ belongs to $BV[0, 1]$ for every $l = 1, 2, \ldots$. Hence by I.4.19 the integral $\int_0^1 |\mathbf{z}_l(t)| \, \mathrm{d}\omega_2(t)$ exists for every $l = 1, 2, \ldots$. I.4.24 implies by (1,12)

$$\lim_{l \to \infty} \int_0^1 |\mathbf{z}_l(t)| \, \mathrm{d}\omega_2(t) = 0$$

and this together with (1,13) leads to the relation

$$\lim_{l\to\infty} \operatorname{var}_0^1 \left(\int_0^1 \mathrm{d}_t[\mathbf{K}(.,t)]\, \mathbf{x}_{k_l}(t) - \int_0^1 \mathrm{d}_t[\mathbf{K}(.,t)]\, \tilde{\mathbf{x}}(t) \right) = 0\,. \tag{1,14}$$

By (I.6.13) we have further

$$\left| \int_0^1 \mathrm{d}_t[\mathbf{K}(0,t)]\, \mathbf{z}_l(t) \right| \le \int_0^1 |\mathbf{z}_l(t)|\, \mathrm{d}[\operatorname{var}_0^t \mathbf{K}(0,.)]$$

and the same argument as above gives by (1,12)

$$\lim_{l\to\infty} \left| \int_0^1 \mathrm{d}_t[\mathbf{K}(0,t)]\, \mathbf{x}_{k_l}(t) - \int_0^1 \mathrm{d}_t[\mathbf{K}(0,t)]\, \tilde{\mathbf{x}}(t) \right| = 0\,. \tag{1,15}$$

Let us now denote $\tilde{\mathbf{y}}(s) = \int_0^1 \mathrm{d}_t[\mathbf{K}(s,t)]\, \tilde{\mathbf{x}}(t)$. By 1,1 evidently $\tilde{\mathbf{y}} \in BV_n$ and by (1,14) and (1,15) we obtain

$$\lim_{l\to\infty} \|\mathbf{K}\mathbf{x}_{k_l} - \tilde{\mathbf{y}}\|_{BV_n} = \lim_{l\to\infty} \{|\mathbf{K}\mathbf{x}_{k_l}(0) - \tilde{\mathbf{y}}(0)| + \operatorname{var}_0^1 (\mathbf{K}\mathbf{x}_{k_l} - \tilde{\mathbf{y}})\} = 0\,,$$

i.e. the sequence $\{\mathbf{K}\mathbf{x}_k\}$ contains a subsequence which converges in BV_n. Hence $\mathbf{K} \in K(BV_n)$.

The operator $\int_0^1 \mathbf{K}(s,t)\, \mathrm{d}\boldsymbol{\varphi}(s)$

Let us assume that $\boldsymbol{\varphi} \in BV_n$ is given, $\boldsymbol{\varphi}(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^*$, $t \in [0,1]$. If

$$\operatorname{var}_0^1 \mathbf{K}(.,0) < \infty\,, \tag{1,16}$$

then by I.6.6 we obtain $\operatorname{var}_0^1 \mathbf{K}(.,t) \le \mathrm{v}_I(\mathbf{K}) + \operatorname{var}_0^1 \mathbf{K}(.,0) < \infty$ for every $t \in [0,1]$ provided (1,1) is fulfilled. In this case by I.4.19 the Perron-Stieltjes integral

$$\int_0^1 \mathbf{K}(s,t)\, \mathrm{d}\boldsymbol{\varphi}(s) = \boldsymbol{\psi}(t) \tag{1,17}$$

exists for every $t \in [0,1]$.

Let us show that the function $\boldsymbol{\psi}\colon [0,1] \to R_n$ defined by (1,17) is of bounded variation on $[0,1]$ if (1,16), (1,1), (1,2) are assumed.

Let $0 = \gamma_0 < \gamma_1 < \ldots < \gamma_l = 1$ be an arbitrary subdivision of $[0,1]$. By I.4.27 we have

$$|\boldsymbol{\psi}(\gamma_i) - \boldsymbol{\psi}(\gamma_{i-1})| = \left| \int_0^1 (\mathbf{K}(s,\gamma_i) - \mathbf{K}(s,\gamma_{i-1}))\, \mathrm{d}\boldsymbol{\varphi}(s) \right|$$

$$\le \sup_{s\in[0,1]} |\mathbf{K}(s,\gamma_i) - \mathbf{K}(s,\gamma_{i-1})| \operatorname{var}_0^1 \boldsymbol{\varphi} \le (\mathrm{v}_{[0,1]\times[\gamma_{i-1},\gamma_i]}(\mathbf{K}) + |\mathbf{K}(0,\gamma_i) - \mathbf{K}(0,\gamma_{i-1})|) \operatorname{var}_0^1 \boldsymbol{\varphi}$$

because for every $s \in [0,1]$

$$\begin{aligned} &|\boldsymbol{K}(s,\gamma_i) - \boldsymbol{K}(s,\gamma_{i-1})| \\ &\le |\boldsymbol{K}(s,\gamma_i) - \boldsymbol{K}(s,\gamma_{i-1}) - \boldsymbol{K}(0,\gamma_i) + \boldsymbol{K}(0,\gamma_{i-1})| + |\boldsymbol{K}(0,\gamma_i) - \boldsymbol{K}(0,\gamma_{i-1})| \\ &\le \mathrm{v}_{[0,1]\times[\gamma_{i-1},\gamma_i]}(\boldsymbol{K}) + |\boldsymbol{K}(0,\gamma_i) - \boldsymbol{K}(0,\gamma_{i-1})| \end{aligned}$$

(cf. I.6). Hence by I.6.5

$$\begin{aligned} &\sum_{i=1}^{l} |\boldsymbol{\psi}(\gamma_i) - \boldsymbol{\psi}(\gamma_{i-1})| \\ &\le \sum_{i=1}^{l} (\mathrm{v}_{[0,1]\times[\gamma_{i-1},\gamma_i]}(\boldsymbol{K}) + |\boldsymbol{K}(0,\gamma_i) - \boldsymbol{K}(0,\gamma_{i-1})|)\,\mathrm{var}_0^1\,\boldsymbol{\varphi} \\ &\le [\mathrm{v}_I(\boldsymbol{K}) + \mathrm{var}_0^1\,\boldsymbol{K}(0,.)]\,\mathrm{var}_0^1\,\boldsymbol{\varphi} \le [\mathrm{v}_I(\boldsymbol{K}) + \mathrm{var}_0^1\,\boldsymbol{K}(0,.)]\,\|\boldsymbol{\varphi}\|_{BV_n} \end{aligned} \tag{1,18}$$

for all subdivisions $0 = \gamma_0 < \gamma_1 < \ldots < \gamma_l = 1$ and so $\mathrm{var}_0^1\,\boldsymbol{\psi} < \infty$. In this way the integral (1,17) defines an operator acting on BV_n; we set

$$\tilde{\boldsymbol{K}}\boldsymbol{\varphi} = \int_0^1 \boldsymbol{K}(s,t)\,\mathrm{d}\boldsymbol{\varphi}(s), \qquad \boldsymbol{\varphi} \in BV_n. \tag{1,19}$$

1.6. Theorem. *If* $\boldsymbol{K}: I \to L(R_n)$ *satisfies* (1,1), (1,2) *and* (1,16), *then the operator* $\tilde{\boldsymbol{K}}$ *defined by* (1,19) *is a bounded linear operator on* BV_n; *i.e.* $\tilde{\boldsymbol{K}} \in B(BV_n)$ *and*

$$\|\tilde{\boldsymbol{K}}\|_{B(BV_n)} \le |\boldsymbol{K}(0,0)| + \mathrm{var}_0^1\,\boldsymbol{K}(.,0) + \mathrm{var}_0^1\,\boldsymbol{K}(0,.) + \mathrm{v}_I(\boldsymbol{K}). \tag{1,20}$$

Proof. The linearity of $\tilde{\boldsymbol{K}}$ is obvious. For any $\boldsymbol{\varphi} \in BV_n$ by I.4.27 we have

$$\left|\int_0^1 \boldsymbol{K}(s,0)\,\mathrm{d}\boldsymbol{\varphi}(s)\right| \le \sup_{s\in[0,1]} |\boldsymbol{K}(s,0)|\,\mathrm{var}_0^1\,\boldsymbol{\varphi} \le (|\boldsymbol{K}(0,0)| + \mathrm{var}_0^1\,\boldsymbol{K}(.,0))\,\|\boldsymbol{\varphi}\|_{BV_n}.$$

Using (1,18) we obtain

$$\begin{aligned} \|\tilde{\boldsymbol{K}}\boldsymbol{\varphi}\|_{BV_n} &= \left|\int_0^1 \boldsymbol{K}(s,0)\,\mathrm{d}\boldsymbol{\varphi}(s)\right| + \mathrm{var}_0^1 \int_0^1 \boldsymbol{K}(s,.)\,\mathrm{d}\boldsymbol{\varphi}(s) \\ &\le [|\boldsymbol{K}(0,0)| + \mathrm{var}_0^1\,\boldsymbol{K}(0,.) + \mathrm{var}_0^1\,\boldsymbol{K}(.,0) + \mathrm{v}_I(\boldsymbol{K})]\,\|\boldsymbol{\varphi}\|_{BV_n}. \end{aligned}$$

Hence $\tilde{\boldsymbol{K}} \in B(BV_n)$ and (1,20) holds.

1.7. Lemma. *Let* $\boldsymbol{M}: [0,1] \to L(R_n)$ *be an* $n \times n$*-matrix valued function such that*

$$\mathrm{var}_0^1\,\boldsymbol{M} < \infty. \tag{1,21}$$

Assume that a fixed $\sigma \in [a,b]$ *is given. Define for* $\boldsymbol{x} \in BV_n$ *the operators*

$$\boldsymbol{M}\boldsymbol{x} = \boldsymbol{M}(t)\,\boldsymbol{x}(\sigma), \qquad \boldsymbol{M}^+\boldsymbol{x} = \boldsymbol{M}(t)\,\Delta^+\boldsymbol{x}(\sigma), \qquad \boldsymbol{M}^-\boldsymbol{x} = \boldsymbol{M}(t)\,\Delta^-\boldsymbol{x}(\sigma)$$

where $\Delta^+\boldsymbol{x}(\sigma) = \boldsymbol{x}(\sigma+) - \boldsymbol{x}(\sigma)$, $\Delta^-\boldsymbol{x}(\sigma) = \boldsymbol{x}(\sigma) - \boldsymbol{x}(\sigma-)$. *The operators* $\boldsymbol{M}, \boldsymbol{M}^+, \boldsymbol{M}^-$ *are compact linear operators on* BV_n, *i.e.* $\boldsymbol{M}, \boldsymbol{M}^+, \boldsymbol{M}^- \in K(BV_n)$.

Proof. Since evidently

$$\|\mathbf{M}\mathbf{x}\|_{BV_n} = |\mathbf{M}(0)\,\mathbf{x}(\sigma)| + \operatorname{var}_0^1(\mathbf{M}(.)\,\mathbf{x}(\sigma)) \le [|\mathbf{M}(0)| + \operatorname{var}_0^1 \mathbf{M}]\,|\mathbf{x}(\sigma)| \le [|\mathbf{M}(0)| + \operatorname{var}_0^1 \mathbf{M}]\,\|\mathbf{x}\|_{BV_n}$$

we have $\mathbf{M} \in B(BV_n)$ and

$$\|\mathbf{M}\|_{B(BV_n)} \le [|\mathbf{M}(0)| + \operatorname{var}_0^1 \mathbf{M}]\,.$$

The same argument gives also $\mathbf{M}^+, \mathbf{M}^- \in B(BV_n)$ and the inequalities

$$\|\mathbf{M}^+\|_{B(BV_n)} \le [|\mathbf{M}(0)| + \operatorname{var}_0^1 \mathbf{M}]\,, \qquad \|\mathbf{M}^-\|_{B(BV_n)} \le [|\mathbf{M}(0)| + \operatorname{var}_0^1 \mathbf{M}]\,.$$

Let us denote by $B = \{\mathbf{x} \in BV_n;\ \|\mathbf{x}\|_{BV_n} \le 1\}$ the unit ball in BV_n. $\mathbf{M}^+(B) = \{\mathbf{y} \in BV_n;\ \mathbf{y} = \mathbf{M}^+\mathbf{x},\ \mathbf{x} \in B\}$ is the image of B under the map $\mathbf{M}^+$. Let $\mathbf{y}_k \in \mathbf{M}^+(B)$, $k = 1, 2, \dots$ be an arbitrary sequence in $\mathbf{M}^+(B)$, i.e. there is a sequence $\mathbf{x}_k \in B$ such that $\mathbf{y}_k = \mathbf{M}^+\mathbf{x}_k$. Since $\mathbf{x}_k \in B$, $k = 1, 2, \dots$ we have

$$|\Delta^+\mathbf{x}_k(\sigma)| \le \operatorname{var}_0^1 \mathbf{x}_k \le \|\mathbf{x}_k\|_{BV_n} \le 1$$

and there is a subsequence $\{\mathbf{x}_{k_l}\}$, $l = 1, 2, \dots$ such that $\lim_{l\to\infty} \Delta^+\mathbf{x}_{k_l}(\sigma) = \mathbf{z} \in R_n$ and $\mathbf{M}(t)\,\mathbf{z} \in BV_n$. Since evidently

$$\|\mathbf{M}^+\mathbf{x}_{k_l} - \mathbf{M}(t)\,\mathbf{z}\|_{BV_n} \le (|\mathbf{M}(0)| + \operatorname{var}_0^1 \mathbf{M})\,|\Delta^+\mathbf{x}_{k_l} - \mathbf{z}|$$

we obtain that

$$\lim_{l\to\infty} \mathbf{y}_{k_l} = \lim_{l\to\infty} \mathbf{M}^+\mathbf{x}_{k_l} = \mathbf{M}(t)\,\mathbf{z} \qquad \text{in} \quad BV_n$$

and $\mathbf{M}^+ \in K(BV_n)$.

