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ADDITIVE GENERATORS OF DISCRETE SEMI-UNINORMS

Ya-Ming Wang, Hang Zhan and Yuan-Yuan Zhao

This work explores commutative semi-uninorms on finite chains by means of strictly increasing unary functions and the usual addition. In this paper, there are three families of additively generated commutative semi-uninorms. We not only study the structures and properties of semi-uninorms in each family but also show the relationship among these three families. In addition, this work provides the characterizations of uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ that are generated by additive generators.

Keywords: aggregation operations, semi-uninorms, additive generators, semi-t-norms, semi-t-conorms, finite chains

Classification: 46F10, 62E86

1. INTRODUCTION

The general evaluation scale in fuzzy set theory and fuzzy logic is the real unit interval $[0,1]$, which is equipped with a rich algebraic structure. However, practical applications supported by computer implementations are often based on arguments that take values in a finite chain $L_n = \{0, 1, 2, \cdots, n\}$ [4]. Since several types of associative aggregation operations (t-norms, t-conorms, uninorms and nullnorms, etc.) play a key role in many aspects, it is important to study the corresponding operations on finite chains. Therefore, the study of increasing operations on a finite chain L_n that are associative has received great attention. For instance, t-norms and t-conorms were characterized in [3, 4, 20, 21], uninorms were studied in [5, 12, 15, 17, 24], t-operators were introduced in [15] and smooth associative operations were discussed in [7, 18]. The researchers also studied increasing operations that are not necessarily associative, such as copulas [11, 22] and bisymmetric operators [16]. Known from these literatures, a discrete copula is associative if and only if its associated matrix is an ordinal sum of Łukasiewicz matrices, and a bisymmetric aggregation operation on a finite chain is associative if it is commutative and smooth.

As introduced in [9], uninorms are associative, commutative, increasing binary operations on $[0,1]$ that have a neutral element e in $[0,1]$. They can be seen as a special kind of binary aggregation operations generalizing both t-norms and t-conorms. On finite chains, discrete uninorms in the common classes of $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ were characterized

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in [15] and a non-commutative version of these uninorms were discussed in [17]. The class of idempotent uninorms was studied in [5, 14] and the characterization of all discrete uninorms with a smooth underlying t-norm and a smooth underlying t-conorm was given in [12, 24]. In addition to the above research about the common operations on finite chains, Mayor and Monreal [19, 20] studied the additive generators of discrete conjunctive and disjunctive aggregation operations. They proposed: the general form of the representable operations is as $F(i,j) = h^{(-1)}(h(i) + h(j))$, where $i, j \in L_n$, h is a strictly monotone function, and $h^{(-1)}$ is a pseudo-inverse of h. If h is strictly decreasing with $h(n) = 0$ and $h^{(-1)}$ is defined by $h^{(-1)}(t) = \min\{i \in L_n \mid h(i) \leq t\}$, then F is a commutative aggregation operation with a neutral element n. If h is strictly increasing with $h(0) = 0$ and $h^{(-1)}$ is defined by $h^{(-1)}(t) = \max\{i \in L_n \mid h(i) \leq t\}$, then F is a commutative aggregation operation with a neutral element 0. Based on this, they studied several properties of these discrete binary operations generated by additive generators. Keeping on the idea of generating a binary operation by a monotone unary functions with the usual addition, we are going to focus on studying commutative semi-uninorms on finite chains. Firstly, we define a strictly increasing unary function h from L_n to $[-\infty, +\infty]$ such that $h(e) = 0$, where $e \in L_n$. In this work, there are three families of commutative semi-uninorms generated by strictly increasing functions with the usual addition. Then we discuss the structures and several basic properties of commutative semi-uninorms in each family. Moreover, we show the relationship between the three families of these representable aggregation operations. In addition, this work provides the characterizations of uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ that are generated by additive generators.

The paper is organized as follows. In Section 2, we present some preliminary notions and results that are necessary for the rest of the paper. In Section 3, we study commutative semi-uninorms on finite chains by strictly increasing unary functions and the usual addition. Section 4 is the conclusion of this work.

