BERNARD BOLZANO AND THE FOUNDATIONS OF MATHEMATICAL ANALYSIS

The discovery of differential and integral calculus by Newton and Leibniz at the end of the seventeenth century opened new scopes to mathematics and at the same time provided a powerful tool, indispensable for further progress of the then developing physics. No wonder that the new branch of mathematics, the so-called mathematical analysis, which made it possible to deal with problems whose successful solution had been unthinkable before, and which at the same time was continuously bringing forward new problems, went through a stormy development in the course of which the mathematicians, dazzled by amazing new prospects, set off on far exploration voyages, not caring much for the firmness of foundations of the new branch of science. Apparently this state was not permanently tenable and in the first half of the nineteenth century the building of mathematical analysis was raised to such a height that continuing its construction without fortifying its foundations was unthinkable. This brought a period of great revision of the foundations of analysis which lasted about fifty years and could be considered consummated by the work of Weierstrass about the year 1870; the development of the other branches of mathematics continued, of course, simultaneously and in mutual interaction.

It seems evident that at this level the revision could not follow other direction than that of consequential arithmetization of analysis. This arithmetization grew gradually only to attain a certain one-sidedness in the work of Weierstrass and was later corrected by the modern development of mathematics; after all, even in the period mentioned the dialectics of this process can be observed: so, for example, B. Riemann who on the one hand contributed considerably to the *arithmetization* of analysis by his theory of integral, was on the other hand the ingenious builder of the *geometric* theory of analytic functions.

However, it would be a mistake to describe this revision as a merely "critical" work in contradistinction to, for example, the "creative" period of the eighteenth century. The revision of the foundations of analysis led not only to the reinforcement of results already known but also to the discovery of new reliable and general methods which often allowed to collect in one theorem numerous results, each of which had required before to be tackled separately – apart from the fact that such a general theorem represents qualitatively a much higher level of knowledge than a thousand of its special cases, even if we must not forget the latter unless we sink into verbalism. These methods provide a much more effective and more easily controlled tool, which can be used to deal with much more difficult and complicated problems than before.

I have already mentioned that this great period of revision may be considered completed about 1870 when the fundamental concepts which were connected with the concept of the limit were established, the main theorems concerning these concepts were deduced and the whole structure was crowned[1]) with the theory of real numbers (Dedekind's theory is probably still the most popular of various theories of real numbers with our mathematical community). It is at the beginning of that period that our compatriot, Bernard Bolzano (1781–1848) made his appearence.

It is well known that Bolzano was not only a mathematician, for his many-sided beneficial activities left a deep trace in the life of our nation. Nonetheless, we shall pay attention here only to Bolzano's mathematical work, which represents an essential part of his activity and which assumes a significant place in the history of mathematics – e.g. in the Great Soviet Encyclopedia the entry on Bolzano is mostly devoted to his work in mathematics. Moreover, we shall not even deal with Bolzano's mathematical work in its full extent but only with those parts which concern the revision of the foundations of mathematical analysis. Indeed it seems that his work in this field incomparably surpasses his other mathematical works, whether those concerning geometry or the theory of numbers.

Actually only two of Bolzano's papers concerning the foundations of analysis – inextensive but very significant – appeared during his lifetime: "Der binomische Lehrsatz und als Folgerung aus ihm der polynomische und die Reihen, die zur Berechnung der Logarithmen und Exponentialgrössen dienen, genauer als bisher erwiesen" from 1816 and "Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege" from 1817 (referred to subsequently as "Analytic proof"). In addition to these, two other papers of a more philosophical character should be mentioned, namely "Beyträge zu einer begründeteren Darstellung der Mathematik, I. Lieferung" from 1810, which contains, so to say, a program of Bolzano's further mathematical work, and "Paradoxien des Unendlichen" written in 1847–48 and published after Bolzano's death in 1851 by F. Přihonský. The most important of these papers is "Analytic proof" to which we shall return later. All these papers remained almost unknown for a long time and only long after Bolzano's death, several outstanding mathematicians pointed out their significance. Since then Bolzano has been appreciated as one of the creators of the foundations of mathematical analysis. However, the above mentioned papers are far from covering Bolzano's activity in this field. At the end of World War I, Professor M. Jašek from Plzeň found an extensive manuscript in Bolzano's inheritance in the Vienna Court Library. This manuscript was published in 1930 by the Royal

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[1]) This development proceeded actually downwards: the theory of real numbers is logically the starting point of analysis in the real domain; historically its creation marks the end of this period, since the revision of the foundations proceeded from the "surface" to the ground.