For an analogous reason the results $\mathbf{M} \in K(BV_n)$, $\mathbf{M}^- \in K(BV_n)$ are derivable.

1.8. Lemma. *Let $\{\sigma_l\}_{l=1}^\infty$ be an arbitrary sequence of real numbers in $[0, 1]$. Suppose that $\mathbf{M}_l\colon [0, 1] \to L(R_n)$, $l = 1, 2, \dots$ is a sequence of $n \times n$-matrix valued functions satisfying*

$$\sum_{l=1}^\infty (|\mathbf{M}_l(0)| + \operatorname{var}_0^1 \mathbf{M}_l) < \infty\,. \tag{1,22}$$

Define for $\mathbf{x} \in BV_n$ the series

$$\mathbf{R}\mathbf{x} = \sum_{l=1}^\infty \mathbf{M}_l(t)\,\Delta^+\mathbf{x}(\sigma_l)\,, \tag{1,23}$$

$$\mathbf{L}\mathbf{x} = \sum_{l=1}^\infty \mathbf{M}_l(t)\,\Delta^-\mathbf{x}(\sigma_l) \tag{1,24}$$

where $\Delta^+\mathbf{x}(\sigma) = \mathbf{x}(\sigma+) - \mathbf{x}(\sigma)$, for $\sigma \in [0, 1)$ $\Delta^-\mathbf{x}(\sigma) = \mathbf{x}(\sigma) - \mathbf{x}(\sigma-)$, for $\sigma \in (0, 1]$ $\Delta^+\mathbf{x}(1) = \mathbf{0}$, $\Delta^-\mathbf{x}(0) = \mathbf{0}$.
Both expressions (1,23) and (1,24) define compact operators on BV_n, i.e. $\mathbf{R}, \mathbf{L} \in K(BV_n)$.

Proof. We prove this lemma only for $\mathbf{R}$; the proof for $\mathbf{L}$ is similar. First let us prove that $\mathbf{R} \in B(BV_n)$. The linearity of the operator $\mathbf{R}$ is evident. Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ be an arbitrary subdivision of $[0, 1]$. We have

$$\sum_{j=1}^{k} \left| \sum_{l=1}^{\infty} (\mathbf{M}_l(\alpha_j) - \mathbf{M}_l(\alpha_{j-1})) \, \Delta^+ \mathbf{x}(\sigma_l) \right| \le \sum_{j=1}^{k} \sum_{l=1}^{\infty} |\mathbf{M}_l(\alpha_j) - \mathbf{M}_l(\alpha_{j-1})| \operatorname{var}_0^1 \mathbf{x}$$

$$= \sum_{l=1}^{\infty} \left(\sum_{j=1}^{k} |\mathbf{M}_l(\alpha_j) - \mathbf{M}_l(\alpha_{j-1})| \right) \operatorname{var}_0^1 \mathbf{x} \le \sum_{l=1}^{\infty} \operatorname{var}_0^1 \mathbf{M}_l \operatorname{var}_0^1 \mathbf{x} .$$

Hence

$$\operatorname{var}_0^1 \mathbf{R}\mathbf{x} \le \left(\sum_{l=1}^{\infty} \operatorname{var}_0^1 \mathbf{M}_l \right) \operatorname{var}_0^1 \mathbf{x} \le \left(\sum_{l=1}^{\infty} \operatorname{var}_0^1 \mathbf{M}_l \right) \|\mathbf{x}\|_{BV_n} .$$

Further

$$\left| \sum_{l=1}^{\infty} \mathbf{M}_l(0) \, \Delta^+ \mathbf{x}(\sigma_l) \right| \le \sum_{l=1}^{\infty} |\mathbf{M}_l(0)| \operatorname{var}_0^1 \mathbf{x} \le \left(\sum_{l=1}^{\infty} |\mathbf{M}_l(0)| \right) \|\mathbf{x}\|_{BV_n}$$

and consequently

$$\|\mathbf{R}\mathbf{x}\|_{BV_n} \le \left[\sum_{l=1}^{\infty} (|\mathbf{M}_l(0)| + \operatorname{var}_0^1 \mathbf{M}_l) \right] \|\mathbf{x}\|_{BV_n}, \quad \text{i.e.} \quad \mathbf{R} \in B(BV_n) .$$

Let us now define for every $N = 1, 2, \ldots$ the operator

$$R_N \mathbf{x} = \sum_{l=1}^{N} \mathbf{M}_l(t) \, \Delta^+ \mathbf{x}(\sigma_l) , \qquad \mathbf{x} \in BV_n .$$

1.7 implies that $\mathbf{R}_N$ is compact for every $N = 1, 2, \ldots$ because $\mathbf{R}_N$ is a finite sum of compact operators. Further for every $\mathbf{x} \in BV_n$ we have

$$\mathbf{R}\mathbf{x} - \mathbf{R}_N \mathbf{x} = \sum_{l=N+1}^{\infty} \mathbf{M}_l(t) \, \Delta^+ \mathbf{x}(\sigma_l)$$

and as above also

$$\|\mathbf{R}\mathbf{x} - \mathbf{R}_N \mathbf{x}\|_{BV_n} \le \left[\sum_{l=N+1}^{\infty} (|\mathbf{M}_l(0)| + \operatorname{var}_0^1 \mathbf{M}_l) \right] \|\mathbf{x}\|_{BV_n} .$$

Hence by the assumption (1,22) we obtain that $\lim_{N \to \infty} \mathbf{R}_N = \mathbf{R}$ in $B(BV_n)$ and therefore by I.3.17 we get $\mathbf{R} \in K(BV_n)$.

1.9. Theorem. *If* $\mathbf{K}: I \to L(R_n)$ *satisfies* (1,1), (1,2) *and* (1,16), *then the operator* $\tilde{\mathbf{K}} \in B(BV_n)$ *defined by* (1,19) *is compact, i.e.* $\tilde{\mathbf{K}} \in K(BV_n)$.

Proof. In 1.6 we have proved that $\tilde{\mathbf{K}} \in B(BV_n)$. The assumptions guarantee by I.6.5 that $\operatorname{var}_0^1 \mathbf{K}(., t) < \infty$ for every $t \in [0, 1]$. Hence by the integration-by-parts formula I.4.33 we get

$$(1,25) \qquad \int_0^1 \mathbf{K}(s, t) \, \mathrm{d}\varphi(s) = - \int_0^1 \mathrm{d}_s[\mathbf{K}(s, t)] \, \varphi(s) + \mathbf{K}(1, t) \, \varphi(1) - \mathbf{K}(0, t) \, \varphi(0)$$
$$- \sum_{0 \le \sigma < 1} \Delta_s^+ \mathbf{K}(\sigma, t) \, \Delta^+ \varphi(\sigma) + \sum_{0 < \sigma \le 1} \Delta_s^- \mathbf{K}(\sigma, t) \, \Delta^- \varphi(\sigma)$$

for any $t \in [0,1]$, where $\Delta_s^+ \mathbf{K}(\sigma, t) = \mathbf{K}(\sigma+, t) - \mathbf{K}(\sigma, t)$, $\Delta_s^- \mathbf{K}(\sigma, t) = \mathbf{K}(\sigma, t) - \mathbf{K}(\sigma-, t)$, $\Delta^+ \varphi(\sigma) = \varphi(\sigma+) - \varphi(\sigma)$, $\Delta^- \varphi(\sigma) = \varphi(\sigma) - \varphi(\sigma-)$.

By 1.5 the integral $\int_0^1 \mathrm{d}_s[\mathbf{K}(s,t)]\,\varphi(s)$ defines a compact operator on BV_n. Further by (1,1) and (1,2) we have $\operatorname{var}_0^1 \mathbf{K}(s, .) < \infty$ for any $s \in [0,1]$ (cf. I.6.6). Hence by 1.7 the expressions $\mathbf{K}(1,t)\,\varphi(1)$, $\mathbf{K}(0,t)\,\varphi(0)$ determine compact operators on BV_n. If we prove that the last two terms on the right-hand side in (1,25) define compact operators on BV_n; then $\tilde{\mathbf{K}} \in B(BV_n)$ is expressed by (1,25) in the form of the finite sum of compact operators and is therefore also compact.

Let us consider the term

$$\sum_{0 \le \sigma < 1} \Delta_s^+ \mathbf{K}(\sigma, t)\, \Delta^+ \varphi(\sigma) = \mathbf{R}\varphi \tag{1,26}$$

from the expression (1,25). Since (1,1) and (1,16) are assumed, the set of discontinuity points of $\mathbf{K}(s,t)$ in the first variable lies on an at most denumerable system of lines parallel to the t-axis (see I.6.8) i.e. there is a sequence σ_l, $l = 1, 2, \ldots$, $\sigma_l \in [0,1]$ such that $\Delta_s^+ \mathbf{K}(\sigma, t) = \mathbf{0}$ whenever $\sigma \neq \sigma_l$, $l = 1, 2, \ldots, \sigma \in [0,1)$, and $t \in [0,1]$ is arbitrary. Hence the sum $\mathbf{R}\varphi$ from (1,26) can be written in the form

$$\mathbf{R}\varphi = \sum_{l=1}^{\infty} \Delta_s^+ \mathbf{K}(\sigma_l, t)\, \Delta^+ \varphi(\sigma_l)\,.$$

By I.6.15 we have

$$\operatorname{var}_0^1 \Delta_s^+ \mathbf{K}(\sigma_l, .) \le \omega_1(\sigma_l +) - \omega_1(\sigma_l)\,,$$

where $\omega_1 \colon [0,1] \to R$ is defined by (I.6,5) for $\mathbf{K} \colon I \to L(R_n)$. Hence (see I.6.7)

$$\sum_{l=1}^{\infty} \operatorname{var}_0^1 \Delta_s^+ \mathbf{K}(\sigma_l, .) \le \sum_{l=1}^{\infty} (\omega_1(\sigma_l +) - \omega_1(\sigma_l)) \le \operatorname{var}_0^1 \omega_1 = \mathrm{v}_I(\mathbf{K})\,.$$

Further evidently

$$\sum_{l=1}^{\infty} |\Delta_s^+ \mathbf{K}(\sigma_l, 0)| \le \operatorname{var}_0^1 \mathbf{K}(., 0) < \infty$$

by (1,16). Hence

$$\sum_{l=1}^{\infty} (|\Delta_s^+ \mathbf{K}(\sigma_l, 0)| + \operatorname{var}_0^1 \Delta_s^+ \mathbf{K}(\sigma_l, .)) < \infty\,.$$

All assumptions of 1.8 being satisfied we obtain that $\mathbf{R}\varphi$ is a compact operator acting on BV_n. In a similar way we can show that the expression $\sum_{0 < \sigma \le 1} \Delta_s^- \mathbf{K}(\sigma, t)\, \Delta^- \varphi(\sigma)$ from (1,25) also defines a compact operator on BV_n and this yields our theorem.

From 1.9 the following can easily be deduced.

