ON BOLZANO'S FUNCTION

The aim of this paper is to prove that Bolzano's function[1]) possesses no derivative for any value of its argument, and to study its derivates.

Bolzano's function is defined in the following way: Let us consider the segment $y = x$ in the interval [0, 1]; let us divide the interval [0, 1] into four intervals [0, 3/8], [3/8, 1/2], [1/2, 7/8], [7/8, 1] and construct the points (0, 0), (3/8, 5/8), (1/2, 1/2), (7/8, 9/8), (1, 1). Now we join every two consecutive points by a segment. Thus we obtain four intervals, which will be called "first subintervals"; the above five points will be called "first nodes" (if there is no danger of misunderstanding I shall use the same term for their projections onto the x-axis); the piecewise linear line which joins the nodes and consists of four segments will be called "first polygonal line" (the dot-and-dashed line in Fig. 1). Now we

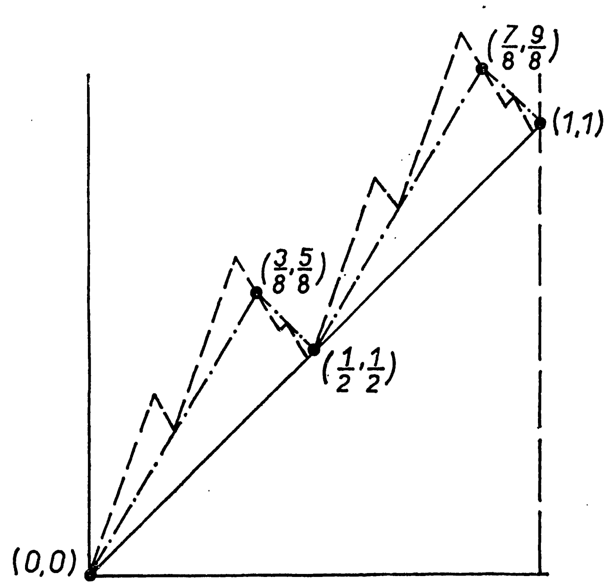

Fig. 1

[1]) See M. Jašek, Bolzano's function (Czech), Časopis pro pěstování matematiky a fysiky 51 (1922), pp. 69–76.

construct in the same way a polygonal line over each of these segments. This is to say, if (a, A), (b, B) are the endpoints of one of these segments, we construct points

$$(a, A), \quad (a + \tfrac{3}{8}(b - a), A + \tfrac{5}{8}(B - A)), \quad (\tfrac{1}{2}(a + b), \tfrac{1}{2}(A + B)),$$
$$(a + \tfrac{7}{8}(b - a), A + \tfrac{9}{8}(B - A)), \quad (b, B)$$

and join every two consecutive points by a segment; thus we obtain the "second polygonal line" consisting of 16 segments (dashed line in Fig. 1), 17 "second nodes" (obviously each first node is simultaneously a second node) and 16 "second subintervals". We continue in the same way; generally, the "n-th polygonal line" consists of 4^n segments, each of them over one of 4^n "n-th subintervals"; the endpoints of these segments represent $4^n + 1$ "n-th nodes". If a node has coordinates x, y then the value of Bolzano's function for the value x of its argument equals y. Thus we have defined Bolzano's function on the set of nodes which is everywhere dense in the interval $[0, 1]$. If this definition can be extended to the interval $[0, 1]$ in such a way that f may be continuous in this closed interval, then this is possible in a unique way.[2])

I offer these remarks, being aware of their incompleteness, only to explain the terminology used subsequently.

In what follows I denote Bolzano's function by f; the symbol $\langle x \rangle$ stands for a point of Bolzano's curve (i.e. of the graph of the function $y = f(x)$ with coordinates $x, f(x)$).

Let us introduce two more notes which will be found useful in what follows.[3])

(A) *In any subinterval* $[a, b]$, *Bolzano's curve is constructed from the segment* $\overline{\langle a \rangle \langle b \rangle}$ *in the same way as in the interval* $[0, 1]$ *from the segment* $\overline{\langle 0 \rangle \langle 1 \rangle}$.

In the interval $[0, 1]$ Bolzano's curve consists of two identical parts which are in the following relation: if x, x' are two values from the interval $[0, 1]$ satisfying $x' - x = 1/2$, then $f(x') - f(x) = 1/2$ as well, i.e. the slope of the line joining $\langle x \rangle$, $\langle x' \rangle$ is one.

(B) Analogously: *If* $[a, b]$ *is an arbitrary* n*-th subinterval and* k *the slope of the* n*-th polygonal line in this interval, then to each value* x *in* $[a, b]$ *there exists a value* x' *in* $[a, b]$ *such that* $|x' - x| = (b - a)/2$ *and the slope of the joining line* $\overline{\langle x \rangle \langle x' \rangle}$ *is equal to* k. We shall call such two points $\langle x \rangle$, $\langle x' \rangle$ "conjugate points with respect to the interval $[a, b]$".

---

[2]) The proof that such an extension of our definition is possible was given by K. Rychlík: Über die Funktion von Bolzano (cf. Věstník Král. české spol. nauk 1921–22). He also proved in a different way that Bolzano's function possesses no derivative. In what follows, I shall use the term "Bolzano's function" to indicate the continuous function defined in this way on the whole closed interval $[0, 1]$.