Bohemian Learned Society under the title "Functionenlehre". The book is provided with an outstanding foreword by Professor K. Petr and with careful, detailed notes by Professor K. Rychlík. In two sections of the book (on continuity and derivative) Bolzano systematically applied the principles that he had used in a special case in "Analytic proof". The whole work, completed about 1831–34, contains a number of fundamental results, which were re-discovered only after several decades. Incidentally, we may ask how much Bolzano's work could have changed the way analysis followed, had it been published at the time. Today, when we reflect more than ever upon the traditions of our national culture, it seems relevant to recall how much we owe to Bolzano's comprehensive and extensive work.

The following remarks have no other aim than to turn general attention again to the importance of Bolzano's work and to some problems which may appear in evaluating its significance.

We have already mentioned that the first works of Bolzano in this field ("Binomial theorem" and "Analytic proof" from 1816 and 1817) appeared at the very beginning of the period of revision of the foundations of analysis. Among his great contemporaries we find several who contributed by their work in various fields to a reliable construction of the foundations of mathematical analysis. However, a systematic treatment of these problems may be found perhaps only in A. Cauchy's "Cours d'Analyse" (1821), "Résumé des leçons ... sur le Calcul Infinitésimal" (1823), "Leçons sur le Calcul différentiel" (1829). Thus Bolzano's "Analytic proof" is older; nevertheless, later Bolzano became acquainted with Cauchy's works and he quoted them in his "Functionenlehre". However, there is one essential difference between the two great scientists. Unlike Cauchy, who was above all a mathematician (and one of the greatest mathematicians of all times), concerned with the fundamental questions only to the extent necessary for his further investigations, Bolzano was at the same time a philosopher. He himself said that he was interested in mathematics above all as in a branch of philosophy and an exercise in correct thinking. This fact was distinctly reflected in his work. First of all, in the analysis of fundamental concepts, in the study of their interrelations and in his striving for exactness of proofs, Bolzano surpassed Cauchy. On the other hand, from the strictly "professional viewpoint" Bolzano as a mathematician was not experienced and skilled enough. The influence of this drawback upon his work will be demonstrated on examples later.[2])

Let us now pass to his "Analytic proof" as the most significant of Bolzano's papers concerning the foundations of analysis and published during his lifetime. This study is devoted to the following theorem: If a function, continuous in a closed interval, assumes

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[2]) Thirdly, Bolzano baffled his own respectable efforts in some cases by mixing obscure metaphysical elements into his considerations. Fortunately, this objection does not apply to his works on the foundations of analysis, where the set of real numbers (though their theory was not yet properly developed then) offered a sufficiently reliable base.

values of opposite signs at the endpoints of this interval, then this function equals zero at one inner point of the interval at least.

Bolzano introduces this theorem in a rather (unnecessarily) complicated form: If f, g are two functions, both continuous in a closed interval $[a, b]$, and if $f(a) < g(a)$, $f(b) > g(b)$, then there is at least one number x inside this interval, such that $f(x) = g(x)$. This theorem is "intuitively" very plausible: a continuous curve which passes partly under, partly above the x-axis, necessarily intersects the x-axis. However, Bolzano points out correctly: this is not sufficient, it is necessary to prove the theorem as a consequence of the definition of continuity. Bolzano defines continuity essentially in the same way as Cauchy does a little later. The proof of the theorem follows. First, Bolzano introduces a necessary and sufficient condition for convergence of a sequence, the so-called *Bolzano-Cauchy condition*, which Cauchy introduces four years later. Bolzano, however, proceeds further than Cauchy and attempts to prove the sufficiency of this condition. Actually, he offers only an argument which makes the sufficiency of the condition plausible instead of a real proof – this failure was unavoidable since the proof of the assertion is based on the theory of real numbers, which was built only some fifty years later. Nevertheless, even the mere formulation of this extremely important theorem in which Bolzano has priority over Cauchy gives evidence of his penetrating insight into the fundamental problems of analysis. Bolzano proves here also the fact that a sequence has at most one limit; it might be interesting to find out who, apart from Bolzano, was the first to realize the need for proof of this theorem (which is of course very easy).

In the next section, Bolzano applies the Bolzano-Cauchy condition in order to prove the following theorem: If all numbers less than a certain number u have a property M and if there is at least one number without the property M, then there is a number U which is the maximum of all numbers v which have the property that every number less than v has the property M. Analyzing this construction, we see that Bolzano proves here the greatest lower bound theorem (the infinum theorem): the number U is in fact the greatest lower bound of those numbers which do not possess the property M. The emphasis with which Bolzano points out the fundamental importance of this theorem in the following section, witnesses his sense for relative importance of mathematical theorems. The main theorem is proved in Section 15 – briefly speaking – as follows: Let x be the greatest lower bound of those numbers y from the interval $[a, b]$ which satisfy $f(y) \geqq g(y)$; then it is easily seen that $f(x) = g(x)$. After that Bolzano proves in Section 17 the continuity of the polynomial which enables him to apply the theorem from Section 15 to the case of the equation $P(x) = 0$, where P is a polynomial (Section 18).