1.10. Theorem. *If* $\mathbf{K} \colon I \to L(R_n)$ *satisfies* (1,1), (1,2) *and* (1,16) *and moreover*

$$\begin{aligned} &\mathbf{K}(s, t+) = \mathbf{K}(s,t) && \text{for any } s \in [0,1]\,, \quad t \in (0,1)\,, \\ &\mathbf{K}(s, 0) = \mathbf{0} && \text{for any } s \in [0,1] \end{aligned} \tag{1,27}$$

then the expression

$$K'\varphi = \int_0^1 K^*(s,t)\,d\varphi(s), \qquad \varphi \in NBV_n \tag{1,28}$$

defines a compact linear operator acting on NBV_n, *i.e.* $K' \in K(NBV_n)$. (*By* $K^*(s,t)$ *the transposition of the matrix* $K(s,t)$ *is denoted.*)

Proof. Using the properties of the norm of a matrix (see I.1.1) we easily obtain that for $K^*: I \to L(R_n)$ we have $\mathrm{var}_0^1 K^*(0,.) < \infty$, $\mathrm{var}_0^1 K^*(.,0) < \infty$, $v_I(K^*) < \infty$, $K^*(s,t+) = K^*(s,t)$ for any $s \in [0,1]$, $t \in (0,1)$ and $K^*(s,0) = 0$ for any $s \in [0,1]$ whenever the assumptions of the theorem are satisfied. By 1.9 the operator $\tilde{K}\psi = \int_0^1 K^*(s,t)\,d\psi(s)$, $\psi \in BV_n$ belongs to $K(BV_n)$. The operator K' given by (1,28) is evidently a restriction of $\tilde{K}$ to the closed subspace $NBV_n \subset BV_n$ (cf. I.5.2). For an arbitrary $\psi \in BV_n$ we have by (1,27)

$$\int_0^1 K^*(s,0)\,d\psi(s) = 0 \qquad \text{and for any} \quad t \in (0,1)$$

$$\lim_{\delta \to 0+} \int_0^1 K^*(s,t+\delta)\,d\psi(s) = \int_0^1 K^*(s,t)\,d\psi(s)$$

since by I.4.27 we have

$$\left|\int_0^1 (K^*(s,t+\delta) - K^*(s,t))\,d\psi(s)\right| \le \sup_{s\in[0,1]} |K^*(s,t+\delta) - K^*(s,t)|\, \|\psi\|_{BV_n}$$

and by I.6.16

$$\lim_{\delta \to 0+} \sup_{s\in[0,1]} |K^*(s,t+\delta) - K^*(s,t)| = 0\,.$$

Hence the above mentioned operator $\tilde{K} \in K(BV_n)$ maps BV_n into NBV_n when (1,27) is satisfied and its restriction K' to the closed subspace $NBV_n \subset BV_n$ consequently belongs to $K(NBV_n)$.

Let us now consider the pair of Banach spaces BV_n, NBV_n which form a dual pair (BV_n, NBV_n) with respect to the bilinear form

$$\langle x, \varphi \rangle = \int_0^1 x^*(t)\,d\varphi(t), \qquad x \in BV_n, \quad \varphi \in NBV_n \tag{1,29}$$

(see I.5.9). By the results from 1.3 we have

$$Kx = \int_0^1 d_t[K(s,t)]\,x(t) = \int_0^1 d_t[\hat{K}(s,t)]\,x(t), \qquad s \in [0,1]$$

for every $x \in BV_n$, where $\hat{K}(s,t)$ is defined by (1,9) and $\hat{K}(s,t)$ evidently satisfies (1,1), (1,2), (1,16) and (1,27) (i.e. the assumptions of 1.10). Hence

$$\langle Kx, \varphi \rangle = \left\langle \int_0^1 d_t[\hat{K}(.,t)]\,x(t), \varphi \right\rangle$$

for every $\boldsymbol{x} \in BV_n$, $\varphi \in NBV_n$. Using I.6.22 we obtain

$$\langle \boldsymbol{K}\boldsymbol{x}, \varphi \rangle = \left\langle \boldsymbol{x}, \int_0^1 \hat{\boldsymbol{K}}^*(s, .)\, \mathrm{d}\varphi(s) \right\rangle \qquad \text{for every} \quad \boldsymbol{x} \in BV_n, \quad \varphi \in NBV_n,$$

i.e.

$$\langle \boldsymbol{K}\boldsymbol{x}, \varphi \rangle = \langle \boldsymbol{x}, \boldsymbol{K}'\varphi \rangle$$

where

(1,30) $$\boldsymbol{K}'\varphi = \int_0^1 \hat{\boldsymbol{K}}^*(s, t)\, \mathrm{d}\varphi(s), \qquad t \in [0, 1], \quad \varphi \in NBV_n$$

and $\boldsymbol{K}'$ is a compact operator acting on the space NBV_n. Resuming these results we have

1.11. Theorem. *If $\boldsymbol{K}: I \to L(R_n)$ satisfies (1,1), (1,2), then for the operator $\boldsymbol{K} \in K(BV_n)$ given by (1,3) we have*

$$\langle \boldsymbol{K}\boldsymbol{x}, \varphi \rangle = \langle \boldsymbol{x}, \boldsymbol{K}'\varphi \rangle$$

for every $\boldsymbol{x} \in BV_n$, $\varphi \in NBV_n$ where $\boldsymbol{K}' \in K(NBV_n)$ is given by (1,30) and the bilinear form $\langle \boldsymbol{x}, \varphi \rangle$ on $BV_n \times NBV_n$ is given by (1,29).

2. Fredholm-Stieltjes integral equations

In this section we consider the Fredholm-Stieltjes integral equation

$$\boldsymbol{x}(t) - \int_0^1 \mathrm{d}_s[\boldsymbol{K}(t, s)]\, \boldsymbol{x}(s) = \boldsymbol{f}(t)$$

in the Banach space $BV_n[0, 1] = BV_n$.

The fundamental results concerning equations of this kind are contained in the following

2.1. Theorem. *If $\boldsymbol{K}: I \to L(R_n)$* $(I = [0, 1] \times [0, 1] \subset R_2)$ satisfies

(2,1) $$\mathrm{v}_I(\boldsymbol{K}) < \infty,$$

(2,2) $$\mathrm{var}_0^1\, \boldsymbol{K}(0, .) < \infty,$$

then either

I. *the Fredholm-Stieltjes integral equation*

(2,3) $$\boldsymbol{x}(t) - \int_0^1 \mathrm{d}_s[\boldsymbol{K}(t, s)]\, \boldsymbol{x}(s) = \boldsymbol{f}(t), \qquad t \in [0, 1]$$

admits a unique solution in BV_n for any $\boldsymbol{f} \in BV_n$ or

II. *the homogeneous Fredholm-Stieltjes integral equation*

$$x(t) - \int_0^1 d_s[K(t, s)]\, x(s) = 0 \tag{2,4}$$

admits r linearly independent solutions $x_1, \dots, x_r \in BV_n$ *where r is a positive integer.*

If moreover it is assumed that

$$\operatorname{var}_0^1 K(., 0) < \infty , \tag{2,5}$$

$$K(t, s+) = K(t, s) \quad \textit{for any} \quad t \in [0, 1], \quad s \in (0, 1) \tag{2,6}$$

and

$$K(t, 0) = 0 \quad \textit{for any} \quad t \in [0, 1], \tag{2,7}$$

then in the case I. *the equation*

$$\varphi(s) - \int_0^1 K^*(t, s)\, d\varphi(t) = \psi(s) \tag{2,8}$$

admits a unique solution in NBV_n *for any* $\psi \in NBV_n$ *and in the case* II. *the corresponding homogeneous equation*

$$\varphi(s) - \int_0^1 K^*(t, s)\, d\varphi(t) = 0 \tag{2,9}$$

admits also r linearly independent solutions $\varphi_1, \varphi_2, \dots, \varphi_r \in NBV_n$.

Proof. Let us denote by

$$Ax = (I - K)\, x = x(t) - \int_0^1 d_s[K(t, s)]\, x(s), \qquad x \in BV_n$$

the linear operator corresponding to the Fredholm-Stieltjes integral equation (2,3). By I we denote the identity operator on BV_n and K is the operator defined by (1,5). Since 1.5 implies $K \in K(BV_n)$, we have by I.3.20 $\operatorname{ind} A = \operatorname{ind}(I - K) = 0$ and this implies the first part of our theorem immediately.

Under the assumptions of the second part we have by 1.11 $\langle Kx, \varphi \rangle = \langle x, K'\varphi \rangle$ for every $x \in BV_n$, $\varphi \in NBV_n$ where $K'\varphi = \int_0^1 K^*(t, s)\, d\varphi(t)$ is a compact operator acting on NBV_n (see 1,10). Hence $\operatorname{ind}(I - K') = 0$ and by I.3.20 we have $\alpha(I - K) = \alpha(I - K') = \beta(I - K) = \beta(I - K')$. This completes the proof.

2.2. Theorem. *If* $K: I \to L(R_n)$ *satisfies* (2,1), (2,2), (2,5), (2,6) *and* (2,7), *then the equation* (2,3) *has a solution in* BV_n *if and only if*

$$\int_0^1 f^*(t)\, d\varphi(t) = 0 \tag{2,10}$$

for any solution $\varphi \in NBV_n$ *of the homogeneous equation* (2,9) *and symmetrically the equation* (2,8) *has a solution in* NBV_n *if and only if*

$$\int_0^1 \mathbf{x}^*(t)\, \mathrm{d}\psi(t) = 0 \tag{2,11}$$

for any solution $\mathbf{x} \in BV_n$ *of the homogeneous equation* (2,4).

Proof. In the proof of 2.1 it was shown that all assumptions of Theorem I.3.2 are satisfied. Hence this statement is only a reformulation of the results from I.3.2.

2.3. Remark. 2.1 and 2.3 represent Fredholm theorems for the Stieltjes integral equations (2,3) and (2,8). It is of interest that the corresponding integral operators occuring in these equations are not connected with one another by the usual concept of adjointness. In this concrete situation the difficulties with the analytic description of the dual BV_n^* obstruct the analytic description of the adjoint $\mathbf{K}^*$. Fortunately the concept of the conjugate operator $\mathbf{K}'$ with respect to suitably described total subspace NBV_n works in our case and the results are given in an acceptable form.

2.4. Remark. Let us mention that in accordance with 1.4 in the same way the conjugate equation (2,8) in NBV_n can be replaced by the same equation working in NBV_n^- when instead of (2,6) we assume that $\mathbf{K}(t, s-) = \mathbf{K}(t, s)$ for any $t \in [0, 1]$, $s \in (0, 1)$.

2.5. Theorem. *Let* $\mathbf{K}$: $I = [0, 1] \times [0, 1] \rightarrow L(R_n)$ *satisfy* (2,1), (2,2) *and* (2,5). *If the homogeneous Fredholm-Stieltjes integral equation*

$$\mathbf{x}(t) - \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \mathbf{0}, \qquad t \in [0, 1] \tag{2,4}$$

has only the trivial solution $\mathbf{x} = \mathbf{0}$ *in* BV_n, *then there exists a unique* $n \times n$*-matrix valued function* $\mathbf{\Gamma}(t, s)$: $I \rightarrow L(R_n)$ *such that*

$$\mathrm{v}_I(\mathbf{\Gamma}) < \infty, \tag{2,12}$$

$$\mathrm{var}_0^1\, \mathbf{\Gamma}(., 0) < \infty, \tag{2,13}$$

$$\mathrm{var}_0^1\, \mathbf{\Gamma}(0, .) < \infty, \tag{2,14}$$

and for all $(t, s) \in I$ *the equation*

$$\mathbf{\Gamma}(t, s) = \mathbf{K}(t, s) + \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)]\, \mathbf{\Gamma}(r, s) \tag{2,15}$$

is satisfied.

Moreover for any $\boldsymbol{f} \in BV_n$ *the unique solution* $\boldsymbol{x} \in BV_n$ *of the Fredholm-Stieltjes integral equation* (2,3) *is given by the formula*

$$\boldsymbol{x}(t) = \boldsymbol{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t,s)]\, \boldsymbol{f}(s), \qquad t \in [0,1]. \tag{2,16}$$

Proof. Let us set $\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{K} \in B(BV_n)$ where $\boldsymbol{Kx} = \int_0^1 \mathrm{d}_s[\boldsymbol{K}(t,s)]\,\boldsymbol{x}(s)$, $\boldsymbol{x} \in BV_n$ and $\boldsymbol{I}$ is the identity operator on BV_n. By assumption we have $N(\boldsymbol{A}) = \{\boldsymbol{0}\}$. Since $\boldsymbol{K} \in B(BV_n)$ is compact by 1.5, we have $0 = \alpha(\boldsymbol{A}) = \beta(\boldsymbol{A}) = \dim (BV_n/R(\boldsymbol{A}))$ by I.3.20. Since $R(\boldsymbol{A})$ is closed, we obtain $R(\boldsymbol{A}) = BV_n$. Hence the Bounded Inverse Theorem I.3.4 implies that the inverse operator $\boldsymbol{A}^{-1} \in B(BV_n)$ exists and for any $\boldsymbol{f} \in BV_n$ the unique solution of (2,3) is given by $\boldsymbol{A}^{-1}\boldsymbol{f}$ and for this solution the estimate

$$\|\boldsymbol{x}\|_{BV_n} \leq C \|\boldsymbol{f}\|_{BV_n} \tag{2,17}$$

holds where $C = \|\boldsymbol{A}^{-1}\|_{B(BV_n)}$ is a constant.

