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STRUCTURE OF THE UNIT GROUP OF THE GROUP ALGEBRAS OF NON-METABELIAN GROUPS OF ORDER 128

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Abstract. We characterize the unit group for the group algebras of non-metabelian groups of order 128 over the finite fields whose characteristic does not divide the order of the group. Up to isomorphism, there are 2328 groups of order 128 and only 14 of them are non-metabelian. We determine the Wedderburn decomposition of the group algebras of these non-metabelian groups and subsequently characterize their unit groups.

Keywords: non-metabelian groups; finite field; group algebra; unit group

MSC 2020: 16U60, 20C05

1. Introduction

Let $\mathbb{K}$ be the finite field of order $q = p^k$, where p is a prime number. Let $\mathbb{K}G$ denote the group algebra of the finite group G of order n over $\mathbb{K}$ and let $\mathcal{U}(\mathbb{K}G)$ denote the collection of units in $\mathbb{K}G$. Here, $\mathcal{U}(\mathbb{K}G)$ is called the unit group of $\mathbb{K}G$. The study of the unit groups is a very interesting and demanding problem since these units are employed in so many fields. For example, convolution codes can be constructed by the units (see [10]–[12]) and unit groups are also used to solve the various combinatorial number theoretical problems in [6].

Throughout this paper, we assume that p does not divide n. This means that the group algebra $\mathbb{K}G$ is semisimple (see [26]). To determine the unit group of a semisimple group algebra is an extensively studied research problem. One of the main advancements in this direction is due to the seminal work of Bakshi et al. (see [2]). The authors completely characterized the unit groups of semisimple group algebras of metabelian groups. We recall that a group is

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metabelian if its derived subgroup is abelian. Therefore, the research in this direction is focused only on the unit groups of semisimple group algebras of non-metabelian groups.

In some of the recent works, Sharma et al. (see [29], [30]) classified the unit group of the group algebras of some non-abelian groups. Tang et al. in [31] also made a contribution in this direction by studying the unit groups of the group algebras of some groups up to order 21. For the alternating group A_4, the unit group of the group algebra $\mathbb{K}A_4$ is characterized in [8], [28]. The unit groups of the group algebras of dihedral groups are studied in [4], [9], [16]–[18] and that of some symmetric/alternating groups are studied in [1], [14], [19], [28].

The unit group of the group algebras of non-metabelian groups of order 24 are discussed in [13], [15]. Recently, Mittal and Sharma (see [20]) studied the unit groups of semisimple group algebras of all non-metabelian groups up to order 72. Furthermore, Mittal and Sharma in [21], [22], [27] also studied the unit groups of semisimple group algebras of all groups of order 108 and 120 (except the symmetric group S_5). The unit group of the semisimple group algebra of S_n for any n is given in [1]. Furthermore, Mittal and Sharma in [23] characterized the unit group of group algebras of some groups of order 144.

Due to the seminal work of Pazderski (see [25]), one can completely isolate the possible orders of non-metabelian groups. Using [25], we observe that the next possible order of a non-metabelian group greater than 120 is 128 (one can manually compute it from the list of possible orders of non-metabelian groups given in [25]). We note that any group of order 128 is strongly monomial. Broche et al. discussed the Wedderburn decomposition of abelian-by-supersolvable group in [3] as the extension work of Olivieri et al., where the authors have computed the Wedderburn decomposition of a rational group algebra, see [24]. Since every abelian-by-supersolvable group is strongly monomial, one can compute the Wedderburn decomposition of groups of order 128 using the method of [3] but the method requires deep knowledge of recently discovered sophisticated algebraic concepts such as Shoda pairs, strong Shoda pairs and tools from character theory, which is a difficult task. In this work, we are studying the unit group of semisimple group algebras of non-metabelian groups of order 128 through an alternative technique, which does not require sophisticated methods and a deep knowledge of algebraic concepts. To be more precise, we use the results of [5] and [26] to compute the unit group via a comparative approach simpler than [3]. Using GAP (see [7]), we note that there are totally 2328 non-isomorphic groups of order 128 out of which only 14 are non-metabelian. We explicitly compute the unit groups of the semisimple group algebras of these 14 groups after computing their respective Wedderburn decompositions.

This paper is organized as follows. Section 2 deals with the preliminaries while Section 3 contains our main results on the structure of the unit group of all 14 non-metabelian groups of order 128. The final section is concluding in nature.

2. Preliminaries

In this paper, $\mathbb{K}$ denotes the finite field of order $q = p^k$ and G denotes the finite group. The definitions given below are as in [5].

Definition 2.1. An element $x \in G$ is called p-regular if $p \nmid |x|$, where $|x|$ is the order of x.

Let the least common multiple of the orders of all p-regular elements in G be denoted by s. Let the primitive sth root of unity over $\mathbb{K}$ be denoted by θ. Therefore, $\mathbb{K}(\theta)$ is the splitting field over $\mathbb{K}$ and $\mathrm{Gal}(\mathbb{K}(\theta)/\mathbb{K})$ denotes the Galois group of $\mathbb{K}(\theta)$ over $\mathbb{K}$. Since the Galois group $\mathrm{Gal}(\mathbb{K}(\theta)/\mathbb{K})$ is a cyclic group, for any $\sigma \in \mathrm{Gal}(\mathbb{K}(\theta)/\mathbb{K})$ there exists some $t \in \mathbb{Z}_s^*$ such that $\sigma(\theta) = \theta^t$. Therefore, we define the set

$$T_{G,\mathbb{K}} = \{t \in \mathbb{Z}_s^* \colon \ \sigma(\theta) = \theta^t, \ \text{where } \sigma \in \mathrm{Gal}\,(\mathbb{K}(\theta)/\mathbb{K})\}.$$

Clearly, $T_{G,\mathbb{K}}$ is the subset of the multiplicative group $\mathbb{Z}_s^*$. Further, using the set $T_{G,\mathbb{K}}$, we define another set $S_{\mathbb{K}}(\gamma_g)$.

Definition 2.2. For any p-regular element $g \in G$, let γ_g denote the sum of all conjugates of $g \in G$. In other words, $\gamma_g = \sum\limits_{h \in C_g} h$, where C_g is the conjugacy class of $g \in G$. Then the cyclotomic $\mathbb{K}$-class of γ_g is the set defined as

$$S_{\mathbb{K}}(\gamma_g) = \{\gamma_{g^t} \colon \ t \in T_{G,\mathbb{K}}\}.$$

The cardinality of cyclotomic $\mathbb{K}$-classes would play an important role in determining the degrees of the extensions of the simple components of $Z(\mathbb{K}G/J(\mathbb{K}G))$ (see Propositions 2.1 and 2.2).

Proposition 2.1 ([5], Proposition 1.2)**.** *The number of simple components of $\mathbb{K}G/J(\mathbb{K}G)$ is equal to the number of cyclotomic $\mathbb{K}$-classes in G, where $J(\mathbb{K}G)$ is the Jacobson radical of $\mathbb{K}G$.*

Proposition 2.2 ([5], Theorem 1.3)**.** *Suppose the Galois group $\mathrm{Gal}\,(\mathbb{K}(\theta) : \mathbb{K})$ is cyclic and t is the number of cyclotomic $\mathbb{K}$-classes in G. If $K_1, K_2, \ldots, K_t$ are the simple components of $Z(\mathbb{K}G/J(\mathbb{K}G))$ and $S_1, S_2, \ldots, S_t$ are the cyclotomic $\mathbb{K}$-classes of G, then $|S_i| = [K_i : \mathbb{K}]$ with a suitable ordering of the indices.*

Lemma 2.1 ([26], Proposition 3.6.11)**.** *Let $\mathbb{K}G$ be a semi-simple group algebra and G' be the commutator subgroup of G. Then*

$$\mathbb{K}G \simeq \mathbb{K}G_{e_{G'}} \oplus \triangle(G, G'),$$

where $\mathbb{K}G_{e_{G'}} = \mathbb{K}(G/G')$ is the sum of all commutative simple components of $\mathbb{K}G$ and $\triangle(G, G')$ is the sum of all others.