Except for the failure (then unavoidable) in the proof of sufficiency of the Bolzano-Cauchy condition, all details of Bolzano's proof are correct, though some unnecessary complications indicate lack of mathematical skill. For example, for reasons unknown in Section 15 Bolzano unnecessarily distinguishes several cases according to the signs of the numbers a, b which complicates the proof considerably. Other unnecessary complications occur in Section 18; however, they may be connected with the first paragraph of the proof in Section 15, the meaning of which I have not been able to decipher. (Anyhow, this little paragraph is superfluous, so that its obscurity does not matter.) But all this is mere lack of skill, not incorrectness. A worse situation occurs in Section 16, which fortunately does not affect the other parts of the paper. Bolzano asserts here that under the assumptions of the main theorem, the number of roots of the equation $f(x) = g(x)$ in the interval $[a, b]$ is odd. However, it follows from the context that Bolzano evidently counted each root only once, regardless of its multiplicity. Of course, this assertion is obviously wrong. If we notice that this section, under the title "Anmerkung" is styled rather superficially – so that it gives the impression of a rather negligent improvisation – and if we consider the novelty and unusual character of Bolzano's methods, we are hardly surprised at his mistake. On the other hand, it is worth noticing that he did not even recognize the incorrectness of his assertion, which a skilled mathematician would have disproved by the simplest examples.

We believe that this brief analysis of Bolzano's paper shows clearly enough his arithmetizing viewpoint, his exceptional sense for the problems of foundations of mathematical analysis and the depth of his methods of proofs, but at the same time his surprising unskilfulness in using the tools of mathematical craft.

Let us turn now to Bolzano's "Functionenlehre" published in 1930. This work, written about fifteen years later than his "Analytic proof", deals also with fundamental problems of analysis, but at a higher level of evolution. The best known result of this book is Bolzano's excellent example of a continuous function which has a derivative at no point of a certain everywhere dense set. (Now we know that this function has a derivative at no point at all, but Bolzano neither proved nor asserted this.) If we realize that the first example of a continuous function which has a derivative at no point was published by K. Weierstrass in 1875[3] (according to H. A. Schwarz, Weierstrass had presented this example in his lectures earlier, since 1861), then we understand the sensation caused by Jašek's report on the disclosure of this function, now known as *Bolzano's function*, in Bolzano's manuscript.

On the other hand, it would be quite incorrect if we saw the importance of "Functionenlehre" merely in this dazzling detail, Bolzano's function. I believe that its main significance consists in a systematic exposition of the theory of continuity and derivative of functions of one real variable.[4]) I gave a more detailed analysis of these parts of the book in my paper

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[3]) More precisely: Weierstrass's example was published by P. du Bois-Reymond in one of his papers in Journal für die reine und angewandte Mathematik, 79 (1875), pp. 29–31.

[4]) I should point out that the parts concerning derivatives, which require more proficiency in mathematical technique, include more incorrect points than those that concern continuity. Besides, the book includes also an account of functions of several variables; however, these parts are technically too complicated and suffer from many defects.

"Bolzano's Functionenlehre" (this volume pp. 43–66), though only from the mathematical point of view, deliberately leaving its definitive evaluation to historians of mathematics. Therefore I restrict myself only to several principal remarks illustrated by examples. Let us mention, for example, Bolzano's account of continuity. The definition of continuity is given here in a formally more perfect way than in "Analytic proof"; moreover, Bolzano defines here continuity at a point (including one-sided continuity) while the earlier definitions (both in "Analytic proof" and by Cauchy) concerned only continuity in an interval. Then Bolzano proceeds to the theory of continuous functions of one variable, derived strictly deductively from the definition. This theory is then developed in two directions.

First of all, Bolzano proves general theorems on continuous functions, that is, he establishes properties which can be derived from the continuity of a function. The main theorem of "Analytic proof", which in the meantime had been proved also by Cauchy (1821), is again one of the main results, its proof being simplified, but besides there is a number of new theorems, for example a fundamental theorem concerning the existence of maximum and minimum of a continuous function in a closed bounded interval. Bolzano points out immediately that the theorem is not valid for open intervals. In modern terms, it is clear that Bolzano realized the importance of compactness of a closed bounded interval. The proofs are built very systematically mainly on two theorems: firstly, on the theorem on the greatest lower bound, which had appeared already in "Analytic proof", secondly, on the following theorem: Every bounded sequence has a point of accumulation. Here an interesting problem arises: Rychlík in his notes to "Functionenlehre" asserts that this theorem cannot be found in Bolzano's papers printed before 1930. Nonetheless, the theorem had been often called "*Bolzano-Weierstrass theorem*" before – how did Bolzano's name get there?