Let us consider the matrix equation (2,15). Evidently the l-th column $\boldsymbol{\Gamma}_l(t,s)$ of $\boldsymbol{\Gamma}(t,s)$: $I \to L(R_n)$, $l = 1, 2, \ldots, n$ satisfies the equation

$$\boldsymbol{\Gamma}_l(t,s) = \boldsymbol{K}_l(t,s) + \int_0^1 \mathrm{d}_r[\boldsymbol{K}(t,r)]\, \boldsymbol{\Gamma}_l(r,s), \tag{2,18}$$

i.e. $\boldsymbol{\Gamma}_l(t,s)$ satisfies in the first variable the equation (2,3) with $\boldsymbol{f}(t) = \boldsymbol{K}_l(t,s)$ for any $s \in [0,1]$. We have $\boldsymbol{f} \in BV_n$ since by I.6.6 $\operatorname{var}_0^1 \boldsymbol{K}(.,s) \leq v_I(\boldsymbol{K}) + \operatorname{var}_0^1 \boldsymbol{K}(.,0)$ and (2,1), (2,5) are assumed. By 2.1 the equation (2,18) has exactly one solution for any fixed $s \in [0,1]$ and consequently the same holds also for the matrix equation (2,15).

Let us now consider the properties of the matrix $\boldsymbol{\Gamma}(t,s)$ defined by (2,15). By (2,17) the inequality

$$\|\boldsymbol{\Gamma}_l(.,s)\|_{BV_n} \leq C \|\boldsymbol{K}_l(.,s)\|_{BV_n}$$

holds for every $s \in [0,1]$, $l = 1, 2, \ldots, n$. Hence (from the definition of the norm in BV_n) we obtain for any $s \in [0,1]$ the inequality

$$|\boldsymbol{\Gamma}(0,s)| + \operatorname{var}_0^1 \boldsymbol{\Gamma}(.,s) \leq C(|\boldsymbol{K}(0,s)| + \operatorname{var}_0^1 \boldsymbol{K}(.,s))$$

which yields (2,13).

Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ be an arbitrary decomposition of $[0,1]$. If $\boldsymbol{\Gamma}(t,s)$ satisfies (2,15), then for any $j = 1, \ldots, k$ and $t \in [0,1]$ we have

$$\boldsymbol{\Gamma}(t,\alpha_j) - \boldsymbol{\Gamma}(t,\alpha_{j-1}) = \boldsymbol{K}(t,\alpha_j) - \boldsymbol{K}(t,\alpha_{j-1}) + \int_0^1 \mathrm{d}_r[\boldsymbol{K}(t,r)]\,(\boldsymbol{\Gamma}(r,\alpha_j) - \boldsymbol{\Gamma}(r,\alpha_{j-1})),$$

i.e. the difference $\boldsymbol{\Gamma}(t,\alpha_j) - \boldsymbol{\Gamma}(t,\alpha_{j-1})$ satisfies a matrix equation of the type (2,15) and consequently by the Bounded Inverse Theorem I.3.4 we have as above

$$\begin{aligned} &|\boldsymbol{\Gamma}(0,\alpha_j) - \boldsymbol{\Gamma}(0,\alpha_{j-1})| + \operatorname{var}_0^1 (\boldsymbol{\Gamma}(.,\alpha_j) - \boldsymbol{\Gamma}(.,\alpha_{j-1})) \\ &\leq C(|\boldsymbol{K}(0,\alpha_j) - \boldsymbol{K}(0,\alpha_{j-1})| + \operatorname{var}_0^1 (\boldsymbol{K}(.,\alpha_j) - \boldsymbol{K}(.,\alpha_{j-1}))\,. \end{aligned} \tag{2,19}$$

Hence

$$\sum_{j=1}^{k} |\Gamma(0, \alpha_j) - \Gamma(0, \alpha_{j-1})| \le C(\mathrm{var}_0^1 \mathbf{K}(0, .) + \sum_{j=1}^{k} \mathrm{var}_0^1 (\mathbf{K}(., \alpha_j) - \mathbf{K}(., \alpha_{j-1}))$$
$$\le C(\mathrm{var}_0^1 \mathbf{K}(0, .) + \mathrm{v}_I(\mathbf{K})),$$

and since the subdivision $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ was arbitrary we get (2,14) by passing to the supremum over all finite subdivisions of $[0, 1]$.

Let now $J_{ij} = [\alpha_{i-1}, \alpha_i] \times [\alpha_{j-1}, \alpha_j]$, $i, j = 1, 2, \ldots, k$ be the net-type subdivision of I corresponding to the arbitrary subdivision $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ of $[0, 1]$. For Γ: $I \to L(R_n)$ satisfying (2,15) we obtain by (2,19) the following inequality

$$\sum_{i,j=1}^{k} |m_\Gamma(J_{ij})| = \sum_{i,j=1}^{k} |\Gamma(\alpha_i, \alpha_j) - \Gamma(\alpha_i, \alpha_{j-1}) - \Gamma(\alpha_{i-1}, \alpha_j) + \Gamma(\alpha_{i-1}, \alpha_{j-1})|$$
$$\le \sum_{i,j=1}^{k} \mathrm{var}_{\alpha_{i-1}}^{\alpha_i} (\Gamma(., \alpha_j) - \Gamma(., \alpha_{j-1})) = \sum_{j=1}^{k} \mathrm{var}_0^1 (\Gamma(., \alpha_j) - \Gamma(., \alpha_{j-1}))$$
$$\le C \sum_{j=1}^{k} (|\mathbf{K}(0, \alpha_j) - \mathbf{K}(0, \alpha_{j-1})| + \mathrm{var}_0^1 (\mathbf{K}(., \alpha_j) - \mathbf{K}(., \alpha_{j-1})))$$
$$\le C(\mathrm{var}_0^1 \mathbf{K}(0, .) + \mathrm{v}_I(\mathbf{K})).$$

This inequality yields evidently (2,12) and the first part of the theorem is proved.

Now we prove that by (2,16) really the unique solution of (2,3) is given. Since Γ: $I \to L(R_n)$ satisfies (2,12) and (2,14), by I.6.18 the integral $\int_0^1 \mathrm{d}_s[\Gamma(t, s)] \mathbf{f}(s)$ exists for any $\mathbf{f} \in BV_n$ and $t \in [0, 1]$. Putting (2,16) into the left-hand side of (2,3) we obtain the expression

$$\mathbf{f}(t) + \int_0^1 \mathrm{d}_s[\Gamma(t, s)]\, \mathbf{f}(s) - \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)] \left(\mathbf{f}(r) + \int_0^1 \mathrm{d}_s[\Gamma(r, s)]\, \mathbf{f}(s)\right) = \mathbf{l}(t).$$

Hence

$$\mathbf{l}(t) = \mathbf{f}(t) + \int_0^1 \mathrm{d}_s[\Gamma(t, s) - \mathbf{K}(t, s)]\, \mathbf{f}(s) - \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)] \int_0^1 \mathrm{d}_s[\Gamma(r, s)]\, \mathbf{f}(s).$$

Using I.6.20 we obtain

$$\int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)] \int_0^1 \mathrm{d}_s[\Gamma(r, s)]\, \mathbf{f}(s) = \int_0^1 \mathrm{d}_s \left(\int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)]\, \Gamma(r, s)\right) \mathbf{f}(s), \quad \text{i.e.}$$

$$\mathbf{l}(t) = \mathbf{f}(t) + \int_0^1 \mathrm{d}_s \left[\Gamma(t, s) - \mathbf{K}(t, s) + \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)]\, \Gamma(r, s)\right] \mathbf{f}(s) = \mathbf{f}(t)$$

since Γ: $I \to L(R_n)$ satisfies (2,15) and consequently (2,16) gives the solution of (2,3). This concludes the proof of our theorem.

2.6. Remark. The matrix valued function $\Gamma(t, s)$: $I \to L(R_n)$ given in 2.6 is the resolvent of the Fredholm-Stieltjes integral equation (2,3). This resolvent gives by (2,16) the unique solution of (2,3) for every $\mathbf{f} \in BV_n$. For the existence of the resolvent $\mathbf{\Gamma}(t, s)$ the assumption $\alpha(\mathbf{A}) = \dim(\mathbf{I} - \mathbf{K}) = 0$ is essential.

Further let us investigate the equation (2,3) when $r = \alpha(\mathbf{A}) = \dim(\mathbf{I} - \mathbf{K}) \neq 0$.

By assertion II. from 2.1 the homogeneous equation (2,4) admits in this case r linearly independent solutions $\mathbf{x}_1, \ldots, \mathbf{x}_r \in BV_n$ and $R(\mathbf{I} - \mathbf{K}) \neq BV_n$, i.e. (2,3) has no solutions for all $\mathbf{f} \in BV_n$

The following theorem holds in this situation. Let $\mathbf{K}$: $I = [0, 1] \times [0, 1] \to L(R_n)$ satisfy (2,1), (2,2), (2,5) and $\mathbf{K}(t, s+) = \mathbf{K}(t, s)$ for any $t \in [0, 1]$, $s \in (0, 1)$, $\mathbf{K}(t, 0) = \mathbf{0}$ for any $t \in [0, 1]$. Then there exists an $n \times n$-matrix valued function $\hat{\mathbf{\Gamma}}(t, s)$: $I \to L(R_n)$ such that $v_I(\hat{\mathbf{\Gamma}}) < \infty$, $\mathrm{var}_0^1 \hat{\mathbf{\Gamma}}(., 0) < \infty$, $\mathrm{var}_0^1 \hat{\mathbf{\Gamma}}(0, .) < \infty$ and if the Fredholm-Stieltjes integral equation (2,3) has solutions for $\mathbf{f} \in BV_n$ (i.e. if $\mathbf{f} \in R(\mathbf{I} - \mathbf{K})$, see also 2.3), then one of them is given by the formula

(2,20) $$\mathbf{x}(t) = \mathbf{f}(t) + \int_0^1 \mathrm{d}_s[\hat{\mathbf{\Gamma}}(t, s)]\, \mathbf{f}(s)\,, \qquad t \in [0, 1]\,.$$

The general form of solutions of (2,3) *is*

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_0^1 \mathrm{d}_s[\hat{\mathbf{\Gamma}}(t, s)]\, \mathbf{f}(s) + \sum_{i=1}^{r} \alpha_i\, \mathbf{x}^i(t)\,,$$

where $\mathbf{x}^i$, $i = 1, \ldots, r$ *are linearly independent solutions of the homogeneous equation* (2,4) (cf. 2.1) *and* $\alpha_1, \ldots, \alpha_r$ *are arbitrary constants.*

The proof of this assertion is based on some pseudoresolvent technique using projections in BV_n. The theorem is completely proved in Schwabik [6].

3. Volterra-Stieltjes integral equations

In this section we consider integral equations of the form

(3,1) $$\mathbf{x}(t) - \int_0^t \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \mathbf{f}(t)\,, \qquad t \in [0, 1]$$

in the Banach space $BV_n[0, 1] = BV_n$ with $\mathbf{f} \in BV_n$. Equations of the form (3,1) are called *Volterra-Stieltjes integral equations.*

Throughout this paragraph it will be assumed that $\mathbf{K}$: $I = [0, 1] \times [0, 1] \to L(R_n)$ satisfies

(3,2) $$v_I(\mathbf{K}) < \infty$$

and

(3,3) $$\mathrm{var}_0^1\, \mathbf{K}(0, .) < \infty\,.$$

Let us mention that (3,3) can be replaced by $\mathrm{var}_0^1\, \mathbf{K}(t_0, .) < \infty$, where $t_0 \in [0, 1]$ is arbitrary. This follows from I.6.6.

Since (3,2) and (3,3) are assumed, for every fixed $t \in [0,1]$ we have $\operatorname{var}_0^1 \boldsymbol{K}(t,.) < \infty$ by I.6.6. Hence for any $\boldsymbol{x} \in BV_n$ and $t \in [0,1]$ the integral $\int_0^t \mathrm{d}_s[\boldsymbol{K}(t,s)]\,\boldsymbol{x}(s)$ exists by I.4.19.

Let us show that the equation (3,1) is a special case of the Fredholm-Stieltjes integral equation considered in the previous Section II.2.