[3]) The reader will find these notes clear and their verification immediate for the nodes, which result from a finite number of elementary geometrical constructions (or, arithmetical operations). For the other points they follow by an easy limiting process. I ask the reader to cooperate by drawing a rough sketch to each of our arguments.

I

Let us proceed to the proof of the assertion that *Bolzano's function possesses neither a finite derivative at any interior point of* $[0, 1]$, *nor a finite derivative from the right or from the left at* $x = 0$ *or* $x = 1$, *respectively.*

To this end, let us proceed as follows: Consider a value x which is not a node. Then x lies in the interior of one of the first subintervals; let us denote this interval by J_1, or more precisely by J_1^{I}, J_1^{II}, J_1^{III} or J_1^{IV} according to whether J_1 is the interval $[0, 3/8]$, $[3/8, 1/2]$, $[1/2, 7/8]$, or $[7/8, 1]$. Let us divide the interval J_1 again into second subintervals; the point x lies in the interior of one and only one of them, say J_2; let $J_1 = [x_1^l, x_1^r]$; then the interval J_2 will be denoted by J_2^{I}, J_2^{II}, J_2^{III}, J_2^{IV} if J_2 coincides with

$$[x_1^l, x_1^l + \tfrac{3}{8}(x_1^r - x_1^l)], \quad [x_1^l + \tfrac{3}{8}(x_1^r - x_1^l), x_1^l + \tfrac{1}{2}(x_1^r - x_1^l)],$$
$$[x_1^l + \tfrac{1}{2}(x_1^r - x_1^l), x_1^l + \tfrac{7}{8}(x_1^r - x_1^l)], \quad [x_1^l + \tfrac{7}{8}(x_1^r - x_1^l), x_1^r],$$

respectively. Proceeding like that we obtain an infinite sequence of intervals

$$(J) \quad J_1, J_2, J_3, \ldots, J_n, \ldots ;$$

each of them is contained in all the preceding ones, their lengths converge to zero as n increases, and x is an inner point of all these intervals.

The sequence (J) is uniquely determined if we know the order in which the intervals J^{I}, J^{II}, J^{III}, J^{IV} follow. At the same time it is evident that the sequence (J) determines uniquely the point x and conversely, if x is not a node, then it determines uniquely the sequence (J).[4])

In what follows we denote by k_n the slope of the n-th polygonal line in the interval J_n[5]), by x_n^l, x_n^r the left and right endpoints of the interval J_n, respectively; finally, J_n will stand also for the length of the interval J_n.

Let us now consider an arbitrary point $\langle x \rangle$ of Bolzano's curve; it is uniquely de-

---

[4]) This argument remains valid for the nodes as well, with the following changes:
a) starting from a certain n, x belongs to the boundary of all J_n;
b) there are two sequences (J) corresponding to the point x; one which – starting from a certain n – consists only of intervals J^{I}, the other, whose intervals – starting from a certain n – are all J^{IV}; only the points $x = 0$ and $x = 1$ are associated with a single sequence each, namely $J_1^{\mathrm{I}}, J_2^{\mathrm{I}}, J_3^{\mathrm{I}}, \ldots, J_n^{\mathrm{I}}, \ldots$ and $J_1^{\mathrm{IV}}, J_2^{\mathrm{IV}}, J_3^{\mathrm{IV}}, \ldots, J_n^{\mathrm{IV}}, \ldots$, respectively.

[5]) Note that if J_{n+1} is either J_{n+1}^{I} or J_{n+1}^{III}, then $k_{n+1} = \frac{5}{3}k_n$; if it is either J_{n+1}^{II} or J_{n+1}^{IV} then $k_{n+1} = -k_n$; since $k_0 = 1$, we have $|k_n| \geqq 1$ for all n.

termined by the corresponding sequence (J) (this is true even for the nodes). Then one of the following two cases occurs:

1) Either this sequence includes only *a finite number* of both J^{II} and J^{IV}, hence it includes infinitely many J^{I} or J^{III} (or both of them); for the point $\langle x \rangle$ there exists a conjugate point $\langle x_n' \rangle$ with respect to J_n such that the line $\overline{\langle x \rangle \langle x_n' \rangle}$ has the slope k_n[6]); if n increases to infinity, then J_n as well as $|x - x_n'| = J_n/2$ tends to zero, $|k_n|$ increases to infinity (cf. footnote 5) and hence *a finite derivative at x cannot exist.*

2) Or the sequence includes *infinitely many* intervals J^{II} or J^{IV} (or both of them); then for every N we can find $n > N$ such that J_{n+1} is either J_{n+1}^{II} or J_{n+1}^{IV}. Hence there exist again conjugate points with $\langle x \rangle$, say $\langle x_n' \rangle$ with respect to J_n, $\langle x_n'' \rangle$ with respect to J_{n+1}, such that

$$\lambda(x, x_n') = k_n\,, \quad \lambda(x, x_n'') = k_{n+1}\,.$$

However, if N increases to infinity then also n increases to infinity, both x_n', x_n'' tend to x, $k_{n+1} = -k_n$ (cf. footnote 5). In this case *there exists neither a finite nor infinite derivative at the point x.*

The nodes (except for $x = 1$) can be included in point 1) (see footnote 4); the conjugate points then lie – starting from a certain n – all to the right from $\langle x \rangle$, so that *the finite derivative from the right does not exist at these nodes.* On the other hand, the nodes (except for $x = 0$) can be included in point 2) as well; then x_n', x_n'' lie – starting from a certain n – all to the left from x, so that *the derivative from the left does not exist at the nodes, neither finite nor infinite.*

Finally, let us notice that if the sequence (J) includes infinitely many J_n^{I} or J_n^{III} as well as infinitely many J_n^{II} or J_n^{IV}, then

$$\limsup_{x' \to x} \lambda(x, x') = +\infty\,, \quad \liminf_{x' \to x} \lambda(x, x') = -\infty\,.$$

This completes the proof of our theorem.

