APPENDIX I.

On the Impossible in Geometry.

SEE Art. 6. i., 10. viii., 17. vi., 47. vi. The following is a revision of a note originally appended to the private reprint of the Abstract of my second Memoir on Plane Stigmatics.

AXIOM. *It is impossible for two points to be, at the same time, separate and coincident.*

Although most of the following propositions are employed without hesitation by eminent mathematicians, they apparently involve this impossibility.

1. *An infinitesimal distance is no distance;* [i.e., infinitesimal separation is coincidence; i. e., as finite length is an aggregate of infinitesimal lengths, each of which is, on this theory, a coincidence or single point, every finite length is a single point, i. e. a point, having no dimensions, may be regarded as a circle, having two dimensions, or a sphere, having three dimensions; assumptions constantly made].

2. *When the ratio of the distances of two points from a third, which is not midway between them, is one of equality, the third point is at an infinite distance from the other two, and conversely;* [i. e. the separation of the two first points is equivalent to a coincidence, or else lengths differing by a finite length may have a ratio of equality]. *Chasles*, Géométrie Supérieure, 15, *et passim. Townsend*, Modern Geometry, art. 15.

3. *Parallel straight lines meet at one single point at infinity;* [i. e., two points, connected by an invariable straight line, and moving each upon a straight line, and hence never approaching, meet.] *Poncelet*, Proj. Persp. p. 52, art. 103. *Chasles*, G. S. *passim*. *Townsend*, M. G. art. 16. [The intersection of two concentric circles is liable to a similar objection.]

4. *A straight line has a single point at infinity;* [i. e., if two points move in opposite directions upon a straight line, they will meet at one point at infinity; i. e., a continually increasing separation promotes coincidence, or else a straight line is an enclosed curve, e. g. a circle with an infinite radius; hence diametrically opposite directions are the same]. *Steiner*, Geometrische Gestalten, p. 2, note. *Chasles*, G. S. 20. *Townsend*, M. G. art. 17. [Similar objections apply to the single point at infinity of a parabola, and the two (and no more) points at infinity of the hyperbola, *Chasles*, Sections Coniques, 13, where he says that the ellipse "n'a *aucun* point à l'infini," whereas in G. S. 719 he speaks of the "double contact imaginaire *à l'infini*" of two concentric circles, which are particular cases of concentric ellipses, so that circles which, according to the first citation, have *no* point at infinity, have also *two* such points, according to the second.]

5. "*The product of nothing by infinity may be finite;*" *Salmon*, Conics, 4th edit. art. 67; [i. e., perpetual doing of nothing may produce something; or, an infinite aggregation of points, each having no length, may produce some length; or, nonentity infinitely repeated may produce an entity]. *Townsend*, M. G. art. 13.

6. *Infinity is a single straight line having an indeterminate direction.* [Salmon's equation to this line, $0 . x + 0 . y + C = 0$, shews that its existence assumes that C is both $=$ and not $= 0$, unless proposition 5 holds. Townsend's proof of the equivalent proposition involves a threefold application of proposition 2.] *Poncelet*, Projections Perspectives, p. 53, art. 107. *Chasles*, G. S. 503, 651, &c. *Salmon*, Conics, 4th edit. art. 67. *Townsend*, M. G. art. 136, 150.

7. *Relations of moving points which hold as they approach a limit, hold also at the limit.* [The limit is a condition not reached: hence this asserts that what is not reached is reached (i. e., that separation is coincidence), or that relations of existence hold for non-existence, under which last form the proposition is continually applied, as when four points reduce to three, or two points to one.] *Chasles*, G. S. 15. *Townsend*, M. G. art. 19, and examples.

8. *Variable relations of moving points which approach a fixed relation as the points approach a fixed limit, assume that fixed relation when the points reach the limit.* [This is liable to the same objection of reaching a limit. The proposition is, however, almost universally applied in the theory of limits, except as laid down by Carnot. See Tract II. above. It generally assumes that what is true for separation is true for coincidence.]

These contradictions are avoided in the above Tracts. The operation of annihilation (o) has been distinguished from those of change (a, b, c, &c.). Incommensurables are treated independently of limits. The essence of a limit is held to consist in its joint approachability and unattainability.