To any given kernel $\boldsymbol{K}: I \to L(R_n)$ satisfying (3,2) and (3,3) we define a new "triangular" kernel $\boldsymbol{K}^\Delta: I \to L(R_n)$ as follows:

$$(3,4) \qquad \begin{aligned} \boldsymbol{K}^\Delta(t,s) &= \boldsymbol{K}(t,s) - \boldsymbol{K}(t,0) && \text{if} \quad 0 \le s \le t \le 1\,, \\ \boldsymbol{K}^\Delta(t,s) &= \boldsymbol{K}(t,t) - \boldsymbol{K}(t,0) = \boldsymbol{K}^\Delta(t,t) && \text{if} \quad 0 \le t < s \le 1\,. \end{aligned}$$

It is obvious that $\boldsymbol{K}^\Delta(t,0) = \boldsymbol{0}$ for any $t \in [0,1]$ and $\boldsymbol{K}^\Delta(0,s) = \boldsymbol{K}^\Delta(0,0) = \boldsymbol{0}$ for any $s \in [0,1]$. Let

$$J_{ij} = [\alpha_{i-1}, \alpha_i] \times [\alpha_{j-1}, \alpha_j]\,, \qquad i,j = 1, \dots, k$$

be an arbitrary net-type subdivision of the interval I corresponding to the subdivision $0 = \alpha_0 < \alpha_1 < \dots < \alpha_k = 1$ of $[0,1]$. By definition (3,4) of $\boldsymbol{K}^\Delta$ we have

$$\begin{aligned} m_{\boldsymbol{K}^\Delta}(J_{ij}) &= m_{\boldsymbol{K}}(J_{ij}) && \text{if} \quad 0 \le j < i \le k\,, \\ m_{\boldsymbol{K}^\Delta}(J_{ij}) &= \boldsymbol{0} && \text{if} \quad 0 \le i < j \le k \end{aligned}$$

and

$$m_{\boldsymbol{K}^\Delta}(J_{ii}) = \boldsymbol{K}(\alpha_i, \alpha_i) - \boldsymbol{K}(\alpha_i, \alpha_{i-1}) \qquad \text{for} \quad i = 1, 2, \dots, k\,.$$

(For $m_{\boldsymbol{K}}(J)$ see I.6.2.) Hence

$$\begin{aligned} \sum_{i,j=1}^{k} |m_{\boldsymbol{K}^\Delta}(J_{ij})| &= \sum_{i=1}^{k} \sum_{j=1}^{i-1} |m_{\boldsymbol{K}}(J_{ij})| + \sum_{i=1}^{k} |\boldsymbol{K}(\alpha_i, \alpha_i) - \boldsymbol{K}(\alpha_i, \alpha_{i-1})| \\ \le \sum_{i=1}^{k} \sum_{j=1}^{i-1} |m_{\boldsymbol{K}}(J_{ij})| &+ \sum_{i=1}^{k} |\boldsymbol{K}(\alpha_i, \alpha_i) - \boldsymbol{K}(\alpha_i, \alpha_{i-1}) - \boldsymbol{K}(0, \alpha_i) + \boldsymbol{K}(0, \alpha_{i-1})| \\ &+ \sum_{i=1}^{k} |\boldsymbol{K}(0, \alpha_i) - \boldsymbol{K}(0, \alpha_{i-1})| \le v_I(\boldsymbol{K}) + \operatorname{var}_0^1 \boldsymbol{K}(0,.)\,. \end{aligned}$$

Consequently we obtain by definition (cf. I.6.1, I.6.3)

$$(3,5) \qquad v_I(\boldsymbol{K}^\Delta) \le v_I(\boldsymbol{K}) + \operatorname{var}_0^1 \boldsymbol{K}(0,.) < \infty$$

Since $\boldsymbol{K}^\Delta(t,s)$ is by definition constant on the interval $[t,1]$ for every $t \in [0,1]$, we have

$$(3,6) \qquad \int_t^1 \mathrm{d}_s[\boldsymbol{K}^\Delta(t,s)]\,\boldsymbol{x}(s) = \boldsymbol{0}$$

for every $\boldsymbol{x} \in BV_n$. Further

$$\int_0^t \mathrm{d}_s[\boldsymbol{K}^\Delta(t,s) - \boldsymbol{K}(t,s)]\,\boldsymbol{x}(s) = -\int_0^t \mathrm{d}_s[\boldsymbol{K}(t,0)]\,\boldsymbol{x}(s) = \boldsymbol{0}\,,$$

i.e.

$$\int_0^t \mathrm{d}_s[\boldsymbol{K}^\Delta(t,s)]\,\boldsymbol{x}(s) = \int_0^t \mathrm{d}_s[\boldsymbol{K}(t,s)]\,\boldsymbol{x}(s)\,.$$

Using (3,6) we obtain for an arbitrary $T \in [0, 1]$ the equality

$$\int_0^t \mathrm{d}_s[\boldsymbol{K}(t,s)]\,\boldsymbol{x}(s) = \int_0^T \mathrm{d}_s[\boldsymbol{K}^\Delta(t,s)]\,\boldsymbol{x}(s) \tag{3,7}$$

for any $\boldsymbol{x} \in BV_n$ and $t \in [0, T]$.

Let us summarize these results.

3.1. Proposition. *Let* $\boldsymbol{K}\colon I \to L(R_n)$ *satisfy* (3,2), (3,3). *Then for the triangular kernel* $\boldsymbol{K}^\Delta\colon I \to L(R_n)$ *defined by the relations* (3, 4) *the following is valid.*

(a) $\mathrm{var}_0^1\, \boldsymbol{K}^\Delta(.,0) = 0, \quad \mathrm{var}_0^1\, \boldsymbol{K}^\Delta(0,.) = 0, \quad \mathrm{v}_I(\boldsymbol{K}^\Delta) < \infty,$

(b) *for every* $\boldsymbol{x} \in BV_n$, $T \in [0, 1]$ *and* $t \in [0, T]$ *the equality* (3,7) *holds, i.e. by the relation*

$$\boldsymbol{Kx} = \int_0^t \mathrm{d}_s[\boldsymbol{K}(t,s)]\,\boldsymbol{x}(s) = \int_0^1 \mathrm{d}_s[\boldsymbol{K}^\Delta(t,s)]\,\boldsymbol{x}(s)\,, \qquad \boldsymbol{x} \in BV_n \tag{3,8}$$

an operator on BV_n *is defined and by* 1.5 *we have* $\boldsymbol{K} \in K(BV_n)$.

3.2. Remark. Proposition 3.1 states that the Volterra-Stieltjes integral equation (3,1) is equivalent to the Fredholm-Stieltjes integral equation

$$\boldsymbol{x}(t) - \int_0^1 \mathrm{d}_s[\boldsymbol{K}^\Delta(t,s)]\,\boldsymbol{x}(s) = \boldsymbol{f}(t)\,, \qquad t \in [0, 1]\,. \tag{3,9}$$

Hence by Theorem 2.1 either the equation (3,1) admits a unique solution in BV_n for every $\boldsymbol{f} \in BV_n$ or the corresponding homogeneous equation

$$\boldsymbol{x}(t) - \int_0^t \mathrm{d}_s[\boldsymbol{K}(t,s)]\,\boldsymbol{x}(s) = \boldsymbol{0}\,, \qquad t \in [0, 1] \tag{3,10}$$

has a finite number of linearly independent solutions in BV_n. Our aim is to give conditions under which the equation (3,1) is really of Volterra type, i.e. when the equation (3,10) admits only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$ in BV_n.

3.3. Theorem. *Let the kernel* $\boldsymbol{K}\colon I \to L(R_n)$ *satisfy* (3,2), (3,3) *and*

$$\lim_{\sigma \to s-} |\boldsymbol{K}(t,\sigma) - \boldsymbol{K}(t,s)| = 0 \tag{3,11}$$

for any $t \in [0, 1]$, $s \in (0, 1]$. *Then the homogeneous Volterra-Stieltjes integral equation* (3,10) *has only the trivial solution* $\boldsymbol{x} = \boldsymbol{0}$ *in* BV_n.

Proof. Let $\boldsymbol{K}^{\Delta}: I \to L(R_n)$ be the triangular kernel corresponding to $\boldsymbol{K}$ by the relations (3,4). Since (3,11) holds we have also

$$\text{(3,12)} \qquad \lim_{\sigma \to s-} |\boldsymbol{K}^{\Delta}(t, \sigma) - \boldsymbol{K}^{\Delta}(t, s)| = 0$$

for any $t \in [0, 1]$, $s \in (0, 1]$. Let us set

$$\omega_2^{\Delta}(0) = 0, \qquad \omega_2^{\Delta}(s) = \mathrm{v}_{[0,1]\times[0,s]}(\boldsymbol{K}^{\Delta}) \qquad \text{for} \quad s \in (0, 1].$$

The function $\omega_2^{\Delta}: [0, 1] \to R$ is evidently nonnegative and nondecreasing (see I.6.7). Since (3,12) holds we have $\omega_2^{\Delta}(s-) = \omega_2^{\Delta}(s)$ for every $s \in (0, 1]$ by I.6.11, i.e. ω_2^{Δ} is left continuous on $[0, 1]$.

Assume that $\boldsymbol{x} \in BV_n$ is a solution of (3,10). Then evidently $\boldsymbol{x}(0) = \boldsymbol{0}$ and

$$|\boldsymbol{x}(s)| \le |\boldsymbol{x}(0)| + \mathrm{var}_0^s\, \boldsymbol{x} = \mathrm{var}_0^s\, \boldsymbol{x}$$

for every $s \in [0, 1]$. Using (b) from 3.1 we get

$$\mathrm{var}_0^{\xi}\, \boldsymbol{x} = \mathrm{var}_0^{\xi}\left(\int_0^t \mathrm{d}_s[\boldsymbol{K}(t, s)]\, \boldsymbol{x}(s)\right) = \mathrm{var}_0^{\xi}\left(\int_0^{\xi} \mathrm{d}_s[\boldsymbol{K}^{\Delta}(., s)]\, \boldsymbol{x}(s)\right)$$

for every $\xi \in [0, 1]$. If (I.6,14) from I.6.18 is used then we obtain

$$\mathrm{var}_0^{\xi}\, \boldsymbol{x} = \mathrm{var}_0^{\xi}\left(\int_0^{\xi} \mathrm{d}_s[\boldsymbol{K}^{\Delta}(., s)]\, \boldsymbol{x}(s)\right) \le \int_0^{s} |\boldsymbol{x}(s)|\, \mathrm{d}\omega_2^{\Delta}(s) \le \int_0^{\xi} \mathrm{var}_0^s\, \boldsymbol{x}\, \mathrm{d}\omega_2^{\Delta}(s)$$

and I.4.30 yields the inequality $\mathrm{var}_0^{\xi}\, \boldsymbol{x} \le 0$ for every $\xi \in [0, 1]$. Hence $\boldsymbol{x}(s) \equiv \boldsymbol{0}$ on $[0, 1]$, i.e. $\boldsymbol{x} = \boldsymbol{0} \in BV_n$.

3.4. Example. Let us define $h(t) = 0$ if $0 \le t < \frac{1}{2}$, $h(t) = 1/t$ if $\frac{1}{2} \le t \le 1$, $g(s) = 0$ if $0 \le s < \frac{1}{2}$, $g(s) = s$ if $\frac{1}{2} \le s \le 1$. Evidently $h, g \in BV$. If we set $k(t, s) = h(t)\, g(s)$ for $(t, s) \in I = [0, 1] \times [0, 1]$, then clearly $\mathrm{v}_I(k) < \infty$ (cf. I.6.4), $\mathrm{var}_0^1\, k(0, .) < \infty$ and $\mathrm{var}_0^1\, k(., 0) < \infty$. Let us consider the homogeneous Volterra-Stieltjes integral equation

$$x(t) = \int_0^t \mathrm{d}_s[k(t, s)]\, x(s) = h(t) \int_0^t x(s)\, \mathrm{d}g(s), \qquad t \in [0, 1].$$

Let us set $y(s) = 0$ for $0 \le s < \frac{1}{2}$, $y(s) = 1$ for $\frac{1}{2} \le s \le 1$. By easy computations using I.4,21 we obtain

$$\int_0^t y(s)\, \mathrm{d}g(s) = 0 \qquad \text{if} \quad 0 < t < \tfrac{1}{2},$$

$$\int_0^t y(s)\, \mathrm{d}g(s) = \tfrac{1}{2}\, y(\tfrac{1}{2}) + \int_{1/2}^t y(s)\, \mathrm{d}s = t \qquad \text{if} \quad \tfrac{1}{2} \le t \le 1$$

and consequently

$$h(t) \int_0^t y(s)\, \mathrm{d}g(s) = y(t)$$

for every $t \in [0, 1]$. Hence $y \in BV$ is a solution of the homogeneous Volterra-Stieltjes integral equation and $y \neq 0$. The condition (3,11) is in this case affected. In fact, $\lim_{\sigma \to 1/2-} (k(t, \sigma) - k(t, \frac{1}{2})) = \frac{1}{2}h(t)$ and $h(t) \neq 0$ e.g. for $t = \frac{1}{2}$.

3.5. Remark. Example 3.4 shows that for $\boldsymbol{K}: I \to L(R_n)$ satisfying (3,2) and (3,3) the corresponding homogeneous Volterra-Stieltjes integral equation (3,10) need not have in general only trivial solutions, i.e. for the corresponding operator $\boldsymbol{K} \in K(BV_n)$ we can obtain in general a nontrivial null space $N(\boldsymbol{I} - \boldsymbol{K})$. If (3,11) is assumed, then this situation cannot occur. The condition (3,11) is too restrictive as will be shown in the following. We shall give necessary and sufficient conditions on $\boldsymbol{K}: I \to L(R_n)$ satisfying (3,2) and (3,3) such that the equation (3,10) has only the trivial solution in BV_n.