II

We shall now prove the following theorem: *At each inner point of the interval $[0, 1]$ there exists neither a finite nor infinite derivative.* To this end, we shall first determine the minimum and maximum of Bolzano's function in the interval $[0, 1]$.

Bolzano's function cannot assume its minimum in the interval $[1/2, 1]$ as $f(x + 1/2) =$

---

[6]) In what follows we denote the slope of a line joining $\langle x \rangle$, $\langle x' \rangle$ briefly by $\lambda(x, x')$.

$= f(x) + 1/2$ for $x \leqq 1/2$. Further, its minimum cannot be positive. Let us assume that the minimum occurs in the interval $[3/8, 1/2]$ at a point x. Defining the point x by the corresponding sequence (J) we have

$$J_1 = \tfrac{1}{8}, \quad x_1^l = \tfrac{3}{8}, \quad f(x_1^l) = \tfrac{5}{8}, \quad k_1 = -1,$$

but

$$|k_n| \leqq \tfrac{5}{3}|k_{n-1}|, \quad J_n \leqq \tfrac{3}{8}J_{n-1};$$

hence

$$|k_n| \leqq (\tfrac{5}{3})^{n-1}, \quad J_n \leqq \tfrac{1}{8}(\tfrac{3}{8})^{n-1}.$$

However,

$$|f(x_{n+1}^l) - f(x_n^l)| \leqq \tfrac{9}{8}|k_n|\, J_n$$

which yields

$$f(x_{n+1}) \geqq f(x_1) - \tfrac{9}{8}\cdot\tfrac{1}{8} - \tfrac{9}{8}\cdot\tfrac{1}{8}\cdot\tfrac{5}{8} - \dots - \tfrac{9}{8}\cdot\tfrac{1}{8}\cdot(\tfrac{5}{8})^{n-1}$$

and, a fortiori,

$$f(x_{n+1}^l) \geqq \tfrac{5}{8} - \tfrac{9}{64}(1 - \tfrac{5}{8})^{-1} = \tfrac{1}{4}$$

for every n, hence also

$$f(x) = \lim_{n\to\infty} f(x_{n+1}^l) \geqq \tfrac{1}{4}$$

which contradicts the assumption $f(x) \leqq 0$. Therefore Bolzano's function assumes its minimum necessarily in the interval $[0, \frac{3}{8}]$. However, the same argument applies to this interval (cf. (A)) etc. Consequently, Bolzano's function assumes its minimum for $x \leqq (3/8)^n$ with n arbitrary, i.e. *it assumes its minimum in* $[0, 1]$ *only for* $x = 0$; *the minimum is equal to zero.*

Using (A) we can state immediately a more general result: in an arbitrary n-th subinterval Bolzano's function assumes its minimum or maximum at the left end of the subinterval provided the n-th polygonal line in this interval is increasing or decreasing, respectively. For example, in the interval $[\frac{7}{8}, 1]$ the maximum of Bolzano's function is $9/8$ and it occurs at $x = 7/8$.

Now it is easy to find the points at which Bolzano's function assumes its maximum. First of all, this occurs necessarily in the interval $[\frac{1}{2}, 1]$ (cf. (B)); however, it cannot occur in the interval $[\frac{7}{8}, 1]$ for in this interval the maximum is $9/8$ while in the interval $[\frac{1}{2}, \frac{7}{8}]$ the function assumes greater values. Hence

$$x_1^l = \tfrac{1}{2}, \quad x_1^r = \tfrac{7}{8} = 1 - \tfrac{1}{8}.$$

Using an analogous argument with respect to this interval $[x_1^l, x_1^r]$ etc. we obtain generally

$$x_n^l = \tfrac{1}{2}[1 + \tfrac{3}{8} + (\tfrac{3}{8})^2 + \ldots + (\tfrac{3}{8})^{n-1}],$$
$$x_n^r = 1 - \tfrac{1}{8}[1 + \tfrac{3}{8} + \ldots + (\tfrac{3}{8})^{n-1}].$$

The common limit of both these sequences is $x = 4/5$, the corresponding value of $f(x)$ is

$$f(x) = \tfrac{1}{2}[1 + \tfrac{5}{8} + (\tfrac{5}{8})^2 + \ldots] = \tfrac{4}{3}.$$