Thus Bolzano proves a number of fundamental theorems on continuous functions which altogether confirm the facts suggested by superficial intuition, and which form the necessary foundation for further study of continuous functions. However, Bolzano investigates the notion of continuity also in another direction, it could be said in the opposite one: he demonstrates the existence of functions which have certain unexpected "paradoxal" properties (now of course we find nothing paradoxal about these properties any more). Thus for example he constructs at the very beginning a function continuous at precisely one point. The most excellent example is doubtlessly Bolzano's function, which is continuous and, as proved by Bolzano, is neither monotone in any interval nor has a finite derivative at the points of a certain everywhere dense set. Thus, while the former direction of investigation concerns properties which every continuous function possesses, the latter, by means of constructing suitable examples, deals with properties which appear at least with some continuous functions, that is, it studies the extent of the notion of continuity.[5] I believe

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[5]) These two directions were emphasized also by K. Petr in his important installation speech as Chancellor of Charles University: "Bernard Bolzano and his significance

that almost nothing had been done in this direction before Bolzano. Therefore, it seems that there is no exaggeration in calling Bolzano an anticipator of the modern general theory of real functions. It would be interesting to compare his work with the not much later developments represented by the names of P. G. Lejeune Dirichlet – B. Riemann – H. Hankel.

If we raise the question whether Bolzano's work in mathematical analysis has been adequately appreciated we have to answer it negatively. Both his "Analytic proof" and "Paradoxes" were published again, with critical comments. A general evaluation was attempted in 1881 by O. Stolz (Bernard Bolzanos Bedeutung in der Geschichte der Infinitesimalrechnung, Math. Annalen 18 (1881), pp. 225–279); however, without the knowledge of "Functionenlehre" this evaluation was far from complete, let alone the other possible reservations against Stolz's paper.

As I have already mentioned, the edition of "Functionenlehre" from 1930 was equipped with a foreword by K. Petr and comments by K. Rychlík. I have also mentioned Petr's installation speech and my own paper. The information provided by the discoverer of Bolzano's manuscript M. Jašek is of high value but his comments should be read – from the mathematician's point of view – with a great deal of reservation. Nonetheless, all this cannot be considered a sufficient general evaluation of Bolzano's work in the foundations of analysis.

Those who will undertake the task of critically evaluating Bolzano's work will face many difficulties. Above all, Bolzano's main work on analysis was published only eighty years after his death, and his other works on analysis were not widely known during his lifetime either. Thus they lacked the direct influence on the author's contemporaries, the examination of which might help to appreciate correctly the work itself.

I have already quoted Bolzano's own words that he was interested in mathematics mainly as a branch of philosophy and an exercise in correct thinking. Consequently, it would be necessary for the evaluation of Bolzano's work – and this seems to me to be an especially difficult task – to find the connection between his philosophical views and his mathematical work. I feel that Bolzano was fully successful in those cases where – though his interest was excited by philosophical considerations, which affected his choice of programme and methods – he avoided purely philosophical arguments and was able to work solely on a mathematical basis – as is the case of his works concerning the foundations of analysis. On the other hand, he did not achieve full success when he directly used some

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for mathematics" (1926). Petr illustrated his opinion by two examples: the main theorem of "Analytic proof" is a general theorem confirming the conjecture which follows by intuition; however, intuition suggests also that every continuous curve has a tangent except for some isolated points – and Bolzano's function is a surprising (now not any more, of course) example of a function that disproves the conjecture.

of his metaphysical ideas — as for example in geometry — nor in the problems which required much knowledge and proficiency of specifically mathematical character. Perhaps this is a far-fetched judgement — but I give at least one example that seems to corroborate it.

Lehrsatz. Wenn eine Reihe von Größen

$$\overset{1}{F}x, \overset{2}{F}x, \overset{3}{F}x, \ldots, \overset{n}{F}x, \ldots, \overset{n+r}{F}x, \ldots$$

von der Beschaffenheit ist, daß der Unterschied zwischen ihrem nten Gliede $\overset{n}{F}x$ und jedem späteren $\overset{n+r}{F}x$, sey dieses von jenem auch noch so weit entfernt, kleiner als jede gegebene Größe verbleibt, wenn man n groß genug angenommen hat: so gibt es jedesmahl eine gewisse *beständige Größe*, und zwar nur *eine*, der sich die Glieder dieser Reihe immer mehr nähern, und der sie so nahe kommen können, als man nur will, wenn man die Reihe weit genug fortsetzt.