3.6. Proposition. *Let* $\boldsymbol{M}: I = [0, 1] \times [0, 1] \to L(R_n)$ *satisfy* $v_I(\boldsymbol{M}) < \infty$, $\operatorname{var}_0^1 \boldsymbol{M}(0, .) < \infty$. *Then for any* $a \in [0, 1]$ *there exists a nondecreasing bounded function* $\xi: [a, 1] \to [0, +\infty)$ *such that for every* $b \in [a, 1]$ *and* $\boldsymbol{x} \in BV_n$ *we have*

$$(3,13) \qquad \operatorname{var}_a^b \left(\int_a^t \mathrm{d}_s[\boldsymbol{M}(t, s)] \, \boldsymbol{x}(s) \right) \le |\boldsymbol{x}(a)| \, (\xi(a+) - \xi(a)) + \|\boldsymbol{x}\|_{BV_n[a,b]}(\xi(b) - \xi(a+)) .$$

Proof. Let $\boldsymbol{M}^\Delta: I \to L(R_n)$ be the triangular kernel which corresponds to $\boldsymbol{M}: I \to L(R_n)$ (see 3.1). For any $t \in [a, b]$ we have (see (3,7))

$$(3,14) \qquad \int_a^t \mathrm{d}_s[\boldsymbol{M}(t, s)] \, \boldsymbol{x}(s) = \int_a^b \mathrm{d}_s[\boldsymbol{M}^\Delta(t, s)] \, \boldsymbol{x}(s) .$$

Let us define the function

$$\xi(t) = v_{[a,1] \times [a,t]}(\boldsymbol{M}^\Delta) \qquad \text{for} \quad t \in (a, 1] , \quad \xi(a) = 0 .$$

$\xi: [a, 1] \to R$ is evidently nondecreasing and bounded on $[a, 1]$ (cf. I.6.7).
From (I.6,14) in I.6.18 we obtain

$$(3,15) \qquad \operatorname{var}_a^b \left(\int_a^t \mathrm{d}_s[\boldsymbol{M}(t, s)] \, \boldsymbol{x}(s) \right) \le \int_a^b |\boldsymbol{x}(s)| \, \mathrm{d}\xi(s) .$$

Using I.4.13 we have

$$(3,16) \qquad \int_a^b |\boldsymbol{x}(s)| \, \mathrm{d}\xi(s) = |\boldsymbol{x}(a)| \, (\xi(a+) - \xi(a)) + \lim_{\delta \to 0+} \int_{a+\delta}^b |\boldsymbol{x}(s)| \, \mathrm{d}\xi(s)$$

and for any $0 < \delta < b - a$ by I.4.27

$$\int_{a+\delta}^b |\boldsymbol{x}(s)| \, \mathrm{d}\xi(s) \le \sup_{s \in [a+\delta, b]} |\boldsymbol{x}(s)| \, (\xi(b) - \xi(a+\delta)) \le \|\boldsymbol{x}\|_{BV_n[a,b]}(\xi(b) - \xi(a+)) .$$

Hence (3,15) and (3,16) imply (3,13).

3.7. Proposition. *Let* $\boldsymbol{H}\colon [0,1] \to L(R_n)$ *be such that*

(i) $\operatorname{var}_0^1 \boldsymbol{H} < \infty$,

(ii) *there is an at most countable set of points* $t_i \in [0,1]$, $i = 1, 2, \dots$ *such that*

$$\boldsymbol{H}(t) = \boldsymbol{0} \quad \text{for} \quad t \in [0,1], \quad t \neq t_i, \quad i = 1, 2, \dots,$$

(iii) *the matrix* $\boldsymbol{I} - \boldsymbol{H}(t)$ *is regular for all* $t \in [0,1]$. *Let us define the linear operator*

$$\boldsymbol{T}\mathbf{z} = [\boldsymbol{I} - \boldsymbol{H}(t)]^{-1}\, \mathbf{z}(t), \qquad t \in [0,1] \;\; \text{for} \quad \mathbf{z} \in BV_n. \tag{3,17}$$

Then there exists a constant $C \geq 0$ *such that*

$$\|\boldsymbol{T}\mathbf{z}\|_{BV_n} \leq C \|\mathbf{z}\|_{BV_n} \qquad \text{for every} \quad \mathbf{z} \in BV_n, \tag{3,18}$$

i.e. $\boldsymbol{T} \in B(BV_n)$.

Proof. By (iii) the inverse matrix $[\boldsymbol{I} - \boldsymbol{H}(t)]^{-1}$ exists for every $t \in [0,1]$ and the operator $\boldsymbol{T}$ from (3,17) is well-defined.

Since $\boldsymbol{I} = (\boldsymbol{I} - \boldsymbol{H}(t))\,[\boldsymbol{I} - \boldsymbol{H}(t)]^{-1} = [\boldsymbol{I} - \boldsymbol{H}(t)]^{-1} - \boldsymbol{H}(t)\,[\boldsymbol{I} - \boldsymbol{H}(t)]^{-1}$ for any $t \in [0,1]$, we have $[\boldsymbol{I} - \boldsymbol{H}(t)]^{-1} = \boldsymbol{I} + \boldsymbol{H}(t)\,[\boldsymbol{I} - \boldsymbol{H}(t)]^{-1}$ and for any $\mathbf{z} \in BV_n$ we have

$$\boldsymbol{T}\mathbf{z} = \mathbf{z} + \boldsymbol{u} \tag{3,19}$$

where

$$\boldsymbol{u}(t) = \boldsymbol{H}(t)\,[\boldsymbol{I} - \boldsymbol{H}(t)]^{-1}\, \mathbf{z}(t), \qquad t \in [0,1]. \tag{3,20}$$

The assumption (ii) implies $\boldsymbol{u}(t) = \boldsymbol{0}$ for any $t \in [0,1]$, $t \neq t_i$, $i = 1, 2, \dots$. Hence evidently

$$\begin{aligned} \operatorname{var}_0^1 \boldsymbol{u} &= 2 \sum_{i=1}^{\infty} |\boldsymbol{u}(t_i)| = 2 \sum_{i=1}^{\infty} |\boldsymbol{H}(t_i)\,[\boldsymbol{I} - \boldsymbol{H}(t_i)]^{-1}\, \mathbf{z}(t_i)| \\ &\leq 2 \|\mathbf{z}\|_{BV_n} \sum_{i=1}^{\infty} |\boldsymbol{H}(t_i)|\; |[\boldsymbol{I} - \boldsymbol{H}(t_i)]^{-1}|. \end{aligned} \tag{3,21}$$

By (i) and (ii) we have

$$\sum_{i=1}^{\infty} |\boldsymbol{H}(t_i)| \leq |\boldsymbol{H}(0)| + |\boldsymbol{H}(1)| + 2 \sum_{t_i \in (0,1)} |\boldsymbol{H}(t_i)| = \operatorname{var}_0^1 \boldsymbol{H} < \infty\,;$$

hence there exists an integer $i_0 > 0$ such that $|\boldsymbol{H}(t_i)| < \frac{1}{2}$ for any $i > i_0$. This implies

$$|[\boldsymbol{I} - \boldsymbol{H}(t_i)]^{-1}| \leq 1 + |\boldsymbol{H}(t_i)| + \dots + |\boldsymbol{H}(t_i)|^k + \dots = \frac{1}{1 - |\boldsymbol{H}(t_i)|} < 2$$

for $i > i_0$ and immediately also the inequality

$$\begin{aligned} \sum_{i=1}^{\infty} |\boldsymbol{H}(t_i)|\; |[\boldsymbol{I} - \boldsymbol{H}(t_i)]^{-1}| &\leq \sum_{i=1}^{i_0} |\boldsymbol{H}(t_i)|\; |[\boldsymbol{I} - \boldsymbol{H}(t_i)]^{-1}| + 2 \sum_{i=i_0+1}^{\infty} |\boldsymbol{H}(t_i)| \\ &\leq \Big(\max_{i=1,2,\dots,i_0} |[\boldsymbol{I} - \boldsymbol{H}(t_i)]^{-1}| + 2 \Big) \sum_{i=1}^{\infty} |\boldsymbol{H}(t_i)| = C_0 < \infty \end{aligned}$$

which yields by (3,21)

$$\operatorname{var}_0^1 \mathbf{u} \le 2C_0 \|\mathbf{z}\|_{BV_n}.$$

Hence (see (3,19))

$$\|\mathbf{Tz}\|_{BV_n} = |[\mathbf{I} - \mathbf{H}(0)]^{-1} \mathbf{z}(0)| + \operatorname{var}_0^1 \mathbf{Tz} \le (|[\mathbf{I} - \mathbf{H}(0)]^{-1}| + 1 + 2C_0) \|\mathbf{z}\|_{BV_n}$$

and (3,18) is satisfied with $C = 1 + 2C_0 + |[\mathbf{I} - \mathbf{H}(0)]^{-1}|$.

3.8. Proposition. *Let us assume that* $\mathbf{K}$: $I = [0,1] \times [0,1] \to L(R_n)$ *satisfies* (3,2) *and* (3,3). *Define* $\mathbf{M}$: $I \to L(R_n)$ *as follows*:

(3,22)
$$\begin{aligned} \mathbf{M}(t,s) &= \mathbf{K}(t,s) && \text{if } (t,s) \in I, \quad t \neq s, \\ \mathbf{M}(t,t) &= \mathbf{K}(t,t-) && \text{if } t \in (0,1], \\ \mathbf{M}(0,0) &= \mathbf{K}(0,0). \end{aligned}$$

Then

(i) $v_I(\mathbf{M}) < \infty$, $\operatorname{var}_0^1 \mathbf{M}(0,.) < \infty$,

(ii) *if* $\mathbf{x} \in BV_n$, *then for any fixed* $t \in [0,1]$ *we have*

$$\lim_{\tau \to t-} \int_0^\tau \mathrm{d}_s[\mathbf{M}(t,s)]\, \mathbf{x}(s) = \int_0^t \mathrm{d}_s[\mathbf{M}(t,s)]\, \mathbf{x}(s),$$

i.e. the integral $\int_0^t \mathrm{d}_s[\mathbf{M}(t,s)]\, \mathbf{x}(s)$ *does not depend on the value* $\mathbf{x}(t) \in R_n$,

(iii) *for every* $\mathbf{x} \in BV_n$ *and* $t \in [0,1]$ *we have*

(3,23)
$$\int_0^t \mathrm{d}_s[\mathbf{K}(t,s)]\, \mathbf{x}(s) = \int_0^t \mathrm{d}_s[\mathbf{M}(t,s)]\, \mathbf{x}(s) + \mathbf{H}(t)\, \mathbf{x}(t)$$

where

(3,24)
$$\begin{aligned} \mathbf{H}(t) &= \mathbf{K}(t,t) - \mathbf{K}(t,t-) && \text{for } t \in (0,1], \\ \mathbf{H}(0) &= \mathbf{0}, \end{aligned}$$

(iv) *for* $\mathbf{H}$: $[0,1] \to L(R_n)$ *given by* (3,24) *there exists an at most countable set of points* $t_i \in [0,1]$, $i = 1,2,\ldots$ *such that* $\mathbf{H}(t) = \mathbf{0}$ *for* $t \in [0,1]$, $t \neq t_i$, $i = 1,2,\ldots$ *and* $\operatorname{var}_0^1 \mathbf{H} < \infty$.

Proof. In order to prove (i) let us mention that $\mathbf{M}(0,s) = \mathbf{K}(0,s)$ for all $s \in [0,1]$ and consequently $\operatorname{var}_0^1 \mathbf{M}(0,.) = \operatorname{var}_0^1 \mathbf{K}(0,.) < \infty$. Further let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ be an arbitrary subdivision of $[0,1]$ and let

$$J_{ij} = [\alpha_{i-1}, \alpha_i] \times [\alpha_{j-1}, \alpha_j], \qquad i,j = 1,\ldots,k$$

be the corresponding net-type subdivision of I. We consider the sum $\sum_{i,j=1}^{k} |m_{\mathbf{M}}(J_{ij})|$ where

$$m_{\mathbf{M}}(J_{ij}) = \mathbf{M}(\alpha_i, \alpha_j) - \mathbf{M}(\alpha_i, \alpha_{j-1}) - \mathbf{M}(\alpha_{i-1}, \alpha_j) + \mathbf{M}(\alpha_{i-1}, \alpha_{j-1})$$

for $i, j = 1, \dots, k$. Usual considerations using the definition of $\boldsymbol{M}$ in (3,22) give

$$\sum_{i,j=1}^{k} |m_{\boldsymbol{M}}(J_{ij})| \leq v_I(\boldsymbol{K}) + 4 \sum_{j=1}^{k} |\boldsymbol{K}(\alpha_j, \alpha_j) - \boldsymbol{K}(\alpha_j, \alpha_j -)| .$$

Since

$$\sum_{j=1}^{k} |\boldsymbol{K}(\alpha_j, \alpha_j) - \boldsymbol{K}(\alpha_j, \alpha_j -)|$$

$$\leq \sum_{j=1}^{k} |\boldsymbol{K}(\alpha_j, \alpha_j) - \boldsymbol{K}(\alpha_j, \alpha_j -) - \boldsymbol{K}(0, \alpha_j) + \boldsymbol{K}(0, \alpha_j -)| + \sum_{j=1}^{k} |\boldsymbol{K}(0, \alpha_j) - \boldsymbol{K}(0, \alpha_j -)|$$

$$\leq v_I(\boldsymbol{K}) + \operatorname{var}_0^1 \boldsymbol{K}(0, .),$$

we obtain

$$\sum_{i,j=1}^{k} |m_{\boldsymbol{M}}(J_{ij})| \leq 5 v_I(\boldsymbol{K}) + 4 \operatorname{var}_0^1 \boldsymbol{K}(0, .) < \infty$$

and (i) holds since J_{ij} was an arbitrary net-type subdivision.