Taking into account (A) we can assert: *In the interval* $[0, 1]$ *Bolzano's function has the absolute minimum zero for* $x = 0$, *the absolute maximum* $4/3$ *for* $x = 4/5$; *local extrema are determined as follows: if* $[a, b]$ *is an arbitrary n-th subinterval and k the slope of the n-th polygonal line in this subinterval, let us construct the point with coordinates*

$$(M)\quad x = a + \tfrac{4}{5}(b - a),$$
$$y = f(a) + \tfrac{4}{3}[f(b) - f(a)] = f(a) + \tfrac{4}{3}k(b - a);$$

at this point a local extremum occurs, viz. maximum or minimum provided k is positive or negative, respectively. There are no other extrema; in the first place we can exclude the nodes (since at them the function oscillates from the left, cf. Sec. I). Further, if a point x_0 were a point of a local extremum of Bolzano's function then it would be a point of an *absolute* extremum in a certain interval $[x_0 - \varepsilon, x_0 + \varepsilon]$. The intervals J_n of the associated sequence (J) are for sufficiently large n included *in the interior* of the interval $[x_0 - \varepsilon, x_0 + \varepsilon]$, therefore Bolzano's function has an absolute extremum in the interval J_n at the point x_0, which lies inside the interval J_n; thus

$$x_0 = x_n^l + \tfrac{4}{5}(x_n^r - x_n^l),$$

i.e. the point x_0 is one of the points of the set M.

Thus the points at which Bolzano's function assumes local extrema form a countable set everywhere dense in $[0, 1]$; we have expressed explicitly all of these points.

The above results enable us to prove that Bolzano's function has not even an infinite derivative at each inner point of the interval $[0, 1]$.

We know from Sec. I that it suffices to consider those points x the sequence (J) of which includes only intervals J^{I}, J^{III} (cf. (A)). Moreover, we can exclude the case when (J) includes (starting from a certain index) only intervals J^{I}, for then x would be a node at which no derivative exists (see Sec. I).

First of all we shall prove that the sequence (J) cannot include infinitely many pairs J_n^{III}, J_{n+1}^{III}. Indeed, if this were the case, we could find for every N such a number $n > N$ that the consecutive intervals would be J_n, J_{n+1}^{III}, J_{n+2}^{III}.

For a point $\langle x \rangle$ there exists a point $\langle x_n' \rangle$ conjugate with respect to J_n and such that

$\lambda(x, x_n') = k_n$. Further,

$$f(x) > f(x_n^l) + \tfrac{1}{2}k_nJ_n + \tfrac{1}{2}\cdot\tfrac{5}{3}k_n\cdot\tfrac{3}{8}J_n\,,$$
$$f(x) > f(x_n^l) + \tfrac{13}{16}k_nJ_n\ \ (k_n = (\tfrac{5}{3})^n)\,.$$

The interval adjacent to J_n from the right is an n-th subinterval J^* of a length $\frac{1}{3}J_n$; the slope of the n-th polygonal line in it is $k^* = -\frac{3}{5}k_n$ (as J_n is either J_n^{I} or J_n^{III}). Hence the minimum in the interval J^* occurs for the value

$$a_n = x_n^r + \tfrac{4}{5}J^* = x_n^l + J_n + \tfrac{4}{5}\cdot\tfrac{1}{3}J_n\,,$$

hence $0 < a_n - x < 19J_n/15$ and we have

$$f(a_n) = f(x_n^r) + \tfrac{4}{3}J^*k^* = f(x_n^l) + k_nJ_n - \tfrac{4}{3}\cdot\tfrac{1}{3}J_n\cdot\tfrac{3}{5}k_n = f(x_n^l) + \tfrac{11}{15}J_nk_n\,.$$

Thus

$$\lambda(x, a_n) = \frac{f(x) - f(a_n)}{x - a_n} < -(\tfrac{13}{16} - \tfrac{11}{15})\cdot\tfrac{15}{19}k_n = -bk_n\,,$$

with the number $b > 0$ independent of n.

Now if N tends to infinity, then the same holds for both n and k_n, the values of both x_n' and a_n converge to x and we have

$\lambda(x, x_n') = k_n$ for the former,

$\lambda(x, a_n) < -bk_n$ for the latter.

It is apparent that in this case there does not exist even an infinite derivative and, moreover,

$$(1)\quad \limsup_{x'\to x}\,(f(x) - f(x'))/(x - x') = +\infty\,,$$
$$\liminf_{x'\to x}\,(f(x) - f(x'))/(x - x') = -\infty\,.$$

We obtain an analogous result assuming that the sequence (J) includes infinitely many pairs J_n^{I}, J_{n+1}^{I} (and obviously infinitely many J^{III} as well).[7]

Thus it remains only to investigate such sequences (J) in which intervals J^{I}, J^{III} alternate from a certain index. After omitting a finite number of intervals the point to deal with is defined by a sequence

$$J_1^{\mathrm{I}},\ J_2^{\mathrm{III}},\ J_3^{\mathrm{I}},\ \ldots,\ J_{2n+1}^{\mathrm{I}},\ J_{2n+2}^{\mathrm{III}},\ \ldots$$

(this point represents of course a whole countable set of points, as in all our considerations).

---

[7]) It suffices to consider the triplet of intervals J_n^{III}, J_{n+1}^{I}, J_{n+2}^{I}.