2. PRELIMINARIES

We begin with some basic definitions and results that will be used throughout this paper. From now on, L_n will denote a finite chain $L_n = \{0, 1, 2, \ldots, n\}$. In order to distinguish this symbol from the standard notations for real intervals, we will use indistinctly the interval notation $L_n = [0, n]_{L_n}$ and also the usual notations $[p, q]_{L_n} = \{p, p+1, \ldots, q\}$, $[p, q[_{L_n} = \{p, p+1, \ldots, q-1\}$, $]p, q]_{L_n} = \{p+1, \ldots, q\}$ and $]p, q[_{L_n} = \{p+1, \ldots, q-1\}$ for the corresponding subsets of L_n with $p, q \in L_n$. Note that in case there is no subscript L_n when we refer to the usual real intervals. In addition, a binary operation is increasing meaning that it is increasing with respect to each variable.

Definition 2.1. (Beliakov et al. [1], Calvo et al. [2], Sander [25]) A binary *aggregation operation* on L_n is an increasing function $A : L_n \times L_n \to L_n$ such that $A(0, 0) = 0$ and $A(n, n) = n$.

Definition 2.2. (Durante et al. [6], Fodor and Keresztfalvi [8], Pradera et al. [23]) Let A be a binary increasing operation on L_n. Then:

(i) A is *conjunctive* if $A(i, j) \leq \min(i, j)$ for any $i, j \in L_n$.

(ii) A is *disjunctive* if $A(i,j) \geq \max(i,j)$ for any $i,j \in L_n$.

(iii) A is *commutative* if $A(i,j) = A(j,i)$ for any $i,j \in L_n$.

(iv) A is *associative* if $A(i, A(j,k)) = A(A(i,j),k)$ for any $i,j,k \in L_n$.

(v) A has a *neutral element* $e \in L_n$ if $A(i,e) = A(e,i) = i$ for any $i \in L_n$.

Definition 2.3. (Liu [13]) A *semi-t-norm* T on L_n is a binary aggregation operation with a neutral element n. A *semi-t-conorm* S on L_n is a binary aggregation operation with a neutral element 0. A *semi-uninorm* U on L_n is a binary aggregation operation with a neutral element $e \in L_n$.

Definition 2.4. (Li et al. [12], Mayor and Torrens [21]) An associative and commutative semi-t-norm on L_n is called a *t-norm*. An associative and commutative semi-t-conorm on L_n is called a *t-conorm*. An associative and commutative semi-uninorm on L_n is called a *uninorm*.

Obviously, any semi-t-norm or t-norm is a conjunctive aggregation operation, any semi-t-conorm or t-conorm is a disjunctive aggregation operation.

It is clear that a semi-uninorm U becomes a semi-t-norm when $e = n$ and a semi-t-conorm when $e = 0$. From Definition 2.3, we can get the structure of semi-uninorms on L_n. In fact, $U|_{[0,e]^2_{L_n}}$ is a semi-t-norm on $[0,e]_{L_n}$, $U|_{[e,n]^2_{L_n}}$ is a semi-t-conorm on $[e,n]_{L_n}$ and $\min \leq U|_{A(e)} \leq \max$, where $A(e) = [0,e[_{L_n} \times]e,n]_{L_n} \cup]e,n]_{L_n} \times [0,e[_{L_n}$. If U is a uninorm on L_n with a neutral element e, then $U|_{[0,e]^2_{L_n}}$ is a t-norm on $[0,e]_{L_n}$, $U|_{[e,n]^2_{L_n}}$ is a t-conorm on $[e,n]_{L_n}$ and $\min \leq U|_{A(e)} \leq \max$. Besides, we have $U(n,0) \in \{0,n\}$ for any uninorm U on L_n.

The notation $\mathcal{U}$ denotes the set of all uninorms defined on L_n. In addition, we denote by $\mathcal{U}_{\min}$ the class of uninorms on L_n that in $A(e)$ behave as the minimum, and by $\mathcal{U}_{\max}$ the class of uninorms on L_n that in $A(e)$ behave as the maximum. In other words,

$$\mathcal{U}_{\min} = \{U \in \mathcal{U} \mid U(i,j) = \min(i,j) \text{ for all } (i,j) \in A(e)\},$$

$$\mathcal{U}_{\max} = \{U \in \mathcal{U} \mid U(i,j) = \max(i,j) \text{ for all } (i,j) \in A(e)\}.$$

At the end of this section, let us review the knowledge about additive generators of aggregation operations.