Let $t \in (0, 1]$ be fixed, $\boldsymbol{x}, \boldsymbol{y} \in BV_n$, $\boldsymbol{x}(s) = \boldsymbol{y}(s)$ for $s \in [0, t)$. By I.4.21 and from the definition of $\boldsymbol{M}$ we obtain

$$\int_0^t \mathrm{d}_s[\boldsymbol{M}(t, s)] (\boldsymbol{x}(s) - \boldsymbol{y}(s)) = (\boldsymbol{M}(t, t) - \boldsymbol{M}(t, t-)) (\boldsymbol{x}(t) - \boldsymbol{y}(t)) = \boldsymbol{0} .$$

Hence

$$\int_0^t \mathrm{d}_s[\boldsymbol{M}(t, s)] \, \boldsymbol{x}(s) = \int_0^t \mathrm{d}_s[\boldsymbol{M}(t, s)] \, \boldsymbol{y}(s)$$

or in other words: for all $\boldsymbol{x} \in BV_n$ we have

$$\lim_{\tau \to t-} \int_0^\tau \mathrm{d}_s[\boldsymbol{M}(t, s)] \, \boldsymbol{x}(s) = \int_0^t \mathrm{d}_s[\boldsymbol{M}(t, s)] \, \boldsymbol{x}(s) .$$

For $t = 0$ the statement is trivial. Hence (ii) is proved.

Further for any $t \in (0, 1]$ and $\boldsymbol{x} \in BV_n$ we have

$$\int_0^t \mathrm{d}_s[\boldsymbol{K}(t, s) - \boldsymbol{M}(t, s)] \, \boldsymbol{x}(s) = [\boldsymbol{K}(t, t) - \boldsymbol{M}(t, t) - \boldsymbol{K}(t, t-) + \boldsymbol{M}(t, t-)] \, \boldsymbol{x}(t)$$

$$= [\boldsymbol{K}(t, t) - \boldsymbol{K}(t, t-)] \, \boldsymbol{x}(t) = \boldsymbol{H}(t) \, \boldsymbol{x}(t),$$

and (3,23) holds. For $t = 0$ the equality (3,24) is evident. Hence (iii) is valid.

By I.6.8 the set of discontinuity points of $\boldsymbol{K}(t, s)$ in the second variable s lies on an at most countable system of lines in I which are parallel to the t axis, i.e. there is an at most countable system t_i, $i = 1, 2, \dots$ of points in $[0, 1]$ such that $\boldsymbol{H}(t) = \boldsymbol{K}(t, t) - \boldsymbol{K}(t, t-) = \boldsymbol{0}$ for all $t \in [0, 1]$, $t \neq t_i$, $i = 1, 2, \dots$.

For any $t \in [0, 1]$ we have evidently

$$|\boldsymbol{H}(t)| = |\boldsymbol{K}(t, t) - \boldsymbol{K}(t, t-)|$$

$$\leq |\boldsymbol{K}(t, t) - \boldsymbol{K}(t, t-) - \boldsymbol{K}(0, t) + \boldsymbol{K}(0, t-)| + |\boldsymbol{K}(0, t) - \boldsymbol{K}(0, t-)| .$$

Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b$ be an arbitrary finite subdivision of $[0, 1]$. Then

$$\sum_{i=1}^{k} |\mathbf{H}(\alpha_i) - \mathbf{H}(\alpha_{i-1})| \le 2\sum_{i=1}^{k} |\mathbf{H}(\alpha_i)|$$

$$\le 2\sum[|\mathbf{K}(\alpha_i, \alpha_i) - \mathbf{K}(\alpha_i, \alpha_i -) - \mathbf{K}(0, \alpha_i) + \mathbf{K}(0, \alpha_i -)| + |\mathbf{K}(0, \alpha_i) - \mathbf{K}(0, \alpha_{i-1})|]$$

$$\le 2(\mathrm{v}_I(\mathbf{K}) + \mathrm{var}_0^1 \mathbf{K}(0, .)) < \infty$$

and (iv) is also proved.

3.9. Theorem. *Let the kernel* $\mathbf{K}: I \to L(R_n)$ $(I = [0, 1] \times [0, 1])$ *satisfy* (3,2) *and* (3,3). *Then the homogeneous Volterra-Stieltjes integral equation* (3,10) *has only the trivial solution* $\mathbf{x} = \mathbf{0}$ *in* BV_n *if and only if the matrix* $\mathbf{I} - (\mathbf{K}(t, t) - \mathbf{K}(t, t-))$ *is regular for any* $t \in (0, 1]$ *).

Proof. By (iii) from 3.8 the equation (3,10) can be written in the equivalent form

$$\mathbf{x}(t) = \int_0^t \mathrm{d}_s[\mathbf{M}(t, s)]\, \mathbf{x}(s) + \mathbf{H}(t)\, \mathbf{x}(t), \qquad t \in [0, 1] \tag{3,25}$$

where $\mathbf{M}: I \to L(R_n)$, $\mathbf{H}: [0, 1] \to L(R_n)$ are defined by (3,22), (3,24) respectively. Hence if we assume that for any $t \in [0, 1]$ the matrix $\mathbf{I} - \mathbf{H}(t) = \mathbf{I} - (\mathbf{K}(t, t) - \mathbf{K}(t, t-))$ is regular, then the inverse $[\mathbf{I} - \mathbf{H}(t)]^{-1}$ exists for any $t \in [0, 1]$ and (3,25) can be rewritten in the equivalent form

$$\mathbf{x}(t) = [\mathbf{I} - \mathbf{H}(t)]^{-1} \int_0^t \mathrm{d}_s[\mathbf{M}(t, s)]\, \mathbf{x}(s), \qquad t \in [0, 1]. \tag{3,26}$$

This equality can be formally written in the form $\mathbf{x} = \mathbf{TMx}$ where

$$\mathbf{Tz} = [\mathbf{I} - \mathbf{H}(t)]^{-1}\, \mathbf{z}(t) \qquad \text{for} \quad \mathbf{z} \in BV_n$$

and

$$\mathbf{Mx} = \int_0^t \mathrm{d}_s[\mathbf{M}(t, s)]\, \mathbf{x}(s) \qquad \text{for} \quad \mathbf{x} \in BV_n.$$

Assume that $\mathbf{x} \in BV_n$ is a solution of (3,10). Then evidently $\mathbf{x}(0) = \mathbf{0}$ and by 3.7, 3.6 we have for any δ $(0 < \delta \le 1)$

$$\|\mathbf{x}\|_{BV_n[0,\delta]} = |\mathbf{x}(0)| + \mathrm{var}_0^\delta\, \mathbf{x} = \|\mathbf{TMx}\|_{BV_n[0,\delta]} \le C\|\mathbf{Mx}\|_{BV_n[0,\delta]} \tag{3,27}$$

$$= C\,\mathrm{var}_0^\delta \left(\int_0^t \mathrm{d}_s[\mathbf{M}(t, s)]\, \mathbf{x}(s)\right) \le C(|\mathbf{x}(0)|\, (\xi(0+) - \xi(0)) + \|\mathbf{x}\|_{BV_n[0,\delta]}(\xi(\delta) - \xi(0+))$$

$$= C(\xi(\delta) - \xi(0+))\, \|\mathbf{x}\|_{BV_n[0,\delta]}$$

*) In this case we have $\mathbf{K}(0, 0) = \mathbf{K}(0, 0-)$ if we use the agreement $\mathbf{K}(0, s) = \mathbf{K}(0, 0)$ for $s < 0$, i.e. in fact $\mathbf{I} - (\mathbf{K}(0, 0) - \mathbf{K}(0, 0-)) = \mathbf{I}$ is also regular. Nevertheless this is not used in the proof of the theorem and the result does not depend on the behaviour of $\mathbf{K}(0, 0) - \mathbf{K}(0, 0-)$.

where ξ: $[0,1] \to [0,+\infty)$ is bounded and nondecreasing by 3.6 and $C \geq 0$ is a constant (cf. 3.7). The function ξ is of bounded variation and has consequently onesided limits at all points of $[0,1]$. Hence we can find a $\delta > 0$ such that $C(\xi(\delta) - \xi(0+)) < \frac{1}{2}$ and by (3,27) we obtain

$$\|\boldsymbol{x}\|_{BV_n[0,\delta]} < \tfrac{1}{2}\|\boldsymbol{x}\|_{BV_n[0,\delta]},$$

i.e. $\boldsymbol{x}(t) = \boldsymbol{0}$ for all $t \in [0,\delta]$.

Let us now assume that $t^* \in [0,1]$ is the supremum of all such positive δ that the solution $\boldsymbol{x} \in BV_n$ of the equation (3,10) equals zero on $[0,\delta]$. Evidently $\boldsymbol{x}(t) = \boldsymbol{0}$ for all $t \in [0,t^*)$. Since by (ii) from 3.8 we have

$$\int_0^{t^*} \mathrm{d}_s[\boldsymbol{M}(t^*,s)]\,\boldsymbol{x}(s) = \lim_{\tau \to t^*-} \int_0^{\tau} \mathrm{d}_s[\boldsymbol{M}(t^*,s)]\,\boldsymbol{x}(s) = \boldsymbol{0}$$

and $[\boldsymbol{I} - \boldsymbol{H}(t^*)]^{-1}$ exists, we have by (3,26) $\boldsymbol{x}(t^*) = \boldsymbol{0}$, i.e. $\boldsymbol{x}(t) = \boldsymbol{0}$ for $t \in [0,t^*]$. Now assuming $t^* < 1$ we have

$$\boldsymbol{x}(t) = [\boldsymbol{I} - \boldsymbol{H}(t)]^{-1} \int_0^t \mathrm{d}_s[\boldsymbol{M}(t,s)]\,\boldsymbol{x}(s) = [\boldsymbol{I} - \boldsymbol{H}(t)]^{-1} \int_{t^*}^t \mathrm{d}_s[\boldsymbol{M}(t,s)]\,\boldsymbol{x}(s)$$

for all $t \in [t^*,1]$. Using the same procedure as above we can determine a $\delta > 0$ such that the inequality

$$\|\boldsymbol{x}\|_{BV_n[t^*,t^*+\delta]} \leq \tfrac{1}{2}\|\boldsymbol{x}\|_{BV_n[t^*,t^*+\delta]}$$

holds and consequently $\boldsymbol{x}(t) = \boldsymbol{0}$ for $t \in [t^*, t^*+\delta]$. Hence we obtain a contradiction to the property of t^*. In this way we have $t^* = 1$, i.e. $\boldsymbol{x}(t) = \boldsymbol{0}$ for all $t \in [0,1]$ and the "if" part of the theorem is proved.

For the proof of the "only if" part of the theorem we refer to the Fredholm alternative included in 2.1. (cf. also 3.2) which states that either (3,10) has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$ in BV_n or there exists $\boldsymbol{f} \in BV_n$ such that the equation (3,1) has no solutions in BV_n.

Let us now assume that the matrix $\boldsymbol{I} - (\boldsymbol{K}(t,t) - \boldsymbol{K}(t,t-)) = \boldsymbol{I} - \boldsymbol{H}(t)$ is not regular for all $t \in (0,1]$. This may occur only for a finite set of points $0 < t_1 < \ldots < t_k$ in $(0,1]$ because $\mathrm{var}_0^1 \boldsymbol{H} < \infty$ by (iv) from 3.8 and consequently $|\boldsymbol{H}(t)| < \frac{1}{2}$ for all $t \in [0,1]$ except for a finite set of points in $(0,1]$. Hence $[\boldsymbol{I} - \boldsymbol{H}(t)]^{-1}$ exists for all $t \in [0,1]$, $t \neq t_i$, $i = 1,2,\ldots,k$, and $\boldsymbol{I} - \boldsymbol{H}(t_i)$, $i = 1,2,\ldots,k$ is not regular. Evidently there exists $\boldsymbol{y} \in R_n$ such that $\boldsymbol{y} \notin R(\boldsymbol{I} - \boldsymbol{H}(t_1))$, i.e. the linear algebraic equation

$$(\boldsymbol{I} - \boldsymbol{H}(t_1))\,\boldsymbol{x} = \boldsymbol{y}$$

has no solutions in R_n. Let us define

$$\boldsymbol{f}(t) = \boldsymbol{0} \quad \text{for} \quad t \in [0,1], \quad t \neq t_1, \quad \boldsymbol{f}(t_1) = \boldsymbol{y}$$

and consider the nonhomogeneous equation (3,1) with this right-hand side. Let us assume that $\boldsymbol{x} \in BV_n$ is a solution of this equation. In the same way as in the proof of the "if" part we can show that $\mathbf{x}(t) = \mathbf{0}$ for all $t \in [0, t_1)$ since $[\mathbf{I} - \mathbf{H}(t)]^{-1}$ exists for all $t \in [0, t_1)$. Using the expression (3,23) and (ii) from 3.8 for $\int_0^{t_1} d_s[\mathbf{K}(t_1, s)]\, \mathbf{x}(s)$ we obtain

$$[\mathbf{I} - \mathbf{H}(t_1)]\, \mathbf{x}(t_1) = \int_0^{t_1} d_s[\mathbf{M}(t_1, s)]\, \mathbf{x}(s) + \mathbf{f}(t_1) = \mathbf{y}$$

and $\mathbf{x}(t_1)$ cannot be determined since $\mathbf{y} \notin R(\mathbf{I} - \mathbf{H}(t_1))$ and consequently there is no $\mathbf{x} \in BV_n$ satisfying (3,1) with the given $\mathbf{f} \in BV_n$. By the above quoted Fredholm alternative the equation (3,10) possesses nontrivial solutions and our theorem is completely proved.