We have (denoting $[0, 1] = J_0$)

$$x_0^l = 0\,, \quad x_1^l = 0\,, \quad x_{2n+1}^l = x_{2n}^l\,, \quad x_{2n+2}^l = x_{2n+1}^l + \tfrac{1}{2}J_{2n+1}\,;$$
$$k_0 = 1\,, \quad k_n = (\tfrac{5}{3})^n\,, \quad J_0 = 1\,, \quad J_n = (\tfrac{3}{8})^n\,;$$

hence

$$x_{2n+2}^l = x_{2n}^l + \tfrac{1}{2}(\tfrac{3}{8})^{2n+1} = \tfrac{1}{2}[\tfrac{3}{8} + (\tfrac{3}{8})^3 + (\tfrac{3}{8})^5 + \ldots + (\tfrac{3}{8})^{2n+1}]\,,$$
$$f(x_{2n+2}^l) = f(x_{2n}^l) + \tfrac{1}{2}k_{2n+1}J_{2n+1} = \tfrac{1}{2}[\tfrac{5}{8} + (\tfrac{5}{8})^3 + (\tfrac{5}{8})^5 + \ldots + (\tfrac{5}{8})^{2n+1}]$$

and passing to the limit, $x = 12/55, f(x) = 20/39$. Of course, we have analogously (by (A))

$$x = x_{2n}^l + \tfrac{12}{55}J_{2n}\,, \quad f(x) = f(x_{2n}^l) + \tfrac{20}{39}k_{2n}J_{2n}\,.$$

Now there exists a point $\langle x'_{2n} \rangle$ conjugate to $\langle x \rangle$ with respect to J_{2n} so that $\lambda(x, x'_{2n}) = k_{2n}$; denoting further by x''_{2n} the point $x_{2n}^l + \frac{1}{2}J_{2n}$, we have

$$f(x''_{2n}) = f(x_{2n}^l) + \tfrac{1}{2}k_{2n}J_{2n}\,.$$

Then

$$\lambda(x, x''_{2n}) = \frac{\frac{20}{39} - \frac{1}{2}}{\frac{12}{55} - \frac{1}{2}}\,k_{2n} = -bk_{2n}$$

where $b > 0$ is again independent of n. Hence there does not exist an infinite derivative here and, moreover, relations (1) hold here as above.

The theorem from the beginning of this section is completely proved.

III

In the previous sections we have shown that neither finite nor even infinite derivative exists at the inner points of the interval $[0, 1]$. Are there perhaps at least some points at which the derivatives from the left or from the right exist?

To answer this question, let us first consider the points where an extremum occurs. Evidently, we can restrict ourselves to the point $x = 4/5$[8]. Then $f(x) = 4/3$ and analogously

$$x = x_n^l + \tfrac{4}{5}J_n\,, \quad f(x) = f(x_n^l) + \tfrac{4}{3}k_nJ_n$$

by (A). Further,

$$x_{n+1}^l = x_n^l + \tfrac{1}{2}J_n\,, \quad x_{n+1}^r = x_n^r - \tfrac{1}{8}J_n\,.$$

---

[8]) See Sec. II.

If x' is in the interval $[x_n^l, x_{n+1}^l]$ then

$$f(x') \leqq f(x) - \tfrac{1}{2}k_n J_n \quad (\text{cf. (B)})\,;$$

if x'' is in the interval $[x_{n+1}^r, x_n^r]$ then

$$f(x'') \leqq f(x_n^r) + \tfrac{1}{8}J_n k_n = f(x_n^l) + \tfrac{9}{8}J_n k_n\,.$$

Since $0 < x - x' < J_n$, $0 < x'' - x < J_n$, we have

$$\frac{f(x) - f(x')}{x - x'} > \tfrac{1}{2}k_n\,, \quad \frac{f(x) - f(x'')}{x - x''} < -(\tfrac{4}{3} - \tfrac{9}{8})\,k_n = -bk_n$$

with $b > 0$ independent of n. However, each $x' < x$ belongs to a certain interval $[x_n^l, x_{n+1}^l]$; if x' approaches x then n and hence also $k_n = (5/3)^n$ increases to infinity. Consequently

$$\lim_{x' \to x-} \frac{f(x) - f(x')}{x - x'} = +\infty$$

and analogously

$$\lim_{x'' \to x+} \frac{f(x) - f(x'')}{x - x''} = -\infty\,.$$

We conclude that at the point $x = 4/5$ the derivatives from the left and from the right exist and equal $+\infty$ and $-\infty$, respectively. Applying the remark (A) we find immediately: *Bolzano's function has the derivatives from the left and from the right equal to* $+\infty$ *and* $-\infty$, *respectively, at the points of local maxima*; *equal to* $-\infty$ *and* $+\infty$, *respectively, at the points of local minima.*

Is it possible that both the one-sided derivatives exist simultaneously at any more points? We can exclude a priori the nodes (derivative from the left does not exist). Further, the sequence considered cannot include infinitely many J^{II}, as is easily seen.

Indeed, considering an interval J_{n+1}^{II} with $k_n > 0$ we obtain

$$f(x_n^l) + \tfrac{1}{2}k_n J_n - \tfrac{1}{3}\cdot\tfrac{1}{8}k_n J_n \leqq f(x) \leqq f(x_n^l) + \tfrac{5}{8}k_n J_n\,.$$

Hence

$$\lambda(x, x_n^l) > (\tfrac{1}{2} - \tfrac{1}{24})\,k_n \quad (0 < x - x_n^l < J_n)\,.$$

Secondly, considering the point

$$x_n'' = x_n^l + \tfrac{4}{5}\cdot\tfrac{3}{8}J_n\,,$$

we have

$$f(x_n'') = f(x_n^l) + \tfrac{4}{3}\cdot\tfrac{5}{8}k_n J_n \quad (0 < x - x_n'' < J_n)$$

and hence

$$\lambda(x, x_n'') < -(\tfrac{4}{3} \cdot \tfrac{5}{8} - \tfrac{5}{8})\, k_n .$$

Analogous inequalities – after interchanging $<$ and $>$ – hold for $k_n < 0$ as well. This yields easily that the derivative from the left does not exist here.