3. Main result

Throughout this section, let $\mathbb{K}$ denote the finite field of characteristic p, where p is an odd prime number. Let the commutator of any two elements x, y of any group G be denoted as $x^{-1}y^{-1}xy = [x, y]$. We study the unit groups of the semisimple group algebras of groups of order 128 in the subsequent 14 subsections. But before that, we explicitly write the structure of 14 non-metabelian groups of order 128 as follows:

(1) $(C_4 \rtimes C_8) \rtimes C_2) \rtimes C_2$,
(2) $((C_8 \rtimes C_4) \rtimes C_2) \rtimes C_2$,
(3) $((C_8 \rtimes C_4) \rtimes C_2) \rtimes C_2$,
(4) $(C_2 \times C_2) \cdot (C_4 \times C_2) \rtimes C_2$,
(5) $(((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_2) \rtimes C_2$,
(6) $(((C_4 \times C_4) \rtimes C_2) \rtimes C_2) \rtimes C_2$,
(7) $(((C_2 \times C_2 \times C_2) \rtimes C_4) \rtimes C_2) \rtimes C_2$,
(8) $(((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_2) \rtimes C_2$,
(9) $(((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_2) \rtimes C_2$,
(10) $(((C_2 \times Q_8) \rtimes C_2) \rtimes C_2) \rtimes C_2$,
(11) $(((C_2 \times Q_8) \rtimes C_2) \rtimes C_2) \rtimes C_2$,
(12) $(((C_4 \times C_4) \rtimes C_2) \rtimes C_2) \rtimes C_2$,
(13) $(((C_2 \times Q_8) \rtimes C_2) \rtimes C_2) \rtimes C_2$,
(14) $(Q_8 \times Q_8) \rtimes C_2$.

The structure description, presentation, conjugacy classes and the commutator subgroups of all the groups are computed through the GAP software (see [7]).

3.1. Unit group of $\mathbb{K}_q G_1$, where $G_1 = ((C_4 \rtimes C_8) \rtimes C_2) \rtimes C_2$. The presentation of the group G_1 is given by

$$\begin{aligned} G_1 = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \colon\ & x_1^2 x_4^{-1}, [x_2, x_1]x_3^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1], [x_5, x_1]x_6^{-1}, \\ & [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2, [x_3, x_2], [x_4, x_2]x_7^{-1}x_5^{-1}, [x_5, x_2]x_7^{-1}, [x_6, x_2]x_7^{-1}, \\ & [x_7, x_2], [x_4, x_3]x_7^{-1}x_6^{-1}, x_3^2, [x_5, x_3]x_7^{-1}, [x_6, x_3], [x_7, x_3], x_4^2, [x_5, x_4]x_7^{-1}, \\ & [x_6, x_4], [x_7, x_4], x_5^2 x_7^{-1}, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2 \rangle. \end{aligned}$$

There are a total of 17 conjugacy classes for group G_1 as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_4	x_2x_4
Size	1	16	8	8	8	4	2	1	8	16	16
Order	1	4	2	2	2	4	2	2	8	4	8

Representative	x_2x_6	x_3x_4	$x_1x_2x_4$	$x_1x_2x_7$	$x_3x_4x_5$	$x_1x_2x_4x_5$
Size	8	4	8	8	4	8
Order	4	4	8	8	4	8

The exponent of the group G_1 is 8 and the commutator subgroup of G_1 is $G_1' = C_2 \times D_8$. Consequently, the factor group $G_1/G_1' \simeq C_4 \times C_2$. Also, as $p > 2$, the group algebra $\mathbb{K}_q G_1$ is semisimple.

Theorem 3.1. *Let G_1 be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

(1) *for k even or $p^k \equiv 1 \bmod 8$, $\mathcal{U}(\mathbb{K}_q G_1) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^7$,*

(2) *for $p^k \equiv \{3,7\} \bmod 8$, $\mathcal{U}(\mathbb{K}_q G_1) \simeq (\mathbb{K}_q^*)^4 \oplus (\mathbb{K}_{q^{*2}})^2 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^5 \oplus GL_4(\mathbb{K}_{q^2})$,*

(3) *For $p^k \equiv 5 \bmod 8$, $\mathcal{U}(\mathbb{K}_q G_1) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^3 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.*

Proof. *Case 1:* k is even in $q = p^k$. Since k is even, $p^k \equiv 1 \bmod 8 \Rightarrow |S_{\mathbb{K}}(\gamma_g)| = 1$ for all $g \in G_1$. The Wedderburn decomposition of $\mathbb{K}_q G_1$ is given by $\mathbb{K}_q G_1 \simeq \bigoplus_{r=1}^{17} M_{n_r}(\mathbb{K}_q)$. By Lemma 2.1, it holds that

$$\mathbb{K}_q G_1 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{9} M_{n_r}(\mathbb{K}_q), \quad n_r \geqslant 2 \Rightarrow 120 = \sum_{r=1}^{9} n_r^2.$$

The values of n_r can be

$$(2^{(6)}, 4^{(2)}, 8), \quad (2^{(5)}, 5^{(4)}), \quad (2^{(4)}, 3^{(2)}, 5^{(2)}, 6), \quad (2^{(3)}, 3^{(4)}, 6^{(2)}), \quad (2^{(3)}, 3^{(3)}, 4^{(2)}, 7),$$
$$(2^{(2)}, 4^{(7)}), \quad (2, 3^{(3)}, 4^{(4)}, 5), \quad \text{and} \quad (3^{(6)}, 4, 5^{(2)}).$$

To find it uniquely, we take the normal subgroup $N_1 = \langle x_7 \rangle$ of G_1. The corresponding factor group $H_1 = G_1/N_1 \simeq ((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$ has the following conjugacy classes. We remark that every element here is a coset.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_1x_2	x_1x_4	x_2x_4	x_2x_6	x_3x_4	$x_1x_2x_4$
Size	1	8	4	4	4	2	1	8	8	8	4	4	8
Order	1	4	2	2	2	2	2	8	4	4	2	4	8

We note that k is even and so, $|S_{\mathbb{K}}(\gamma_h)| = 1$ for all $h \in H_1$. The commutator subgroup of H_1 is $H_1' = C_2 \times C_2 \times C_2$ and $H_1/H_1' \simeq C_4 \times C_2$. By Lemma 2.1, it holds that

$$\mathbb{K}_q H_1 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{K}_q) \Rightarrow 56 = \sum_{r=1}^{5} n_r^2 \Rightarrow n_r = (2^{(2)}, 4^{(3)}), (2, 3^{(3)}, 5).$$

This reduces the choices of n_r's in the decomposition of $\mathbb{K}_q G_1$ only to $(2^{(2)}, 4^{(7)})$, $(2, 3^{(3)}, 4^{(4)}, 5)$. Again, to find the uniqueness, we consider the normal subgroup $N_2 = \langle x_6, x_7 \rangle$. The conjugacy classes of the factor group $H_2 = G_1/N_2 = (C_2 \times C_2 \times C_2) \rtimes C_4$ are given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_4	x_2x_4	x_3x_4	$x_1x_2x_4$
Size	1	4	4	2	2	1	4	4	4	2	4
Order	1	4	2	2	2	2	4	4	4	2	4

Here, $|S_{\mathbb{K}}(\gamma_h)| = 1$ for all $h \in H_2$. The commutator subgroup of H_2 is $H_2' = C_2 \times C_2$ and $H_2/H_2' \simeq C_4 \times C_2$. By Lemma 2.1, we see that

$$\mathbb{K}_q H_2 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{3} M_{n_r}(\mathbb{K}_q) \Rightarrow 24 = \sum_{r=1}^{3} n_r^2 \Rightarrow n_r = (2^{(2)}, 4).$$

This uniquely tells the choice of n_r's for G_1 to be $(2^{(2)}, 4^{(7)})$. Consequently, we have

$$\mathbb{K}_q G_1 \simeq (\mathbb{K}_q)^8 \oplus (M_2(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_q))^7.$$

It is straightforward to deduce the unit group from the knowledge of Wedderburn decomposition.

Case 2: k is odd and $p^k \equiv 1 \bmod 8$. The result is the same as in Case 1.