Definition 2.5. (Mayor and Monreal [19]) An *additive generator* $f : L_n \to [0,+\infty[$ *of a conjunctive aggregation operation* C on L_n is a strictly decreasing function with $f(n) = 0$ such that

$$C(i,j) = f^{(-1)}(f(i) + f(j)) \qquad \text{for all } i,j \in L_n,$$

where $f^{(-1)} : [0,+\infty[\to L_n$ is the pseudo-inverse of f, defined by $f^{(-1)}(t) = \min\{i \in L_n \mid f(i) \leq t\}$, $t \in [0,+\infty[$.

If $C : L_n^2 \to L_n$ is a conjunctive aggregation operation of the form in Definition 2.5 for some f, we say that C is additively generated by f. We can write $C = \langle f \rangle$.

Definition 2.6. (Mayor and Monreal [20]) An *additive generator* $g : L_n \to [0, +\infty[$ *of a disjunctive aggregation operation* D on L_n is a strictly increasing function with $g(0) = 0$ such that

$$D(i, j) = g^{(-1)}(g(i) + g(j)) \qquad \text{for all } i, j \in L_n,$$

where $g^{(-1)} : [0, +\infty[\to L_n$ is the pseudo-inverse of g, defined by $g^{(-1)}(t) = \max\{i \in L_n \mid g(i) \leq t\}$, $t \in [0, +\infty[$.

If $D : L_n^2 \to L_n$ is a disjunctive aggregation operation of the form in Definition 2.6 for some g, we say that D is additively generated by g. We also can write $D = \langle g \rangle$.

3. ADDITIVE GENERATORS OF DISCRETE SEMI-UNINORMS

In this section, we will study commutative semi-uninorms on finite chains by means of strictly increasing unary functions and the usual addition. First, we will introduce a family of commutative semi-uninorms on L_n generated by the usual addition, strictly increasing unary functions and their pseudo-inverses. The notation $\mathcal{H}$ denotes the set of all strictly increasing functions from L_n to $[-\infty, +\infty]$.

Theorem 3.1. Let $e \in L_n$ and $h \in \mathcal{H}$ with $h(e) = 0$. If the binary operation $U : L_n^2 \to L_n$ is given by

$$U(i, j) = h^{(-1)}(h(i) + h(j)) \qquad \text{for all } i, j \in L_n, \tag{1}$$

then U is a commutative semi-uninorm with a neutral element e on L_n, where $h^{(-1)}$ is the pseudo-inverse of h and given by

$$h^{(-1)}(t) = \begin{cases} \min\{k \in L_n \mid h(k) \geq t\} & \text{if } t \in [-\infty, 0], \\ \max\{k \in L_n \mid h(k) \leq t\} & \text{if } t \in]0, +\infty]. \end{cases} \tag{2}$$

Proof. It is obvious that U is commutative. From Equation (2) and the fact that h is strictly increasing, it follows that $h^{(-1)}$ is an increasing function. Thus, U is increasing with respect to each variable. Moreover, $U(i, e) = h^{(-1)}(h(i) + h(e)) = h^{(-1)}(h(i) + 0) = h^{(-1)}(h(i)) = i$ for any $i \in L_n$. That is, e is a neutral element of U. □

If $U : L_n^2 \to L_n$ is a semi-uninorm of the form (1) for some h, we say that U is additively generated by h and h is an additive generator of U. We can write $U = \langle h \rangle$. Additionally, we denote the family of these commutative semi-uninorms in Theorem 3.1 by $\mathcal{F}_1$.

Han and Liu [10] proposed two families of commutative semi-uninorms on L_n generated by strictly increasing unary functions and the usual addition (shown as Theorem 3.2 and Theorem 3.3).

Theorem 3.2. (Han and Liu [10]) Let $e \in L_n$ and $h \in \mathcal{H}$ with $h(e) = 0$. Then, the operation $U : L_n^2 \to L_n$ given by

$$U(i,j) = \begin{cases} \min\{k \in L_n \mid h(k) \geq h(i) + h(j)\} & \text{if } \max(i,j) \leq e, \\ \max\{k \in L_n \mid h(k) \leq h(i) + h(j)\} & \text{if } \max(i,j) > e, \end{cases} \tag{3}$$

is a commutative semi-uninorm on L_n with a neutral element e.