We can prove similarly that the derivative from the left does not exist if (J) includes infinitely many intervals J^{IV}.

It remains only to deal with sequences (J) including only J^{I} and J^{III} but not all J^{I}. However, in this case (1) holds and if there existed both the derivatives from the right and from the left, then an extremum would occur at the point.

We conclude: *Both one-sided derivatives exist simultaneously only at the points of the countable set* M (cf. Sec. II); *both are infinite, with opposite signs.*

IV

The one-sided derivatives found in the preceding section were infinite. Is it possible that there exists a point with a *finite* derivative from the left or from the right?

Let us first consider the derivative from the left. As we know from Sec. III, the sequence (J) may include in this case – starting from a certain index – only J^{I} and J^{III} (but not all J^{I}). If the $(n + 1)$-st interval is J^{III}_{n+1} and $\langle x_n' \rangle$ is the point conjugate with $\langle x \rangle$ with respect to J_n then

$$\lambda(x, x_n') = k_n ;$$

but $x_n' < x$, $|k_n|$ tends to infinity with n tending to infinity, hence a finite derivative from the left cannot exist.

Further, let us discuss the derivative from the right. If (J) includes infinitely many J^{I} then $\lambda(x, x_n') = k_n$ again for infinitely many n, where $x_n' > x$ (since if $\langle x_n' \rangle$ is a conjugate point with $\langle x \rangle$ with respect to J_n and the $(n + 1)$-st interval is J^{I}_{n+1} then $x_n' > x$).

Hence the remaining case is that of sequences (J) which include (starting from a certain index) only J^{II}, J^{III} and J^{IV} but not only J^{IV} (then the corresponding point would be a node where a finite derivative from the right does not exist, cf. Sec. I).

Let n be such a number that either J^{II}_{n+1} or J^{III}_{n+1}. Then the point $\langle x \rangle$ belongs to the rectangle whose sides have the equations

$$x = x_n^l + \tfrac{3}{8} J_n , \quad x = x_n^l + \tfrac{7}{8} J_n ,$$
$$y = y_n^l + (\tfrac{1}{2} - \tfrac{1}{3} \cdot \tfrac{1}{8})\, k_n J_n , \quad y = y_n^l + \tfrac{4}{3} k_n J_n .$$

Now let (x', y') be an arbitrary point of this rectangle and let us join it with the points of

Bolzano's curve $\langle\alpha_1\rangle$, $\langle\alpha_2\rangle$, $\langle\alpha_3\rangle$ with coordinates α_i, β_i (see Fig. 2), where

$$\alpha_1 = x_n^r, \quad \beta_1 = y_n^r, \quad \alpha_2 = x_n^r - \tfrac{1}{16}J_n, \quad \beta_2 = y_n^r + \tfrac{1}{16}k_nJ_n,$$
$$\alpha_3 = x_n^r - \tfrac{1}{64}J_n, \quad \beta_3 = y_n^r - \tfrac{1}{64}k_nJ_n.$$

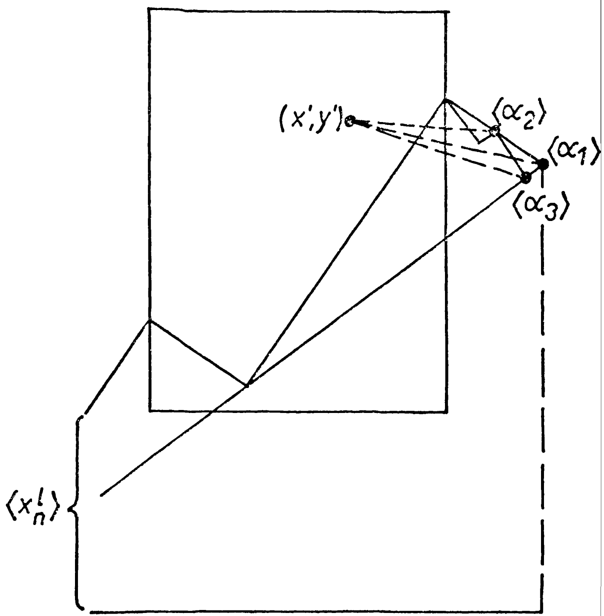

Fig. 2

Denote $x' = x_n^l + \xi J_n$, $y' = y_n^l + \eta k_n J_n$; then the slopes of these joining lines are

$$\frac{y' - \beta_1}{x' - \alpha_1} = \frac{(\eta - 1)\,k_nJ_n}{(\xi - 1)\,J_n}, \quad \frac{y' - \beta_2}{x' - \alpha_2} = \frac{(\eta - 1 - \frac{1}{16})\,k_nJ_n}{(\xi - 1 + \frac{1}{16})\,J_n},$$
$$\frac{y' - \beta_3}{x' - \alpha_3} = \frac{(\eta - 1 + \frac{1}{64})\,k_nJ_n}{(\xi - 1 + \frac{1}{64})\,J_n}.$$