Case 3: k is odd; $p^k \equiv 3 \bmod 8$ or $p^k \equiv 7 \bmod 8$. We have

- ▷ for $p^k \equiv 3 \bmod 8$, $S_{\mathbb{K}}(\gamma_{x_1}) = \{\gamma_{x_1}, \gamma_{x_1x_4}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4x_5}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_2x_4}) = \{\gamma_{x_1x_2x_4}, \gamma_{x_1x_2x_7}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_1$,
- ▷ for $p^k \equiv 7 \bmod 8$, $S_{\mathbb{K}}(\gamma_{x_1}) = \{\gamma_{x_1}, \gamma_{x_1x_4}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_2x_7}) = \{\gamma_{x_1x_2x_4x_5}, \gamma_{x_1x_2x_7}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_1$.

In both cases, by Lemma 2.1, the Wedderburn decomposition is given by

$$\mathbb{K}_q G_1 \simeq (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \bigoplus_{r=1}^{7} M_{n_r}(\mathbb{K}_q) \oplus M_{n_8}(\mathbb{K}_{q^2}), \quad n_r \geqslant 2 \Rightarrow 120 = \sum_{r=1}^{7} n_r^2 + 2 \cdot n_8^2.$$

The values of n_r can be

$$(2^{(6)}, 4^{(2)}, 8), \quad (2^{(5)}, 5^{(4)}), \quad (2^{(4)}, 3^{(2)}, 5^{(2)}, 6), \quad (2^{(3)}, 3^{(4)}, 6^{(2)}), \quad (2^{(3)}, 3^{(3)}, 4^{(2)}, 7),$$
$$(2^{(2)}, 4^{(7)}), \quad (2, 3^{(3)}, 4^{(4)}, 5), \quad \text{and} \quad (3^{(6)}, 4, 5^{(2)}).$$

To find the unique choice, we consider the factor groups corresponding to the normal subgroups N_1 and N_2 as in Case 1. For H_1, $S_{\mathbb{K}}(\gamma_{x_1}) = \{\gamma_{x_1}, \gamma_{x_1x_4}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in H_1$. This means that

$$\mathbb{K}_qH_1 \simeq (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{K}_q), \quad n_r \geqslant 2 \Rightarrow 56 = \sum_{r=1}^{5} n_r^2.$$

The values of n_r can be $(2^{(2)}, 4^{(3)})$, $(2, 3^{(3)}, 5)$. This reduces the choices of n_r's in the decomposition of $\mathbb{K}G_1$ as $(2^{(2)}, 4^{(7)})$, $(2, 3^{(3)}, 4^{(4)}, 5)$. Next, for H_2, $S_{\mathbb{K}}(\gamma_{x_1}) = \{\gamma_{x_1}, \gamma_{x_1x_4}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in H_2$. This provides that

$$\mathbb{K}_qH_2 \simeq (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \bigoplus_{r=1}^{3} M_{n_r}(\mathbb{K}_q), \quad n_r \geqslant 2 \Rightarrow 24 = \sum_{r=1}^{3} n_r^2.$$

The values of n_r's are $(2^{(2)}, 4)$. This reduces the choices of n_r's in the decomposition of $\mathbb{K}G_1$ as $(2^{(2)}, 4^{(7)})$. Hence, we have

$$\mathbb{K}_qG_1 \simeq (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \oplus (M_2(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_q))^5 \oplus M_4(\mathbb{K}_{q^2}).$$

Case 4: k is odd and $p^k \equiv 5 \bmod 8$. In this case, we have $S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_7}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_2x_4}) = \{\gamma_{x_1x_2x_4}, \gamma_{x_1x_2x_4x_5}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_1$. By Lemma 2.1, the Wedderburn decomposition is given by

$$\mathbb{K}_qG_1 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{K}_q) \bigoplus_{r=6}^{7} M_{n_r}(\mathbb{K}_{q^2}), \quad n_r \geqslant 2 \Rightarrow 120 = \sum_{r=1}^{5} n_r^2 + 2 \cdot n_6^2 + 2 \cdot n_7^2.$$

To find the values of n_r's uniquely, we again consider the factor groups corresponding to normal subgroups N_1 and N_2 as in Case 1. For H_1, $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$ for all $g \in H_1$. Therefore, we have

$$\mathbb{K}_qH_1 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{K}_q) \Rightarrow 56 = \sum_{r=1}^{5} n_r^2.$$

This gives the possibilities $(2^{(2)}, 4^{(3)})$ and $(2, 3^{(3)}, 5)$. Again, for H_2, $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$ for all $g \in H_2$. Consequently, we have

$$\mathbb{K}_qH_2 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{3} M_{n_r}(\mathbb{K}_q) \Rightarrow 24 = \sum_{r=1}^{3} n_r^2.$$

This gives the only possibility $(2^{(2)}, 4)$. Thus, we have

$$\mathbb{K}_qG_1 \simeq (\mathbb{K}_q)^8 \oplus (M_2(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_q))^3 \oplus (M_4(\mathbb{K}_{q^2}))^2.$$

This completes the proof. □

3.2. Unit group of $\mathbb{K}_qG_2$, where $G_2 = ((C_8 \rtimes C_4) \rtimes C_2) \rtimes C_2$. The presentation of the group G_2 is given by

$$G_2 = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \colon \, x_1^2x_4^{-1}, [x_2, x_1]x_3^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1], [x_5, x_1]x_6^{-1}, [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2, [x_3, x_2], [x_4, x_2]x_7^{-1}x_5^{-1}, [x_5, x_2]x_7^{-1}, [x_6, x_2]x_7^{-1}, [x_7, x_2], x_3^2, [x_4, x_3]x_7^{-1}x_6^{-1}, [x_5, x_3]x_7^{-1}, [x_6, x_3], [x_7, x_3], x_4^2x_7^{-1}, [x_5, x_4]x_7^{-1}, x_6^2, [x_6, x_4], [x_7, x_4], x_5^2x_7^{-1}, [x_6, x_5], [x_7, x_5], [x_7, x_6], x_7^2 \rangle.$$

There are totally 17 conjugacy classes for group G_2 as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_4	x_2x_4	x_2x_6	x_3x_4	$x_1x_2x_4$	$x_1x_2x_7$	$x_3x_4x_5$	$x_1x_2x_4x_5$
Size	1	16	8	8	8	4	2	1	8	16	16	8	4	8	8	4	8
Order	1	8	2	2	4	4	2	2	8	8	8	4	4	8	8	4	8

The exponent of the group G_2 is 8 and the commutator subgroup of G_2 is $G_2' = C_2 \times D_8$. The factor group $G_2/G_2' \simeq C_4 \times C_2$. Also, we observe that the group algebra $\mathbb{K}_qG_2$ is semisimple.

Theorem 3.2. *Let G_2 be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

(1) *for k even or $p^k \equiv 1 \bmod 8$ or $p^k \equiv 5 \bmod 8$, $\mathcal{U}(\mathbb{K}_qG_2) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^7$.*

(2) *for $p^k \equiv 3 \bmod 8$ or $p^k \equiv 7 \bmod 8$, $\mathcal{U}(\mathbb{K}_qG_2) \simeq (\mathbb{K}_q^*)^4 \oplus (\mathbb{K}_{q^{*2}})^2 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^3 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.*

Proof. *Case 1:* k is even in $q = p^k$. Since k is even, $p^k \equiv 1 \bmod 8 \Rightarrow |S_{\mathbb{K}}(\gamma_g)| = 1$ for all $g \in G_2$. The Wedderburn decomposition of $\mathbb{K}_qG_2$ is given by

$$\mathbb{K}_qG_2 \simeq \bigoplus_{r=1}^{17} M_{n_r}(\mathbb{K}_q).$$

By Lemma 2.1, we have $\mathbb{K}_qG_2 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{9} M_{n_r}(\mathbb{K}_q)$, $n_r \geqslant 2 \Rightarrow 120 = \sum_{r=1}^{9} n_r^2$. The values of n_r can be $(2^{(6)}, 4^{(2)}, 8)$, $(2^{(5)}, 5^{(4)})$, $(2^{(4)}, 3^{(2)}, 5^{(2)}, 6)$, $(2^{(3)}, 3^{(4)}, 6^{(2)})$, $(2^{(3)}, 3^{(3)}, 4^{(2)}, 7)$, $(2^{(2)}, 4^{(7)})$, $(2, 3^{(3)}, 4^{(4)}, 5)$ and $(3^{(6)}, 4, 5^{(2)})$. For uniqueness, we consider the normal subgroup $N_1 = \langle x_7 \rangle$ and the factor group is $H_1 = G_2/N_1 \simeq$