Lehrsatz. Wenn eine Eigenschaft M nicht *allen* Werthen einer veränderlichen Größe x, wohl aber *allen*, die *kleiner* sind, als ein gewisser u, zukömmt: so gibt es allemahl eine Größe U, welche die größte derjenigen ist, von denen behauptet werden kann, daß alle kleineren x die Eigenschaft M besitzen.

Two theorems from Bolzano's treatise "Rein analytischer Beweis..."

In his "Paradoxes" Bolzano mentions (with great emphasis) the fact that the sets *M* of all numbers between zero and five and *N* of all numbers between zero and twelve are, as we say today, equivalent. This means that there is a one-to-one correspondence between the numbers from the sets *M* and *N* — obtained for example by associating each number *x* between zero and five with the number $12x/5$, which obviously lies between zero and twelve. This notion of equivalence was later re-discovered by G. Cantor who used it as the base of his ingenious theory of cardinal numbers. Unlike him, Bolzano, instead of studying this equivalence further, proceeds like this: Though these two sets are equivalent, one of them is part of the other, hence they cannot have the same "Vielheit". This led Bolzano evidently to the conclusion that the notion of equivalence is not essential for evaluating "Vielheit" of infinite sets and to an immediate dismissal of the concept. The fact that Bolzano rejected the important notion of equivalence as soon as he discovered it was caused probably by the obscure character of the notion "Vielheit" and by Bolzano's misuse of metaphysical ideas concerning the relation between "part" and "entirety".

Bolzano's theorems from the foundations of analysis are today an indispensable part of the professional equipment of any mathematician and they are lectured on in introductory courses of analysis at every university; however, in his time, they were great philosophical and ideological achievements; they resulted from a new approach to fundamental problems of mathematics, which in its most developed form is found namely in Bolzano. It was necessary to assume the new standpoint and persist in it consistently even if it led to unexpected, seemingly even paradoxical consequences. Here Bolzano's situation was similar to that of Lobachevskiĭ; he was saved from narrow-minded bawlers probably only by the fact that his "Functionenlehre" remained unpublished.

Let me introduce at least one example. As mentioned by M. Jašek, Bolzano presented the manuscript of his "Functionenlehre" to his favourite student A. Slivka from Slivice; Bolzano's inheritance includes Slivka's extensive critical answer. Slivka argues with Bolzano and claims among other things that he believes it possible to prove a theorem that every continuous function has a derivative everywhere except at some isolated points. Here we meet a wonderful proof how prejudice survives persistently in one's mind: Slivka saw with his own eyes the construction of Bolzano's function – and yet he did not believe in its existence because it contradicted the contemporary (scientifically unjustifiable) ideas.[6]) After all, we meet a similar example – concerning precisely the same problem – many years later. The prominent French mathematician Ch. Hermite (who, of course, was not acquainted with Bolzano's function) mentioned Weierstrass's continuous function which has nowhere a derivative in a letter to Stieltjes. Naturally he did not deny its existence – he was too good a mathematician to do that – but he wrote that he "abandoned with horror" this deplorable wound of continuous functions which do not possess any derivative. And this opinion was shared by other great mathematicians of the second half of the nineteenth century, who thought that similar investigations into the foundations of mathematics can only harm the beautiful building of mathematics.[7]) They were wrong, of course:

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[6]) Another evidence of how Bolzano had to fight the old viewpoints in himself seems to be his exposition on distinguishing of functions of the first and the second kind; cf. p. 59. As concerns Slivka's case, let us mention that there is really one incomplete point in Bolzano's investigation of his function; however, I do not believe that Slivka would have noticed it.

[7]) It is true that the existence of such a "paradoxical" function as Bolzano's or Weierstrass's functions must have worked as a warning signal, but in further development

the bold revision of the foundations of analysis destroyed only the prejudices of the previous period and so levelled the ground for raising the building of modern mathematics,[8]) among the anticipators of which one of the foremost places – both historically and by his importance – belongs to our Bernard Bolzano.

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of science it naturally led to such questions as these: which classes of functions may be asserted to have a derivative "almost everywhere"? And what precise meaning should be assigned to the words "almost everywhere"? Everyone who knows at least something from the modern theory of functions is aware of the significant results to which these problems led.

[8]) See for example the very expressive foreword to S. Saks, Théorie de l'intégrale, Warsaw 1933.