Although the semi-uninorms in Theorem 3.2 do not conform to the form (1), we still write it as $U = \langle h \rangle$ for convenience in this work. We denote the family of these commutative semi-uninorms in Theorem 3.2 by $\mathcal{F}_2$.

Theorem 3.3. (Han and Liu [10]) Let $e \in L_n$ and $h \in \mathcal{H}$ with $h(e) = 0$. Then, the operation $U : L_n^2 \to L_n$ given by

$$U(i,j) = \begin{cases} \min\{k \in L_n \mid h(k) \geq h(i) + h(j)\} & \text{if } \min(i,j) < e, \\ \max\{k \in L_n \mid h(k) \leq h(i) + h(j)\} & \text{if } \min(i,j) \geq e, \end{cases} \tag{4}$$

is a commutative semi-uninorm on L_n with a neutral element e.

Similarly, for the sake of convenience, we also write the semi-uninorm U from Theorem 3.3 as $U = \langle h \rangle$. In addition, we denote by $\mathcal{F}_3$ the family of these commutative semi-uninorms in Theorem 3.3.

The following theorem shows the structure of semi-uninorms in $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$.

Theorem 3.4. Let U be a binary operation on L_n. If $U \in \mathcal{F}_1$ (resp. $U \in \mathcal{F}_2$ or $U \in \mathcal{F}_3$) with a neutral element $e \in L_n$, then there exist a commutative semi-t-norm T on $[0,e]_{L_n}$, a commutative semi-t-conorm S on $[e,n]_{L_n}$ and a commutative and increasing binary operation $C : A(e) \to L_n$ such that

$$U(i,j) = \begin{cases} T(i,j) & \text{if } (i,j) \in [0,e]_{L_n}, \\ S(i,j) & \text{if } (i,j) \in [e,n]_{L_n}, \\ C(i,j) & \text{if } (i,j) \in A(e), \end{cases} \tag{5}$$

where $\min(i,j) < C(i,j) < \max(i,j)$ (resp. $\min(i,j) \leq C(i,j) < \max(i,j)$ or $\min(i,j) < C(i,j) \leq \max(i,j)$) for any $(i,j) \in A(e)$.

Proof. If $U \in \mathcal{F}_1$ with a neutral element $e \in L_n$, then there is at least one function $h \in \mathcal{H}_0$ with $h(e) = 0$ such that $U(i,j) = h^{(-1)}(h(i)+h(j))$ for all $i,j \in L_n$. Let $h_1 = h|_{[0,e]_{L_n}}$ and $h_2 = h|_{[e,n]_{L_n}}$, then $h_1 : [0,e]_{L_n} \to [-\infty, 0]$ is a strictly increasing function with $h_1(e) = 0$ and $h_2 : [e,n]_{L_n} \to [0, +\infty]$ is a strictly increasing function with $h_2(e) = 0$. Besides, $h_1^{(-1)}(t) = \min\{k \in L_n \mid h(k) \geq t\}$ for $t \in [-\infty, 0]$ and $h_2^{(-1)}(t) = \max\{k \in L_n \mid h(k) \leq t\}$ for $t \in [0, +\infty]$. Then, it is evident that $T(i,j) = h_1^{(-1)}(h_1(i) + h_1(j))$ is a commutative semi-t-norm on the finite chain $[0,e]_{L_n}$ and $S(i,j) = h_2^{(-1)}(h_2(i) + h_2(j))$

is a commutative semi-t-conorm on the finite chain $[e, n]_{L_n}$. In addition, it is easy for us to know that $\min(i, j) \leq U(i, j) \leq \max(i, j)$ for any $(i, j) \in A(e)$.