$((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$, which is the same as in the case of H_1 already in Theorem 3.1. This implies that

$$\mathbb{K}_q H_1 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{K}_q) \Rightarrow 56 = \sum_{r=1}^{5} n_r^2 \Rightarrow n_r = (2^{(2)}, 4^{(3)}), (2, 3^{(3)}, 5).$$

This reduces the choices of n_r's in the decomposition of $\mathbb{K}_q G_2$ to $(2^{(2)}, 4^{(7)})$, $(2, 3^{(3)}, 4^{(4)}, 5)$. Again, to find the unique solution, we consider the normal subgroup $N_2 = \langle x_6, x_7 \rangle$. We observe that the factor group $H_2 = G_2/N_2 = (C_2 \times C_2 \times C_2) \rtimes C_4$ is the same as H_2 in Theorem 3.1. Hence, we have

$$\mathbb{K}_q H_2 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{3} M_{n_r}(\mathbb{K}_q) \Rightarrow 24 = \sum_{r=1}^{3} n_r^2 \Rightarrow n_r = (2^{(2)}, 4).$$

Therefore, the choices of n_r's for G_2 are reduced to $(2^{(2)}, 4^{(7)})$, which provides that

$$\mathbb{K}_q G_2 \simeq (\mathbb{K}_q)^8 \oplus (M_2(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_q))^7.$$

Case 2: k is odd and $p^k \equiv 1 \bmod 8$. The result is the same as Case 1.

Case 3: k is odd; $p^k \equiv 3 \bmod 8$ and $p^k \equiv 7 \bmod 8$. We note that

$$S_{\mathbb{K}}(\gamma_{x_1}) = \{\gamma_{x_1}, \gamma_{x_1x_4}\}, \quad S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4x_5}\},$$
$$S_{\mathbb{K}}(\gamma_{x_3x_4}) = \{\gamma_{x_3x_4}, \gamma_{x_3x_4x_5}\}, \quad S_{\mathbb{K}}(\gamma_{x_1x_2x_4}) = \{\gamma_{x_1x_2x_7}, \gamma_{x_1x_2x_4}\}$$

and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_2$. We use Lemma 2.1 to write the Wedderburn decomposition as

$$\mathbb{K}_q G_2 \simeq (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{K}_q) \bigoplus_{r=6}^{7} M_{n_r}(\mathbb{K}_{q^2}), \quad n_r \geqslant 2$$
$$\Rightarrow 120 = \sum_{r=1}^{5} n_r^2 + 2 \cdot n_6^2 + 2 \cdot n_7^2.$$

The values of n_r can be

$(2^{(6)}, 4^{(2)}, 8)$, $(2^{(5)}, 5^{(4)})$, $(2^{(4)}, 3^{(2)}, 5^{(2)}, 6)$, $(2^{(3)}, 3^{(4)}, 6^{(2)})$, $(2^{(3)}, 3^{(3)}, 4^{(2)}, 7)$, $(2^{(2)}, 4^{(7)})$, $(2, 3^{(3)}, 4^{(4)}, 5)$, and $(3^{(6)}, 4, 5^{(2)})$.

To uniquely find the Wedderburn decomposition, we consider the factor groups H_1 and H_2 as in Case 1. For H_1, we have $S_{\mathbb{K}}(\gamma_{x_1}) = \{\gamma_{x_1}, \gamma_{x_1x_4}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4}\}$, and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in H_1$. This immediately implies that

$$\mathbb{K}_q H_1 \simeq (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{K}_q), \quad n_r \geqslant 2 \Rightarrow 56 = \sum_{r=1}^{5} n_r^2.$$

The values of n_r can be $(2^{(2)}, 4^{(3)}), (2, 3^{(3)}, 5)$. This reduces the choices of n_r's in the decomposition of $\mathbb{K}G_2$ to $(2^{(2)}, 4^{(7)})$, $(2, 3^{(3)}, 4^{(4)}, 5)$. Now, for H_2, $S_{\mathbb{K}}(\gamma_{x_1}) = \{\gamma_{x_1}, \gamma_{x_1x_4}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in H_2$. Consequently, we get

$$\mathbb{K}_qH_2 \simeq (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \bigoplus_{r=1}^{3} M_{n_r}(\mathbb{K}_q), \quad n_r \geqslant 2 \Rightarrow 24 = \sum_{r=1}^{3} n_r^2.$$

The values of n_r's are $(2^{(2)}, 4)$. This reduces the choices of n_r's in the decomposition of $\mathbb{K}G_2$ to $(2^{(2)}, 4^{(7)})$. Hence, we have

$$\mathbb{K}_qG_2 \simeq (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \oplus (M_2(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_q))^3 \oplus (M_4(\mathbb{K}_{q^2}))^2.$$

Case 4: k is odd and $p^k \equiv 5 \bmod 8$. In this case, $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$ for all $g \in G_2$. Therefore, the result is the same as in Case 1. This completes the proof. □

3.3. Unit group of $\mathbb{K}_qG_3$, where $G_3 = ((C_8 \rtimes C_4) \rtimes C_2) \rtimes C_2$. The presentation of the group G_3 is given as follows.

$$\begin{aligned} G_3 = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \colon\ & x_1^2x_4^{-1}, [x_2, x_1]x_3^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1], [x_5, x_1]x_6^{-1}, \\ & [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2, [x_3, x_2]x_7^{-1}, [x_4, x_2]x_5^{-1}, [x_5, x_2]x_7^{-1}, [x_6, x_2]x_7^{-1}, \\ & [x_7, x_2], x_3^2x_7^{-1}, [x_4, x_3]x_7^{-1}x_6^{-1}, [x_5, x_3]x_7^{-1}, [x_6, x_3], [x_7, x_3], x_4^2, \\ & [x_5, x_4]x_7^{-1}, [x_6, x_4], [x_7, x_4], x_5^2x_7^{-1}, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2\rangle. \end{aligned}$$

It can be seen that the groups G_2 and G_3 are non-isomorphic to each other due to different actions of the group C_2 onto the group $(C_8 \rtimes C_4) \rtimes C_2$ (see [7]). More specifically, the group G_3 is different (non-isomorphic) from G_2 because of the following reasons:

- ▷ In $G_2, [x_3, x_2] = e$, whereas in $G_3, [x_3, x_2] = x_7$.
- ▷ In $G_2, [x_4, x_2] = x_5x_7$, whereas in $G_3, [x_4, x_2] = x_5$.
- ▷ In $G_2, x_3^2 = e$, whereas in $G_3, x_3^2 = x_7$.
- ▷ In $G_2, x_4^2 = x_7$, whereas in $G_3, x_4^2 = e$.

There are totally 17 conjugacy classes for group G_3 as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_4	x_2x_4
Size	1	16	8	8	8	4	2	1	8	16	16
Order	1	4	2	4	2	4	2	2	8	4	8

Representative	x_2x_6	x_3x_4	$x_1x_2x_4$	$x_1x_2x_5$	$x_3x_4x_5$	$x_1x_2x_4x_6$
Size	8	4	8	8	4	8
Order	4	4	8	8	4	8

The exponent of the group G_3 is 8 and the commutator subgroup of G_3 is $G_3' = C_2 \times Q_8$. So, the factor group $G_3/G_3' \simeq C_4 \times C_2$. Also, the group algebra $\mathbb{K}_q G_3$ is semisimple.