3.10. Theorem. *Assume that* $\mathbf{K}$: $I = [0, 1] \times [0, 1] \to L(R_n)$ *satisfies* (3,2), (3,3) *and the matrix* $\mathbf{I} - \mathbf{K}(t, t) - \mathbf{K}(t, t-))$ *is regular for any* $t \in (0, 1]$.

Then there exists a uniquely determined $\mathbf{\Gamma}$: $I \to L(R_n)$ *such that the unique solution in* BV_n *to the Volterra-Stieltjes integral equation* (3,1) *with* $\mathbf{f} \in BV_n$ *is given by the relation*

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_0^t d_s[\mathbf{\Gamma}(t, s)]\, \mathbf{f}(s), \qquad t \in [0, 1]. \tag{3,28}$$

The matrix $\mathbf{\Gamma}(t, s)$ *satisfies the integral equation*

$$\mathbf{\Gamma}(t, s) = \mathbf{K}(t, s) - \mathbf{K}(t, 0) + \int_0^t d_r[\mathbf{K}(t, r)]\, \mathbf{\Gamma}(r, s) \qquad \text{for} \quad 0 \le s \le t \le 1. \tag{3,29}$$

We have $\mathbf{\Gamma}(t, s) = \mathbf{\Gamma}(t, t)$ *for* $0 \le t < s \le 1$, $\mathbf{\Gamma}(t, 0) = \mathbf{0}$, $\mathrm{var}_0^1\, \mathbf{\Gamma}(0, .) < \infty$ *and* $\mathrm{v}_I(\mathbf{\Gamma}) < \infty$.

Proof. By 3.1 the equation (3,1) can be written in the equivalent Fredholm-Stieltjes form

$$\mathbf{x}(t) - \int_0^1 d_s[\mathbf{K}^\Delta(t, s)]\, \mathbf{x}(s) = \mathbf{f}(t), \qquad t \in [0, 1] \tag{3,30}$$

where $\mathbf{K}^\Delta$: $I \to L(R_n)$ is the corresponding triangular kernel given by (3,4). By 3.9 the homogeneous equation

$$\mathbf{x}(t) - \int_0^1 d_s[\mathbf{K}^\Delta(t, s)]\, \mathbf{x}(s) = \mathbf{0}, \qquad t \in [0, 1]$$

has only the trivial solution $\mathbf{x} = \mathbf{0}$ in BV_n. Since $\mathbf{K}^\Delta$ satisfies evidently all assumptions of 2.6, we obtain by this theorem the existence of $\mathbf{\Gamma}$: $I \to L(R_n)$ such that the solution of (3,30) and consequently also of the equivalent equation (3,1) is given by

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_0^1 d_t[\mathbf{\Gamma}(t, s)]\, \mathbf{f}(s), \qquad t \in [0, 1] \tag{3,31}$$

where $\boldsymbol{\Gamma}(t,s)$ satisfies the matrix integral equation

$$\boldsymbol{\Gamma}(t,s) = \boldsymbol{K}^{\Delta}(t,s) + \int_0^1 \mathrm{d}_r[\boldsymbol{K}^{\Delta}(t,r)]\,\boldsymbol{\Gamma}(r,s) \qquad \text{for all} \quad (t,s) \in I\,.$$

Using the definition (3,4) of $\boldsymbol{K}^{\Delta}(t,s)$ and (3,8) we have

$$\boldsymbol{\Gamma}(t,s) = \boldsymbol{K}(t,s) - \boldsymbol{K}(t,0) + \int_0^t \mathrm{d}_r[\boldsymbol{K}(t,r)]\,\boldsymbol{\Gamma}(r,s) \qquad \text{for} \quad 0 \le s \le t \le 1$$

and (3,29) is satisfied. For $0 \le t < s \le 1$ we have similarly

$$\boldsymbol{\Gamma}(t,s) = \boldsymbol{K}^{\Delta}(t,t) + \int_0^t \mathrm{d}_r[\boldsymbol{K}(t,r)]\,\boldsymbol{\Gamma}(r,s)$$

and

$$\boldsymbol{\Gamma}(t,t) = \boldsymbol{K}^{\Delta}(t,t) + \int_0^t \mathrm{d}_r[\boldsymbol{K}(t,r)]\,\boldsymbol{\Gamma}(r,t)\,.$$

Hence

$$\boldsymbol{\Gamma}(t,s) - \boldsymbol{\Gamma}(t,t) = \int_0^t \mathrm{d}_r[\boldsymbol{K}(t,r)]\,(\boldsymbol{\Gamma}(r,s) - \boldsymbol{\Gamma}(r,t))\,,$$

i.e. $\boldsymbol{\Gamma}(t,s) = \boldsymbol{\Gamma}(t,t)$ since Theorem 3.9 yields that the homogeneous equation $\boldsymbol{x}(t) - \int_0^t \mathrm{d}_s[\boldsymbol{K}(t,s)]\,\boldsymbol{x}(s) = \boldsymbol{0}$ has only the trivial solution $\boldsymbol{x} = \boldsymbol{0} \in BV_n$. Similarly we obtain $\boldsymbol{\Gamma}(t,0) = \boldsymbol{0}$ for all $t \in [0,1]$. The inequalities $\mathrm{var}_0^1\,\boldsymbol{\Gamma}(0,.) < \infty$, $\mathrm{v}_I(\boldsymbol{\Gamma}) < \infty$ are immediate consequences of 2.5.

From the equality $\boldsymbol{\Gamma}(t,s) = \boldsymbol{\Gamma}(t,t)$ valid for $t < s$ we get

$$\int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t,s)]\,\boldsymbol{x}(s) = \int_0^t \mathrm{d}_s[\boldsymbol{\Gamma}(t,s)]\,\boldsymbol{x}(s)$$

for all $\boldsymbol{x} \in BV_n$ and hence by (3,31) we obtain (3,28).

3.11. Theorem. *Let* $\boldsymbol{K}\colon I = [0,1] \times [0,1] \to L(R_n)$ *satisfy* (3,2), (3,3) *and let* $t_0 \in [0,1]$ *be fixed. Then the integral equation*

$$\text{(3,32)} \qquad \boldsymbol{x}(t) = \int_{t_0}^t \mathrm{d}_s[\boldsymbol{K}(t,s)]\,\boldsymbol{x}(s)\,, \qquad t \in [0,1]$$

possesses only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$ *in* BV_n *if and only if for any* $t \in (t_0, 1]$ *the matrix* $\boldsymbol{I} - (\boldsymbol{K}(t,t) - \boldsymbol{K}(t,t-))$ *is regular and for any* $t \in [0, t_0)$ *the matrix* $\boldsymbol{I} + \boldsymbol{K}(t,t+) - \boldsymbol{K}(t,t)$ *is regular.*

The proof of this statement can be given by a modification of the proof of 3.9. Since serious technical troubles do not occur we add only a few remarks on this proof. It is evident that $\boldsymbol{x}(t_0) = \boldsymbol{0}$ for any solution of (3,32). The proof of the fact that $\boldsymbol{x}(t) = \boldsymbol{0}$ for $t \in (t_0, 1]$ if and only if $\boldsymbol{I} - (\boldsymbol{K}(t,t) - \boldsymbol{K}(t,t-))$ is a regular matrix for $t \in (t_0, 1]$, follows exactly the line of the proof of 3.9. For proving "$\mathbf{x}(t) = \mathbf{0}$ for $t \in [0, t_0)$ if and only if $\mathbf{I} + \mathbf{K}(t, t+) - \mathbf{K}(t, t)$ is regular for all $t \in [0, t_0)$", the decomposition $(t \in [0, t_0))$

$$\int_{t_0}^{t} \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \int_{t_0}^{t} \mathrm{d}_s[\mathbf{M}(t, s)]\, \mathbf{x}(s) - (\mathbf{K}(t, t+) - \mathbf{K}(t, t))\, \mathbf{x}(t)$$

valid for any $\mathbf{x} \in BV_n$ can be used where the integral $\int_{t_0}^{t} \mathrm{d}_s[\mathbf{M}(t, s)]\, \mathbf{x}(s)$ does not depend on the value $\mathbf{x}(t)$. This can be done in the same way as in 3.8 when it is assumed that $\mathbf{M}(t, s) = \mathbf{K}(t, s)$ if $(t, s) \in I$, $t \neq s$, $\mathbf{M}(t, t) = \mathbf{K}(t, t+)$ if $t \in [0, 1)$, $\mathbf{M}(1, 1) = \mathbf{K}(1, 1)$. Using the above decomposition of $\int_{t_0}^{t} \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s)$ the approach from 3.9 can be used in order to prove the result.

3.12. Corollary. *Let $\mathbf{K}: I \to L(R_n)$ satisfy (3,2), (3,3) and let $t_0 \in [0, 1]$ be fixed. Then the integral equation*

$$\mathbf{x}(t) = \int_{t_0}^{t} \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) + \mathbf{f}(t), \qquad t \in [0, 1] \tag{3,33}$$

has a unique solution for every $\mathbf{f} \in BV_n$ if and only if for any $t \in (t_0, 1]$ the matrix $\mathbf{I} - (\mathbf{K}(t, t) - \mathbf{K}(t, t-))$ is regular and for any $t \in [0, t_0)$ the matrix $\mathbf{I} + \mathbf{K}(t, t+) - \mathbf{K}(t, t)$ is regular.

Proof. Let us define a new kernel $\mathbf{K}^{t_0}: I \to L(R_n)$ as follows.

If $t_0 \le t \le 1$, then

$$\begin{aligned} \mathbf{K}^{t_0}(t, s) &= \mathbf{K}(t, s) && \text{for } t_0 \le s \le t, \\ \mathbf{K}^{t_0}(t, s) &= \mathbf{K}(t, t) && \text{for } t < s \le 1, \\ \mathbf{K}^{t_0}(t, s) &= \mathbf{K}(t, t_0) && \text{for } 0 \le s < t_0 \end{aligned}$$

and if $0 \le t < t_0$, then

$$\begin{aligned} \mathbf{K}^{t_0}(t, s) &= -\mathbf{K}(t, s) && \text{for } t \le s \le t_0, \\ \mathbf{K}^{t_0}(t, s) &= -\mathbf{K}(t, t) && \text{for } 0 \le s < t, \\ \mathbf{K}^{t_0}(t, s) &= -\mathbf{K}(t, t_0) && \text{for } t_0 \le s \le 1. \end{aligned}$$

It is a matter of routine to show that $\mathrm{v}_I(\mathbf{K}^{t_0}) < \infty$, $\mathrm{var}_0^1 \mathbf{K}^{t_0}(0, .) < \infty$ and

$$\int_{t_0}^{t} \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \int_0^1 \mathrm{d}_s[\mathbf{K}^{t_0}(t, s)]\, \mathbf{x}(s),$$

for every $t \in [0, 1]$ and $\mathbf{x} \in BV_n$. Hence the equation (3,33) can be rewritten in the equivalent Fredholm-Stieltjes form

$$\mathbf{x}(t) - \int_0^1 \mathrm{d}_s[\mathbf{K}^{t_0}(t, s)]\, \mathbf{x}(s) = \mathbf{f}(t), \qquad t \in [0, 1].$$

By 3,11 the corresponding homogeneous equation

$$\mathbf{x}(t) - \int_0^1 \mathrm{d}_s[\mathbf{K}^{t_0}(t, s)]\,\mathbf{x}(s) = \mathbf{0}, \qquad t \in [0, 1]$$

has only the trivial solution if and only if the regularity conditions given in the corollary are satisfied. The corollary follows now immediately from 2.1.

Notes

The Fredholm-Stieltjes integral equation theory is based on the investigations due to Schwabik [2], [5].

The case of Volterra-Stieltjes integral equations was considered by many authors in terms of product integrals, the left and right Cauchy integral or other types of integrals. See e.g. Bitzer [1], Helton [1], Herod [1], Hönig [1], Mac Nerney [2].