If the point (x', y') moves over the given rectangle then ξ, η satisfy the inequalities

$$\tfrac{3}{8} \leqq \xi \leqq \tfrac{7}{8}, \quad \tfrac{1}{2} - \tfrac{1}{24} \leqq \eta \leqq \tfrac{4}{3},$$

which are independent of n. The slopes are continuous functions of ξ, η in the above described domain. Let us construct at each point (x', y') the difference between the greatest and the least of these three slopes; it will be of the form $\varphi(\xi, \eta)\,.\,|k_n|$ where $\varphi(\xi, \eta)$ is a continuous function of ξ, η independent of n. Since all the three slopes coincide for no pair of values ξ, η it is $\varphi(\xi, \eta) > 0$ in the domain and hence also its minimum in the domain is a number $A > 0$ obviously independent of n. Consequently, we obtain: if J_n is an arbitrary interval of the sequence (J) such that it is succeeded by J^{II}_{n+1} or J^{III}_{n+1} then in the interval J_n to the right from x there exist two points x'_n, x''_n such that

$$\lambda(x, x'_n) - \lambda(x, x''_n) \geqq A|k_n| .$$

Evidently there are infinitely many such values of n; let n pass to infinity by these values. Then x'_n, x''_n converge to x with n increasing and $|k_n| \geqq 1$ holds; therefore the limit

$$\lim_{x' \to x+} \lambda(x, x')$$

cannot exist and be finite.

This proves the following assertion: *Bolzano's function has neither finite derivative from the left nor from the right at any point.*

V

Now let us deal with the derivates. The reader easily deduces, in a way analogous to Sec. III, the following theorem:

At the nodes the derivative from the right exists (and is infinite); the left derivates are finite, different and with the same absolute value.

We shall prove: *There exist points at which all four derivates are finite.*

Let us consider a point defined by a sequence $J^{\mathrm{II}}_1, J^{\mathrm{II}}_2, \ldots, J^{\mathrm{II}}_n, \ldots$ Then

$$k_n = (-1)^n, \quad J_n = (\tfrac{1}{8})^n, \quad x^l_{n+1} = x^l_n + \tfrac{3}{8} J_n ,$$
$$f(x^l_{n+1}) = f(x^l_n) + (-1)^n . \tfrac{5}{8} J_n$$

which yields easily $x = 3/7, f(x) = 5/9$; analogously

$$x = x^l_n + \tfrac{3}{7} J_n , \quad f(x) = f(x^l_n) + (-1)^n . \tfrac{5}{9} J_n$$

(cf. (A)). If $x^l_{2n} \leqq x' \leqq x^l_{2n+1}$ then

$$f(x^l_{2n}) \leqq f(x') \leqq f(x^l_{2n}) + \tfrac{5}{6} J_{2n} ,$$
$$(\tfrac{3}{7} - \tfrac{3}{8}) \, J_{2n} \leqq x - x' \leqq J_{2n} .$$

Hence

$$|f(x) - f(x')| < aJ_{2n} , \quad |x - x'| > bJ_{2n}$$

where a, b are positive numbers independent of n; analogous inequalities hold provided

$$x^l_{2n+1} \leqq x' \leqq x^l_{2n+2} .$$

However, every point x' to the left from x belongs to one of the intervals $[x_n^l, x_{n+1}^l]$; thus it holds for all $x' < x$ that

$$\left|\frac{f(x) - f(x')}{x - x'}\right| < C$$

with $C > 0$ independent of x'. An analogous result holds for points $x'' > x$. Hence indeed all derivates at the point $x = 3/7$ are finite (and by Sec. IV the left derivates as well as the right ones are different).

On the other hand, we shall prove the following theorem:

There exists a point set which has the cardinality of continuum, such that $\Lambda(x) = +\infty$, $\lambda(x) = -\infty$, $\Lambda'(x) = +\infty$, $\lambda'(x) = -\infty$ *at every point of this set.*[9])

To this end, let us consider sequences (J) including only intervals J_n^{III} and triplets of consecutive intervals $(J_n^{\mathrm{II}}, J_{n+1}^{\mathrm{I}}, J_{n+2}^{\mathrm{IV}})$, both of these cases occurring infinitely many times. Each such sequence defines a certain point x, and the family F of these sequences defines a certain point set E. The points of the set E are not nodes, consequently, the correspondence between the points x and sequences (J) is one-to-one; hence the sets E, F have the same cardinality. On the other hand, a sequence (J) from F is uniquely determined by the order of the intervals J^{III} and of the above triplets. Hence we can write

$$(J) \equiv A_1, A_2, A_3, \ldots, A_n, \ldots,$$

where A_n stands either for an interval J^{III} or the above-mentioned triplet. Let us assign each A_n the number $\alpha_n = 0$ or $\alpha_n = 1$ if $A_n = J^{\mathrm{III}}$ or A_n is a triplet, respectively; then the set F corresponds in a one-to-one manner to the set E^* of numbers

$$\alpha = \frac{\alpha_1}{2^1} + \frac{\alpha_2}{2^2} + \ldots + \frac{\alpha_n}{2^n} + \ldots,$$

where each α_n is either 0 or 1 with the only condition that the right hand side includes infinitely many of both zeros and ones. Hence α is an arbitrary number in the interval $[0, 1]$ except numbers of the form $p/2^n$ (p, n integers). Thus the set E^* has the cardinality of continuum and the same holds for E.