Theorem 3.3. *Let G_3 be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

(1) *for k even or $p^k \equiv 1 \bmod 8$ or $p^k \equiv 5 \bmod 8$,* $\mathcal{U}(\mathbb{K}_q G_3) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^7$,

(2) *for $p^k \equiv 3 \bmod 8$ or $p^k \equiv 7 \bmod 8$,* $\mathcal{U}(\mathbb{K}_q G_3) \simeq (\mathbb{K}_q^*)^4 \oplus (\mathbb{K}_{q^{*2}})^2 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^3 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.

Proof. The proof is the same as that of Theorem 3.2 and we skip it here. □

3.4. Unit group of $\mathbb{K}_q G_4$, where $G_4 = ((C_2 \times C_2).(C_4 \times C_2) \rtimes C_2)$. The presentation of the group G_4 is given by

$$G_4 = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 :\ x_1^2 x_4^{-1}, [x_2, x_1]x_3^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1], [x_5, x_1]x_6^{-1}, [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2, [x_3, x_2]x_7^{-1}, [x_4, x_2]x_5^{-1}, [x_5, x_2]x_7^{-1}, [x_6, x_2]x_7^{-1}, [x_7, x_2], x_3^2 x_7^{-1}, [x_4, x_3]x_7^{-1}x_6^{-1}, [x_5, x_3]x_7^{-1}, [x_6, x_3], [x_7, x_3], x_4^2 x_7^{-1}, [x_5, x_4]x_7^{-1}, [x_6, x_4], [x_7, x_4], x_5^2 x_7^{-1}, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2\rangle.$$

There are totally 17 conjugacy classes for group G_4 as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_4	x_2x_4
Size	1	16	8	8	8	4	2	1	8	16	16
Order	1	8	2	4	4	4	2	2	8	8	8

Representative	x_2x_6	x_3x_4	$x_1x_2x_4$	$x_1x_2x_5$	$x_3x_4x_5$	$x_1x_2x_4x_6$
Size	8	4	8	8	4	8
Order	4	4	8	8	4	8

The exponent of the group G_4 is 8 and the commutator subgroup of G_4 is $G_4' = C_2 \times Q_8$. Thus, the factor group $G_4/G_4' \simeq C_4 \times C_2$. Also, we observe that the group algebra $\mathbb{K}_q G_4$ is semisimple.

Theorem 3.4. *Let G_4 be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

(1) *for k even or $p^k \equiv 1 \bmod 8$,* $\mathcal{U}(\mathbb{K}_q G_4) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^7$,

(2) *for $p^k \equiv 3 \bmod 8$ or $p^k \equiv 7 \bmod 8$,* $\mathcal{U}(\mathbb{K}_q G_4) \simeq (\mathbb{K}_q^*)^4 \oplus (\mathbb{K}_{q^{*2}})^2 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^5 \oplus GL_4(\mathbb{K}_{q^2})$,

(3) *For $p^k \equiv 5 \bmod 8$,* $\mathcal{U}(\mathbb{K}_q G_4) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^3 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.

Proof. The proof is the same as that of Theorem 3.1 and we skip it here. □

3.5. Unit group of $\mathbb{K}_q G_5$, where $G_5 = (((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_2) \rtimes C_2$. The presentation of the group G_5 is given by

$$G_5 = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \colon\ x_1^2, [x_2, x_1]x_4^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1], [x_5, x_1], [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2, [x_3, x_2], [x_4, x_2], [x_5, x_2]x_7^{-1}x_6^{-1}, [x_6, x_2], [x_7, x_2], x_3^2, [x_4, x_3]x_6^{-1}, [x_5, x_3], [x_6, x_3], [x_7, x_3], x_4^2, [x_5, x_4]x_7^{-1}, [x_6, x_4], [x_7, x_4], x_5^2, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2\rangle.$$

There are 20 conjugacy classes for group G_5 as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_3	x_1x_6	x_2x_3
Size	1	8	4	4	4	4	2	1	16	16	8	8
Order	1	2	2	2	2	2	2	2	4	4	4	2

	x_2x_5	x_2x_6	x_3x_4	x_3x_7	x_4x_5	$x_1x_2x_3$	$x_2x_3x_4$	$x_2x_3x_4x_7$
Size	8	4	8	4	4	16	4	4
Order	4	2	4	2	4	8	4	4

The exponent of the group G_5 is 8 and the commutator subgroup of G_5 is $G_5' = C_2 \times D_8$. So, the factor group $G_5/G_5' \simeq C_2 \times C_2 \times C_2$. Also, we observe that the group algebra $\mathbb{K}_q G_5$ is semisimple.

Theorem 3.5. *Let G_5 be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

$$\mathcal{U}(\mathbb{K}_q G_5) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^6.$$

P r o o f. It can be verified that for any p^k, $|S_{\mathbb{K}}(\gamma_g)| = 1$ for all $g \in G_5$. Consequently, the Wedderburn decomposition of $\mathbb{K}_q G_5$ is given by

$$\mathbb{K}_q G_5 \simeq \bigoplus_{r=1}^{20} M_{n_r}(\mathbb{K}_q).$$

By Lemma 2.1, we have $\mathbb{K}_q G_5 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{12} M_{n_r}(\mathbb{K}_q)$, $n_r \geqslant 2 \Rightarrow 120 = \sum_{r=1}^{12} n_r^2$. The values of n_r can be

$$(2^{(10)}, 4, 8),\quad (2^{(7)}, 3^{(3)}, 4, 7),\quad (2^{(6)}, 4^{(6)}),\quad (2^{(5)}, 3^{(3)}, 4^{(3)}, 5),$$
$$(2^{(4)}, 3^{(6)}, 5^{(2)}),\quad \text{and}\quad (2^{(3)}, 3^{(8)}, 6).$$

To find it uniquely, we take the normal subgroup $N = \langle x_7\rangle$ and the factor group $H = G_5/N \simeq ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2) \rtimes C_2$. This factor group has the following conjugacy classes.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_1x_2	x_1x_3	x_1x_6	x_2x_3	x_2x_5	x_3x_4	x_4x_5	$x_1x_2x_3$	$x_2x_3x_4$
Size	1	4	4	4	2	2	1	8	8	4	4	4	4	2	8	4
Order	1	2	2	2	2	2	2	4	4	2	2	4	4	2	4	4

For all p^k, it can be verified that $|S_{\mathbb{K}}(\gamma_g)| = 1$ for all $g \in G_5$. Also, $H/H' \simeq C_2 \times C_2 \times C_2$. Therefore, we have

$$\mathbb{K}_q H \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{8} M_{n_r}(\mathbb{K}_q) \Rightarrow 56 = \sum_{r=1}^{8} n_r^2 \Rightarrow n_r = (2^{(6)}, 4^{(2)}).$$

This reduces the choices of n_r's in the decomposition of $\mathbb{K}_q G_5$ to $(2^{(6)}, 4^{(6)})$. Hence, we have

$$\mathbb{K}_q G_5 \simeq (\mathbb{K}_q)^8 \oplus (M_2(\mathbb{K}_q))^6 \oplus (M_4(\mathbb{K}_q))^6.$$

This completes the proof. □

3.6. Unit group of $\mathbb{K}_q G_6$, where $G_6 = (((C_4 \times C_4) \rtimes C_2) \rtimes C_2) \rtimes C_2$. The presentation of the group G_6 is given by

$$\begin{aligned} G_6 = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \colon\ & x_1^2, [x_2, x_1]x_4^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1], [x_5, x_1], \\ & [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2 x_7^{-1}, [x_3, x_2], [x_4, x_2], [x_5, x_2]x_7^{-1}x_6^{-1}, [x_6, x_2], \\ & [x_7, x_2], x_3^2, [x_4, x_3]x_6^{-1}, [x_5, x_3], [x_6, x_3], [x_7, x_3], x_4^2, [x_5, x_4]x_7^{-1}, \\ & [x_6, x_4], [x_7, x_4], x_5^2, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2 \rangle. \end{aligned}$$

There are 20 conjugacy classes for group G_6 as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_3	x_1x_6	x_2x_3	x_2x_5	x_2x_6	x_3x_4	x_3x_7	x_4x_5	$x_1x_2x_3$	$x_2x_3x_4$	$x_2x_3x_4x_7$
Size	1	8	4	4	4	4	2	1	16	16	8	8	8	4	8	4	4	16	4	4
Order	1	2	4	2	2	2	2	2	4	4	4	4	4	4	4	2	4	8	4	4

The exponent of the group G_6 is 8 and the commutator subgroup of G_6 is $G_6' = C_2 \times D_8$ and so, the factor group $G_6/G_6' \simeq C_2 \times C_2 \times C_2$. Also, we observe that the group algebra $\mathbb{K}_q G_6$ is semisimple.