Suppose that there exist a point $(i_0, j_0) \in A(e)$ such that $U(i_0, j_0) = \min(i_0, j_0)$, then $h^{(-1)}(h(i_0) + h(j_0)) = \min(i_0, j_0)$. Without loss of generality, we assume that $i_0 < j_0$, that is, $i_0 \in [0, e[_{L_n}$ and $j_0 \in]e, n]_{L_n}$. Then we have $h(j_0) > 0$ and $h^{(-1)}(h(i_0) + h(j_0)) = i_0$. There are two cases that need to be discussed. If $h(i_0) + h(j_0) \leq 0$, then $\min\{k \in L_n \mid h(k) \geq h(i_0) + h(j_0)\} = i_0$. Thus, $h(i_0) \geq h(i_0) + h(j_0)$, which means, $h(j_0) \leq 0$. This contradicts the fact $h(j_0) > 0$. If $h(i_0) + h(j_0) > 0$, then $\max\{k \in L_n \mid h(k) \leq h(i_0)+h(j_0)\} = i_0$. Thus, $0 = h(e) \geq h(i_0+1) > h(i_0)+h(j_0) > 0$, which is a contradiction. Therefore, $U(i, j) > \min(i, j)$ for any $(i, j) \in A(e)$. In an analogous way, we can also obtain that $U(i, j) < \max(i, j)$ for any $(i, j) \in A(e)$. As a result, the fact that $\min(i, j) < C(i, j) < \max(i, j)$ for any $(i, j) \in A(e)$ is true.

If $U \in \mathcal{F}_2$ with a neutral element $e \in L_n$, then we have $U|_{[0,e]^2_{L_n}}(i, j) = \min\{k \in L_n \mid h(k) \geq h(i) + h(j)\} = h^{(-1)}(h(i) + h(j))$, where $h^{(-1)}(t) = \min\{k \in L_n \mid h(k) \geq t\}$. Thus, $T = U|_{[0,e]^2_{L_n}}$ is a commutative semi-t-norm on the finite chain $[0, e]_{L_n}$. Similarly, $U|_{[e,n]^2_{L_n}}(i, j) = \max\{k \in L_n \mid h(k) \leq h(i)+h(j)\} = h^{(-1)}(h(i)+h(j))$, where $h^{(-1)}(t) = \max\{k \in L_n \mid h(k) \leq t\}$. Thus, $S = U|_{[e,n]^2_{L_n}}$ is a commutative semi-t-conorm on the finite chain $[e, n]_{L_n}$. It is clear that $\min \leq C = U|_{A(e)} \leq \max$. Suppose that there exist a point $(i_0, j_0) \in A(e)$ such that $U(i_0, j_0) = \max(i_0, j_0)$, then $\max\{k \in L \mid h(k) \leq h(i_0) + h(j_0)\} = \max(i_0, j_0)$. Without loss of generality, we assume that $i_0 < j_0$, that is, $i_0 \in [0, e[_{L_n}$ and $j_0 \in]e, n[_{L_n}$. Then we have $j_0 = \max\{k \in L \mid h(k) \leq h(i_0) + h(j_0)\}$. Thus, $h(j_0) \leq h(i_0)+h(j_0)$, which means, $h(i_0) \geq 0$. This contradicts the fact $h(i_0) < 0$.

The proof of case $U \in \mathcal{F}_3$ is similar to that of case $U \in \mathcal{F}_2$. □

Here we also emphasize that there may be a point $(i_0, j_0) \in A(e)$ such that $U(i_0, j_0) = \min(i_0, j_0)$ for $U \in \mathcal{F}_2$. Without loss of generality, we assume that $i_0 \in [0, e[_{L_n}$ and $j_0 \in]e, n]_{L_n}$. Then, $i_0 = \min(i_0, j_0) = U(i_0, j_0) = \max\{k \in L_n \mid h(k) \leq h(i) + h(j)\}$. Thus, we have $h(i_0 + 1) > h(i_0) + h(j_0)$. Alternatively, if there exists some point $(i_0, j_0) \in [0, e[_{L_n} \times]e, n]_{L_n}$ such that $h(j_0) < h(i_0+1)-h(i_0)$, then $U(i_0, j_0) = \min(i_0, j_0)$. Furthermore, we provide the characterization of discrete uninorms in $\mathcal{U}_{\min}$ that are generated by additive generators.

Theorem 3.5. Let $e \in L_n$, $h \in \mathcal{H}$ with $h(e) = 0$ satisfying $h(i) + h(j) \in [-\infty, h(0)] \cup \mathrm{Ran}(h)$ for any $i, j \in [0, e]_{L_n}$ and $h(i)+h(j) \in [h(n), +\infty] \cup \mathrm{Ran}(h)$ for any $i, j \in [e, n]_{L_n}$. If $U = \langle h \rangle \in \mathcal{F}_2$, then $U \in \mathcal{U}_{\min}$ if and only if $h(n) < \min\{h(i + 1) - h(i) \mid i \in [0, e[_{L_n}\}$.