It remains to prove that each point of the set E has the above-stated property. For

---

[9]) I use Dini's notation:

$$\Lambda(x) = \limsup_{x' \to x+} \frac{f(x) - f(x')}{x - x'}, \quad \lambda(x) = \liminf_{x' \to x+} \frac{f(x) - f(x')}{x - x'},$$

$$\Lambda'(x) = \limsup_{x' \to x-} \frac{f(x) - f(x')}{x - x'}, \quad \lambda'(x) = \liminf_{x' \to x-} \frac{f(x) - f(x')}{x - x'}.$$

every N there exists $n > N$ such that we have

$$J^{\mathrm{III}}_{n+1}, J^{\mathrm{II}}_{n+2}, J^{\mathrm{I}}_{n+3}, J^{\mathrm{IV}}_{n+4};$$

hence

$$k_{n+1} = \tfrac{5}{3}k_n, \quad k_{n+2} = -\tfrac{5}{3}k_n, \quad k_{n+3} = -(\tfrac{5}{3})^2 k_n.$$

Then there exist points $\langle x_n^i \rangle$, $\langle x_n^{ii} \rangle$, $\langle x_n^{iii} \rangle$, $\langle x_n^{iv} \rangle$ conjugate with $\langle x \rangle$ with respect to the intervals J_n, J_{n+1}, J_{n+2}, J_{n+3} so that

$$\lambda(x, x_n^i) = k_n, \quad \lambda(x, x_n^{ii}) = \tfrac{5}{3}k_n, \quad \lambda(x, x_n^{iii}) = -\tfrac{5}{3}k_n,$$
$$\lambda(x, x_n^{iv}) = -(\tfrac{5}{3})^2 k_n.$$

Evidently $x_n^i < x$, $x_n^{ii} > x$, $x_n^{iii} > x$, $x_n^{iv} < x$.

As both n and $|k_n|$ tend to infinity with N increasing to infinity, x_n^{ii}, x_n^{iii} approach x from the right and x_n^i, x_n^{iv} from the left, the theorem is proved.

Let us notice that the set E can be essentially extended and that some more details can be given concerning the derivates of Bolzano's function. Indeed, some corollaries of the above theorems can be seen immediately from the proofs presented here.

VI

Finally, I should like to make the following remark:

Let a point x be defined by a sequence (J); then $x_{n+1}^l = x_n^l + a_n J_n/8$, where $a_n = 0, 3, 4, 7$; further $J_{n+1} = b_n J_n/8$ where the values of b_n corresponding respectively to the above values of a_n are 3, 1, 3, 1. Further, $k_{n+1} = c_n' k_n$ where the values of c_n' are respectively $5/3$, -1, $5/3$, -1; and finally $y_{n+1}^l = y_n^l + d_n k_n J_n/8$ where d_n assumes the values 0, 5, 4, 9, respectively.

These recurrent formulae easily yield: Each point of the interval $[0, 1]$ can be written in the form

$$x = \frac{a_0}{8} + \frac{b_0 a_1}{8^2} + \frac{b_0 b_1 a_2}{8^3} + \frac{b_0 b_1 b_2 a_3}{8^4} + \dots;$$

then, introducing $b_n c_n' = c_n$, we obtain

$$f(x) = \frac{d_0}{8} + \frac{c_0 d_1}{8^2} + \frac{c_0 c_1 d_2}{8^3} + \frac{c_0 c_1 c_2 d_3}{8^4} + \dots,$$

where the corresponding values of the numbers a_n, b_n, c_n, d_n are given in the following table:

a_n	0	3	4	7
b_n	3	1	3	1
c_n	5	−1	5	−1
d_n	0	5	4	9

At the points of extrema we have $a_n = 4$ starting from a certain index n so that all terms starting from the same index form a geometrical series; we obtain

$$x_e = \frac{a_0}{8} + \frac{b_0 a_1}{8^2} + \dots + \frac{b_0 b_1 b_2 \dots b_{n-2} a_{n-1}}{8^n} + \frac{4 b_0 b_1 b_2 \dots b_{n-1}}{5.8^n},$$

$$f(x_e) = \frac{d_0}{8} + \frac{c_0 d_1}{8^2} + \dots + \frac{c_0 c_1 c_2 \dots c_{n-2} d_{n-1}}{8^n} + \frac{4 c_0 c_1 c_2 \dots c_{n-1}}{3.8^n}.$$

These formulae yield analytical expressions for all extrema except $x = 0$ provided $n, a_0, a_1, \dots, a_{n-1}$ assume successively all the admissible values ($n = 1, 2, 3, \dots, a_i = 0, 3, 4, 7$). It is evident that a maximum or a minimum occurs if $c_0 c_1 c_2 \dots c_{n-1}$ is positive or negative, respectively, or – which is the same – if $a_0 + a_1 + \dots + a_{n-1}$ (or $d_0 + d_1 + \dots + d_{n-1}$) is even or odd, respectively.

Introducing the parameter

$$\zeta = \frac{\alpha_0}{4} + \frac{\alpha_1}{4^2} + \dots + \frac{\alpha_{n-1}}{4^n} + \dots$$

and setting $a_n = 0, 3, 4, 7$ for $\alpha_n = 0, 1, 2, 3$, respectively, we obtain a parametric expression for Bolzano's function which was found by K. Petr and K. Rychlík.