Theorem 3.6. *Let G_6 be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

- ▹ *for k even or $p^k \equiv \{1,5\} \bmod 8$, $\mathcal{U}(\mathbb{K}_q G_6) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^6$,*
- ▹ *for $p^k \equiv \{3,7\} \bmod 8$, $\mathcal{U}(\mathbb{K}_q G_6) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.*

Proof. *Case 1:* k is even or $p^k \equiv 1 \bmod 8$ or $p^k \equiv 5 \bmod 8$. For all these possibilities, we have $|S_{\mathbb{K}}(\gamma_g)| = 1$ for all $g \in G_6$. The Wedderburn decomposition of $\mathbb{K}_q G_6$ is given by

$$\mathbb{K}_q G_6 \simeq \bigoplus_{r=1}^{20} M_{n_r}(\mathbb{K}_q).$$

By Lemma 2.1, we get $\mathbb{K}_q G_6 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{12} M_{n_r}(\mathbb{K}_q)$, $n_r \geqslant 2 \Rightarrow 120 = \sum_{r=1}^{12} n_r^2$. The values of n_r can be

$$(2^{(10)}, 4, 8), \quad (2^{(7)}, 3^{(3)}, 4, 7), \quad (2^{(6)}, 4^{(6)}), \quad (2^{(5)}, 3^{(3)}, 4^{(3)}, 5),$$
$$(2^{(4)}, 3^{(6)}, 5^{(2)}) \quad \text{and} \quad (2^{(3)}, 3^{(8)}, 6).$$

To find it uniquely, we take the normal subgroup $N = \langle x_7 \rangle$ and the corresponding factor group as $H = G_6/N \simeq ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2) \rtimes C_2$, which is the same as in Theorem 3.5. Here, we observe that $|S_{\mathbb{K}}(\gamma_g)| = 1$ for all $g \in G_6$. Also, $H/H' \simeq C_2 \times C_2 \times C_2$. Consequently, we have

$$\mathbb{K}_q H \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{8} M_{n_r}(\mathbb{K}_q) \Rightarrow 56 = \sum_{r=1}^{8} n_r^2 \Rightarrow n_r = (2^{(6)}, 4^{(2)}).$$

This reduces the choices of n_r's in the decomposition of $\mathbb{K}_q G_6$ to $(2^{(6)}, 4^{(6)})$. Hence, it holds that

$$\mathbb{K}_q G_6 \simeq (\mathbb{K}_q)^8 \oplus (M_2(\mathbb{K}_q))^6 \oplus (M_4(\mathbb{K}_q))^6.$$

Case 2: $p^k \equiv 3 \bmod 8$ or $p^k \equiv 7 \bmod 8$. For these possibilities, we have $S_{\mathbb{K}}(\gamma_{x_2}) = \{\gamma_{x_2}, \gamma_{x_2 x_6}\}$, $S_{\mathbb{K}}(\gamma_{x_2 x_3 x_4}) = \{\gamma_{x_2 x_3 x_4}, \gamma_{x_2 x_3 x_4 x_7}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_6$. We engage Lemma 2.1 to write the Wedderburn decomposition as:

$$\mathbb{K}_q G_6 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{8} M_{n_r}(\mathbb{K}_q) \bigoplus_{r=9}^{10} M_{n_r}(\mathbb{K}_{q^2}), \quad n_r \geqslant 2 \Rightarrow 120 = \sum_{r=1}^{8} n_r^2 + 2 \cdot n_9^2 + 2 \cdot n_{10}^2.$$

The values of n_r can be

$$(2^{(10)}, 4, 8), \quad (2^{(7)}, 3^{(3)}, 4, 7), \quad (2^{(6)}, 4^{(6)}), \quad (2^{(5)}, 3^{(3)}, 4^{(3)}, 5),$$
$$(2^{(4)}, 3^{(6)}, 5^{(2)}) \quad \text{and} \quad (2^{(3)}, 3^{(8)}, 6).$$

To find it uniquely, we consider the factor group H as in Case 1. For H, $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$ for all $g \in H$. This gives that

$$\mathbb{K}_q H \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{8} M_{n_r}(\mathbb{K}_q), \quad n_r \geqslant 2 \Rightarrow 56 = \sum_{r=1}^{8} n_r^2.$$

The values of n_r must be $(2^{(6)}, 4^{(2)})$. This reduces the choices of n_r's in the decomposition of $\mathbb{K}G_6$ to $(2^{(6)}, 4^{(6)})$. Hence, we have

$$\mathbb{K}_q G_6 \simeq (\mathbb{K}_q)^8 \oplus (M_2(\mathbb{K}_q))^6 \oplus (M_4(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_{q^2}))^2.$$

This completes the proof. □

3.7. Unit group of $\mathbb{K}_q G_7$, where $G_7 = (((C_2 \times C_2 \times C_2) \rtimes C_4) \rtimes C_2) \rtimes C_2$. The presentation of the group G_7 is given by

$$\begin{aligned} G_7 = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \colon\ & x_1^2, [x_2, x_1]x_4^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1], [x_5, x_1], \\ & [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2 x_7^{-1}, [x_3, x_2], [x_4, x_2], [x_5, x_2]x_7^{-1}x_6^{-1}, [x_6, x_2], \\ & [x_7, x_2], x_3^2 x_7^{-1}, [x_4, x_3]x_6^{-1}, [x_5, x_3], [x_6, x_3], [x_7, x_3], x_4^2, [x_5, x_4]x_7^{-1}, \\ & [x_6, x_4], [x_7, x_4], x_5^2, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2 \rangle. \end{aligned}$$

There are 20 conjugacy classes for group G_7 as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_3	x_1x_6	x_2x_3
Size	1	8	4	4	4	4	2	1	16	16	8	8
Order	1	2	4	4	2	2	2	2	4	4	4	2

Representative	x_2x_5	x_2x_6	x_3x_4	x_3x_7	x_4x_5	$x_1x_2x_3$	$x_2x_3x_4$	$x_2x_3x_4x_7$
Size	8	4	8	4	4	16	4	4
Order	4	4	4	4	4	8	4	4

The exponent of the group G_7 is 8 and the commutator subgroup of G_7 is $G_7' = C_2 \times D_8$. So, the factor group $G_7/G_7' \simeq C_2 \times C_2 \times C_2$. Also, we observe that the group algebra $\mathbb{K}_q G_7$ is semisimple.

Theorem 3.7. *Let G_7 be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

- ▷ *for k even or $p^k \equiv \{1, 5\} \bmod 8$, $\mathcal{U}(\mathbb{K}_q G_7) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^6$,*
- ▷ *for $p^k \equiv \{3, 7\} \bmod 8$, $\mathcal{U}(\mathbb{K}_q G_7) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.*

P r o o f. *Case 1:* k is even or $p^k \equiv 1 \bmod 8$ or $p^k \equiv 5 \bmod 8$. The proof for these possibilities is the same as that of Case 1 in Theorem 3.6.