Proof. If $U \in \mathcal{U}_{\min}$, then $\max\{k \in L_n \mid h(k) \leq h(i) + h(j)\} = U(i, j) = \min(i, j) = i$ for any $i \in [0, e[_{L_n}$ and $j \in]e, n]_{L_n}$. So we have $h(i) \leq h(i)+h(j)$ and $h(i+1) > h(i)+h(j)$ for any $i \in [0, e[_{L_n}$ and $j \in]e, n]_{L_n}$. Now let $j = n$, then we get $h(i + 1) - h(i) > h(n)$ for any $i \in [0, e[_{L_n}$. Thus, $h(n) < \min\{h(i + 1) - h(i) \mid i \in [0, e[_{L_n}\}$.

If $h(n) < \min\{h(i + 1) - h(i) \mid i \in [0, e[_{L_n}\}$, then $h(j) < h(i + 1) - h(i)$ for any $i \in [0, e[_{L_n}$ and $j \in]e, n]_{L_n}$ because h is strictly increasing on L_n. For any $i \in [0, e[_{L_n}$ and $j \in]e, n]_{L_n}$, we have $U(i, j) = \max\{k \in L_n \mid h(k) \leq h(i) + h(j)\} = i = \min(i, j)$ since $h(i) \leq h(i) + h(j)$ and $h(i+1) > h(i) + h(j)$. Next, let us verify that $T = U|_{[0,e]^2_{L_n}}$ is a t-norm. Before starting, we assume that i, j, k are any three elements in $[0, e]_{L_n}$.

Case 1. If $h(i)+h(j)\in[-\infty,h(0)]$ and $h(j)+h(k)\in[-\infty,h(0)]$, then $U(i,j)=\min\{r\in L_n\mid h(r)\geq h(i)+h(j)\}=0$ and $U(j,k)=\min\{r\in L_n\mid h(r)\geq h(j)+h(k)\}=0$. Thus,

$$\begin{aligned}U(i,U(j,k)) &= \min\{r\in L_n\mid h(r)\geq h(i)+h(U(j,k))\}\\ &= \min\{r\in L_n\mid h(r)\geq h(i)+h(0)\}\\ &= 0\\ &= \min\{r'\in L_n\mid h(r)\geq h(0)+h(k)\}\\ &= \min\{r'\in L_n\mid h(r)\geq h(U(i,j))+h(k)\}\\ &= U(U(i,j),k).\end{aligned}$$

Case 2. If $h(i)+h(j)\in[-\infty,h(0)]$ and $h(j)+h(k)\in\mathrm{Ran}(h)$, then $h(U(j,k))=h\circ\min\{r\in L_n\mid h(r)\geq h(j)+h(k)\}=\min\{h(r)\in\mathrm{Ran}(h)\mid h(r)\geq h(j)+h(k)\}=h(j)+h(k)$. Thus,

$$\begin{aligned}U(i,U(j,k)) &= \min\{r\in L_n\mid h(r)\geq h(i)+h(U(j,k))\}\\ &= \min\{r\in L_n\mid h(r)\geq h(i)+h(j)+h(k)\}\\ &= 0\\ &= \min\{r'\in L_n\mid h(r)\geq h(0)+h(k)\}\\ &= \min\{r'\in L_n\mid h(r)\geq h(U(i,j))+h(k)\}\\ &= U(U(i,j),k).\end{aligned}$$

Case 3. If $h(i)+h(j)\in\mathrm{Ran}(h)$ and $h(j)+h(k)\in\mathrm{Ran}(h)$, then

$$\begin{aligned}U(i,U(j,k)) &= \min\{r\in L_n\mid h(r)\geq h(i)+h(U(j,k))\}\\ &= \min\{r\in L_n\mid h(r)\geq h(i)+h(j)+h(k)\}\\ &= \min\{r'\in L_n\mid h(r)\geq h(i)+h(j)+h(k)\}\\ &= \min\{r'\in L_n\mid h(r)\geq h(U(i,j))+h(k)\}\\ &= U(U(i,j),k).\end{aligned}$$

From Theorem 3.4, we know that $T=U|_{[0,e]^2_{L_n}}$ is a commutative semi-t-norm on $[0,e]_{L_n}$. Therefore, $T=U|_{[0,e]^2_{L_n}}$ is a t-norm on $[0,e]_{L_n}$. Finally, let us prove that $S=U|_{[e,n]^2_{L_n}}$ is a t-conorm on $[e,n]_{L_n}$. Assume that i,j,k are any three elements in $[e,n]_{L_n}$.