Case 2: $p^k \equiv 3 \bmod 8$ or $p^k \equiv 7 \bmod 8$. For these possibilities, we have $S_{\mathbb{K}}(\gamma_{x_2}) = \{\gamma_{x_2}, \gamma_{x_2x_6}\}$, $S_{\mathbb{K}}(\gamma_{x_3}) = \{\gamma_{x_3}, \gamma_{x_3x_7}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_7$. We note that, even though the sets $S_{\mathbb{K}}(\gamma_g)$ are different for these possibilities, the proof follows on the lines of Case 2 of Theorem 3.6. This completes the proof. $\square$

3.8. Unit group of $\mathbb{K}_qG_8$, where $G_8 = (((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_2) \rtimes C_2$. The presentation of the group G_8 is given by

$$G_8 = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \colon x_1^2, [x_2, x_1]x_4^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1], [x_5, x_1], [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2, [x_3, x_2]x_7^{-1}, [x_4, x_2], [x_5, x_2]x_7^{-1}x_6^{-1}, [x_6, x_2], [x_7, x_2], x_3^2, [x_4, x_3]x_6^{-1}, [x_5, x_3], [x_6, x_3], [x_7, x_3], x_4^2, [x_5, x_4]x_7^{-1}, [x_6, x_4], [x_7, x_4], x_5^2, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2\rangle.$$

There are 17 conjugacy classes for group G_8 as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_3	x_1x_6
Size	1	8	8	8	4	4	2	1	16	16	8
Order	1	2	2	2	2	2	2	2	4	4	4

Representative	x_2x_3	x_2x_5	x_3x_4	x_4x_5	$x_1x_2x_3$	$x_2x_3x_4$
Size	8	8	8	4	16	8
Order	4	4	4	4	8	4

The exponent of the group G_8 is 8 and the commutator subgroup of G_8 is $G_8' = C_2 \times D_8$. So, the factor group $G_8/G_8' \simeq C_2 \times C_2 \times C_2$. Also, we observe that the group algebra $\mathbb{K}_qG_8$ is semisimple.

Theorem 3.8. *Let G_8 be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

$$\mathcal{U}(\mathbb{K}_qG_8) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^2 \oplus GL_8(\mathbb{K}_q).$$

P r o o f. It can be verified that for any p^k, $|S_{\mathbb{K}}(\gamma_g)| = 1$ for all $g \in G_8$. Consequently, the Wedderburn decomposition of $\mathbb{K}_qG_8$ is given by

$$\mathbb{K}_qG_8 \simeq \bigoplus_{r=1}^{17} M_{n_r}(\mathbb{K}_q).$$

By Lemma 2.1, we see that $\mathbb{K}_qG_8 \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{9} M_{n_r}(\mathbb{K}_q)$, $n_r \geqslant 2 \Rightarrow 120 = \sum_{r=1}^{9} n_r^2$. The values of n_r can be

$$(2^{(6)}, 4^{(2)}, 8),\quad (2^{(5)}, 5^{(4)}),\quad (2^{(4)}, 3^{(2)}, 5^{(2)}, 6),\quad (2^{(3)}, 3^{(4)}, 6^{(2)}),\quad (2^{(3)}, 3^{(3)}, 4^{(2)}, 7),$$
$$(2^{(2)}, 4^{(7)}),\quad (2, 3^{(3)}, 4^{(4)}, 5) \quad \text{and} \quad (3^{(6)}, 4, 5^{(2)}).$$

To find it uniquely, we take the normal subgroup $N = \langle x_7 \rangle$. The factor group $H = G_8/N \simeq ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2) \rtimes C_2$ is the same as in Theorem 3.5. Further, for all p^k, we observe that $|S_{\mathbb{K}}(\gamma_g)| = 1$ for all $g \in G_8$. Also, $H/H' \simeq C_2 \times C_2 \times C_2$. Therefore, it holds that

$$\mathbb{K}_qH \simeq (\mathbb{K}_q)^8 \bigoplus_{r=1}^{8} M_{n_r}(\mathbb{K}_q) \Rightarrow 56 = \sum_{r=1}^{8} n_r^2 \Rightarrow n_r = (2^{(6)}, 4^{(2)}).$$

This reduces the choices of n_r's in the decomposition of $\mathbb{K}_qG_8$ to $(2^{(6)}, 4^{(2)}, 8)$. Hence, we have

$$\mathbb{K}_qG_8 \simeq (\mathbb{K}_q)^8 \oplus (M_2(\mathbb{K}_q))^6 \oplus (M_4(\mathbb{K}_q))^2 \oplus M_8(\mathbb{K}_q).$$

This completes the proof. □

3.9. Unit group of $\mathbb{K}_qG_9$, where $G_9 = (((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_2) \rtimes C_2$. The presentation of the group G_9 is given by

$$\begin{aligned} G_9 = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \colon\ & x_1^2, [x_2, x_1]x_4^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1], [x_5, x_1], \\ & [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2x_7^{-1}, [x_3, x_2]x_7^{-1}, [x_4, x_2], [x_5, x_2]x_7^{-1}x_6^{-1}, \\ & [x_6, x_2], [x_7, x_2], x_3^2, [x_4, x_3]x_6^{-1}, [x_5, x_3], [x_6, x_3], [x_7, x_3], x_4^2, \\ & [x_5, x_4]x_7^{-1}, [x_6, x_4], [x_7, x_4], x_5^2, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2 \rangle. \end{aligned}$$

There are 17 conjugacy classes for group G_9 as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_3	x_1x_6
Size	1	8	8	8	4	4	2	1	16	16	8
Order	1	2	4	2	2	2	2	2	4	4	4

Representative	x_2x_3	x_2x_5	x_3x_4	x_4x_5	$x_1x_2x_3$	$x_2x_3x_4$
Size	8	8	8	4	16	8
Order	2	4	4	4	8	4

The exponent of the group G_9 is 8 and the commutator subgroup of G_9 is $G_9' = C_2 \times D_8$. So, the factor group $G_9/G_9' \simeq C_2 \times C_2 \times C_2$. Also, we observe that the group algebra $\mathbb{K}_qG_9$ is semisimple.

Theorem 3.9. *Let G_9 be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

$$\mathcal{U}(\mathbb{K}_qG_9) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^2 \oplus GL_8(\mathbb{K}_q).$$

P r o o f. The proof is the same as that of Theorem 3.8. □

3.10. Unit group of $\mathbb{K}_qG_{10}$, where $G_{10} = (((C_2 \times Q_8) \rtimes C_2) \rtimes C_2) \rtimes C_2$. The presentation of the group G_{10} is given by

$$G_{10} = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7\colon\ x_1^2, [x_2, x_1]x_4^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1], [x_5, x_1], [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2x_7^{-1}, [x_3, x_2]x_7^{-1}, [x_4, x_2], [x_5, x_2]x_7^{-1}x_6^{-1}, [x_6, x_2], [x_7, x_2], x_3^2x_7^{-1}, [x_4, x_3]x_6^{-1}, [x_5, x_3], [x_6, x_3], [x_7, x_3], x_4^2, [x_5, x_4]x_7^{-1}, [x_6, x_4], [x_7, x_4], x_5^2, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2\rangle.$$

There are 17 conjugacy classes for group G_{10} as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_3	x_1x_6
Size	1	8	8	8	4	4	2	1	16	16	8
Order	1	2	4	4	2	2	2	2	4	4	4

Representative	x_2x_3	x_2x_5	x_3x_4	x_4x_5	$x_1x_2x_3$	$x_2x_3x_4$
Size	8	8	8	4	16	8
Order	4	4	4	4	8	4

The exponent of the group G_{10} is 8 and the commutator subgroup of G_{10} is $G'_{10} = C_2 \times D_8$. So, the factor group $G_{10}/G'_{10} \simeq C_2 \times C_2 \times C_2$. Also, we observe that the group algebra $\mathbb{K}_qG_{10}$ is semisimple.