Case (1). If $h(i)+h(j)\in[h(n),+\infty]$ and $h(j)+h(k)\in[h(n),+\infty]$, then $U(i,j)=\max\{r\in L_n\mid h(r)\leq h(i)+h(j)\}=n$ and $U(j,k)=\max\{r\in L_n\mid h(r)\leq h(j)+h(k)\}=n$. Thus,

$$\begin{aligned}U(i,U(j,k)) &= \max\{r\in L_n\mid h(r)\leq h(i)+h(U(j,k))\}\\ &= \max\{r\in L_n\mid h(r)\leq h(i)+h(n)\}\\ &= n\\ &= \max\{r'\in L_n\mid h(r)\leq h(n)+h(k)\}\\ &= \max\{r'\in L_n\mid h(r)\leq h(U(i,j))+h(k)\}=U(U(i,j),k).\end{aligned}$$

Case (2). If $h(i)+h(j)\in[h(n),+\infty]$ and $h(j)+h(k)\in\mathrm{Ran}(h)$, then $h(U(j,k)) = h\circ\max\{r\in L_n \mid h(r)\le h(j)+h(k)\} = \max\{h(r)\in\mathrm{Ran}(h)\mid h(r)\le h(j)+h(k)\} = h(j)+h(k)$. Thus,

$$\begin{aligned}
U(i,U(j,k)) &= \max\{r\in L_n \mid h(r)\le h(i)+h(U(j,k))\}\\
&= \max\{r\in L_n \mid h(r)\le h(i)+h(j)+h(k)\}\\
&= n\\
&= \max\{r'\in L_n \mid h(r)\le h(n)+h(k)\}\\
&= \max\{r'\in L_n \mid h(r)\le h(U(i,j))+h(k)\}\\
&= U(U(i,j),k).
\end{aligned}$$

Case (3). If $h(i)+h(j)\in\mathrm{Ran}(h)$ and $h(j)+h(k)\in\mathrm{Ran}(h)$, then

$$\begin{aligned}
U(i,U(j,k)) &= \max\{r\in L_n \mid h(r)\le h(i)+h(U(j,k))\}\\
&= \max\{r\in L_n \mid h(r)\le h(i)+h(j)+h(k)\}\\
&= \max\{r'\in L_n \mid h(r)\le h(i)+h(j)+h(k)\}\\
&= \max\{r'\in L_n \mid h(r)\le h(U(i,j))+h(k)\}\\
&= U(U(i,j),k).
\end{aligned}$$

It follows from Theorem 3.4 that $S=U|_{[e,n]^2_{L_n}}$ is a t-conorm on $[e,n]_{L_n}$. In summary, we get $U\in\mathcal{U}_{\min}$. □

The following theorem shows the characterization of discrete uninorms in $\mathcal{U}_{\max}$ that are generated by additive generators.

Theorem 3.6. Let $e\in L_n$, $h\in\mathcal{H}$ with $h(e)=0$ satisfying $h(i)+h(j)\in[-\infty,h(0)]\cup\mathrm{Ran}(h)$ for any $i,j\in[0,e]_{L_n}$ and $h(i)+h(j)\in[h(n),+\infty]\cup\mathrm{Ran}(h)$ for any $i,j\in[e,n]_{L_n}$. If $U=\langle h\rangle\in\mathcal{F}_3$, then $U\in\mathcal{U}_{\max}$ if and only if $h(0)>\max\{h(j-1)-h(j)\mid j\in]e,n]_{L_n}\}$.

Proof. The proof is similar to that of Theorem 3.5. □

The following example shows two additively generated uninorms U' and U'' on the finite chain L_7, where U' is in $\mathcal{U}_{\min}$ and U'' is in $\mathcal{U}_{\max}$.