Theorem 3.10. *Let G_{10} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

$$\mathcal{U}(\mathbb{K}_qG_{10}) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^2 \oplus GL_8(\mathbb{K}_q).$$

P r o o f. The proof is the same as that of Theorem 3.8. □

3.11. Unit group of $\mathbb{K}_qG_{11}$, where $G_{11} = (((C_2 \times Q_8) \rtimes C_2) \rtimes C_2) \rtimes C_2$. The presentation of the group G_{11} is given by

$$G_{11} = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7\colon\ x_1^2, [x_2, x_1]x_4^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1]x_7^{-1}, [x_5, x_1]x_7^{-1}, [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2, [x_3, x_2], [x_4, x_2]x_7^{-1}, [x_5, x_2]x_7^{-1}x_6^{-1}, [x_6, x_2], [x_7, x_2], x_3^2, [x_4, x_3]x_6^{-1}, [x_5, x_3]x_7^{-1}, [x_6, x_3], [x_7, x_3], x_4^2x_7^{-1}, [x_5, x_4]x_7^{-1}, [x_6, x_4], [x_7, x_4], x_5^2x_7^{-1}, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2\rangle.$$

There are 17 conjugacy classes for group G_{11} as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_3	x_1x_6
Size	1	8	8	8	4	4	2	1	16	16	8
Order	1	2	2	2	4	4	2	2	8	8	4

Representative	x_2x_3	x_2x_5	x_3x_4	x_4x_5	$x_1x_2x_3$	$x_2x_3x_4$
Size	8	8	8	4	16	8
Order	2	4	4	4	8	4

The exponent of the group G_{11} is 8 and the commutator subgroup of G_{11} is $G'_{11} = C_2 \times Q_8$. So, the factor group $G_{11}/G'_{11} \simeq C_2 \times C_2 \times C_2$. Also, we observe that the group algebra $\mathbb{K}_q G_{11}$ is semisimple.

Theorem 3.11. *Let G_{11} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

$$\mathcal{U}(\mathbb{K}_q G_{11}) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^2 \oplus GL_8(\mathbb{K}_q).$$

Proof. The proof is the same as that of Theorem 3.8. □

3.12. Unit group of $\mathbb{K}_q G_{12}$, where $G_{12} = (((C_4 \times C_4) \rtimes C_2) \rtimes C_2) \rtimes C_2$. The presentation of the group G_{12} is given by

$$G_{12} = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \colon x_1^2, [x_2, x_1]x_4^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1]x_7^{-1}, [x_5, x_1]x_7^{-1}, [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2x_7^{-1}, [x_3, x_2], [x_4, x_2]x_7^{-1}, [x_5, x_2]x_7^{-1}x_6^{-1}, [x_6, x_2], [x_7, x_2], x_3^2, [x_4, x_3]x_6^{-1}, [x_5, x_3]x_7^{-1}, [x_6, x_3], [x_7, x_3], x_4^2x_7^{-1}, [x_5, x_4]x_7^{-1}, [x_6, x_4], [x_7, x_4], x_5^2x_7^{-1}, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2\rangle.$$

There are 17 conjugacy classes for group G_{12} as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_3	x_1x_6
Size	1	8	8	8	4	4	2	1	16	16	8
Order	1	2	4	2	4	4	2	2	8	8	4

Representative	x_2x_3	x_2x_5	x_3x_4	x_4x_5	$x_1x_2x_3$	$x_2x_3x_4$
Size	8	8	8	4	16	8
Order	4	4	4	4	8	4

The exponent of the group G_{12} is 8 and the commutator subgroup of G_{12} is $G'_{12} = C_2 \times Q_8$. So, the factor group $G_{12}/G'_{12} \simeq C_2 \times C_2 \times C_2$. Also, we observe that the group algebra $\mathbb{K}_q G_{12}$ is semisimple.

Theorem 3.12. *Let G_{12} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

$$\mathcal{U}(\mathbb{K}_q G_{12}) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^2 \oplus GL_8(\mathbb{K}_q).$$

P r o o f. The proof is the same as that of Theorem 3.8. □

3.13. Unit group of $\mathbb{K}_q G_{13}$, where $G_{13} = (((C_2 \times Q_8) \rtimes C_2) \rtimes C_2) \rtimes C_2$. The presentation of the group G_{13} is given by

$$G_{13} = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \colon x_1^2, [x_2, x_1]x_4^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1]x_7^{-1}, [x_5, x_1]x_7^{-1}, [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2, [x_3, x_2]x_7^{-1}, [x_4, x_2]x_7^{-1}, [x_5, x_2]x_7^{-1}x_6^{-1}, [x_6, x_2], [x_7, x_2], x_3^2, [x_4, x_3]x_6^{-1}, [x_5, x_3]x_7^{-1}, [x_6, x_3], [x_7, x_3], x_4^2x_7^{-1}, [x_5, x_4]x_7^{-1}, [x_6, x_4], [x_7, x_4], x_5^2x_7^{-1}, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2\rangle.$$

There are 20 conjugacy classes for group G_{13} as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_3	x_1x_6	x_2x_3
Size	1	8	8	8	4	4	2	1	16	16	8	8
Order	1	2	2	2	4	4	2	2	8	8	4	4

Representative	x_2x_5	x_3x_4	x_4x_5	$x_1x_2x_3$	$x_2x_3x_4$	$x_2x_3x_5$	$x_2x_4x_5$	$x_3x_4x_7$
Size	4	4	4	16	4	4	4	4
Order	4	4	4	8	4	4	4	4

The exponent of the group G_{13} is 8 and the commutator subgroup of G_{13} is $G'_{13} = C_2 \times Q_8$. So, the factor group $G_{13}/G'_{13} \simeq C_2 \times C_2 \times C_2$. Also, we observe that the group algebra $\mathbb{K}_q G_{13}$ is semisimple.

Theorem 3.13. *Let G_{13} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

- ▷ *for k even or $p^k \equiv \{1, 5\} \bmod 8$, $\mathcal{U}(\mathbb{K}_q G_{13}) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^6$,*
- ▷ *for $p^k \equiv \{3, 7\} \bmod 8$, $\mathcal{U}(\mathbb{K}_q G_{13}) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.*

P r o o f. The proof is the same as that of Theorem 3.6. □

3.14. Unit group of $\mathbb{K}_q G_{14}$, where $G_{14} = (Q_8 \times Q_8) \rtimes C_2$. The presentation of the group G_{14} is given by

$$G_{14} = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \colon x_1^2, [x_2, x_1]x_4^{-1}, [x_3, x_1]x_5^{-1}, [x_4, x_1]x_7^{-1}, [x_5, x_1]x_7^{-1}, [x_6, x_1]x_7^{-1}, [x_7, x_1], x_2^2 x_7^{-1}, [x_3, x_2]x_7^{-1}, [x_4, x_2]x_7^{-1}, [x_5, x_2]x_7^{-1}x_6^{-1}, [x_6, x_2], [x_7, x_2], x_3^2 x_7^{-1}, [x_4, x_3]x_6^{-1}, [x_5, x_3]x_7^{-1}, [x_6, x_3], [x_7, x_3], x_4^2 x_7^{-1}, [x_5, x_4]x_7^{-1}, [x_6, x_4], [x_7, x_4], x_5^2 x_7^{-1}, [x_6, x_5], [x_7, x_5], x_6^2, [x_7, x_6], x_7^2 \rangle.$$

There are 20 conjugacy classes for group G_{14} as given below.

Representative	e	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_1x_2	x_1x_3	x_1x_6	x_2x_3
Size	1	8	8	8	4	4	2	1	16	16	8	8
Order	1	2	4	4	4	4	2	2	8	8	4	4

Representative	x_2x_5	x_3x_4	x_4x_5	$x_1x_2x_3$	$x_2x_3x_4$	$x_2x_3x_5$	$x_2x_4x_5$	$x_3x_4x_7$
Size	4	4	4	16	4	4	4	4
Order	4	4	4	8	4	4	4	4

The exponent of the group G_{14} is 8 and the commutator subgroup of G_{14} is $G'_{14} = C_2 \times Q_8$. So, the factor group $G_{14}/G'_{14} \simeq C_2 \times C_2 \times C_2$. Also, we observe that the group algebra $\mathbb{K}_q G_{14}$ is semisimple.

Theorem 3.14. *Let G_{14} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic p not equal to 2. Then*

$$\mathcal{U}(\mathbb{K}_q G_{14}) \simeq (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_4(\mathbb{K}_q))^6.$$

Proof. The proof is the same as that of Theorem 3.5. □

4. Conclusion

We have explicitly given the characterization of the unit groups $\mathcal{U}(\mathbb{K}_q G)$ of semisimple group algebras of 14 non-metabelian groups of order 128. With this paper, the study of characterization of unit groups of $\mathcal{U}(\mathbb{K}_q G)$ for all groups G up to order 128 is complete (except that of order 96). Finally, this paper motivates the researchers to come up with new techniques to uniquely deduce the structure of the unit groups of the group algebras of non-metabelian groups of order greater than 128.

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