(*a*) $U'\in\mathcal{U}_{\min}$

U'	0	1	2	3	4	5	6	7
0	0	0	0	**0**	0	0	0	0
1	0	0	0	**1**	1	1	1	1
2	0	0	1	**2**	2	2	2	2
3	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
4	0	1	2	**4**	5	7	7	7
5	0	1	2	**5**	7	7	7	7
6	0	1	2	**6**	7	7	7	7
7	0	1	2	**7**	7	7	7	7

(*b*) $U''\in\mathcal{U}_{\max}$

U''	0	1	2	3	4	5	6	7
0	0	0	0	**0**	4	5	6	7
1	0	0	0	**1**	4	5	6	7
2	0	0	1	**2**	4	5	6	7
3	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
4	4	4	4	**4**	6	7	7	7
5	5	5	5	**5**	7	7	7	7
6	6	6	6	**6**	7	7	7	7
7	7	7	7	**7**	7	7	7	7

Tab. 1. There are two uninorms U' and U'' on L_7.

Example 3.7. Let $h_1, h_2 \in \mathcal{H}$ with $\mathrm{Ran}(h_1) = \{-50, -40, -20, 0, 3, 6, 8, 9\}$ and $\mathrm{Ran}(h_2) = \{-9, -7, -3, 0, 25, 40, 50, 65\}$. The operations $U' = \langle h_1 \rangle \in \mathcal{F}_2$ and $U'' = \langle h_2 \rangle \in \mathcal{F}_3$ shown in Table 1 are two commutative semi-uninorms with the neutral element $e = 3$ on L_7. Let $R_{11} = \{h_1(i) + h_1(j) \mid i, j \in [0, e]_{L_n}\}$ and $R_{12} = \{h_1(i) + h_1(j) \mid i, j \in [e, n]_{L_n}\}$, then $R_{11} = \{0, -20, -40, -50, -60, -70, -80, -90, -100\} \subset [-\infty, -50] \cup \mathrm{Ran}(h_1)$ and $R_{12} = \{0, 3, 6, 8, 9, 11, 12, 14, 15, 16, 17, 18\} \subset [9, +\infty] \cup \mathrm{Ran}(h_1)$. Besides, $h_1(1) - h_1(0) = 10$, $h_1(2) - h_1(1) = 20$, $h_1(3) - h_1(2) = 20$ and $h_7 = 9 < 10 = \min\{10, 20\}$. Thus, we know that $U' \in \mathcal{U}_{\min}$ from Theorem 3.5. Similarly, let $R_{21} = \{h_2(i) + h_2(j) \mid i, j \in [0, e]_{L_n}\}$ and $R_{22} = \{h_2(i) + h_2(j) \mid i, j \in [e, n]_{L_n}\}$, then $R_{21} = \{0, -3, -6, -7, -9, -10, -12, -14, -16, -18\} \subset [-\infty, -9] \cup \mathrm{Ran}(h_2)$ and $R_{22} = \{0, 25, 40, 50, 65, 75, 80, 90, 100, 105, 115, 130\} \subset [65, +\infty] \cup \mathrm{Ran}(h_2)$. Moreover, $h_2(6) - h_2(7) = -15$, $h_2(5) - h_2(6) = -10$, $h_2(4) - h_2(5) = -15$, $h_2(3) - h_2(4) = -25$ and $h_2(0) = -9 > -10 = \max\{-15, -10, -25\}$. Thus, we know that $U'' \in \mathcal{U}_{\max}$ from Theorem 3.6.

Now we show some basic properties of additively generated commutative semi-uninorms on finite chains. We indicate $h = (a_0, a_1, \ldots, a_{n-1}, a_n)$, where $a_i = h(i)$ for all $i \in L_n$. Of course, $a_0 < a_1 < \ldots < a_{n-1} < a_n$.

Proposition 3.8. Let $e \in L_n$, $h \in \mathcal{H}$ with $h(e) = 0$. If $U = \langle h \rangle \in \mathcal{F}_1$, then the following statements are true.

(i) $U(i, j) = k \in]0, e[_{L_n}$ if and only if $a_{k-1} < a_i + a_j \leq a_k$;
$U(i, j) = k \in]e, n[_{L_n}$ if and only if $a_k \leq a_i + a_j < a_{k+1}$.

(ii) $U(i, j) = e$ if and only if $a_{e-1} < a_i + a_j < a_{e+1}$.

(iii) $U(i, j) = 0$ if and only if $a_0 \geq a_i + a_j$; $U(i, j) = n$ if and only if $a_n \leq a_i + a_j$.

Proof. (i) If $U(i, j) = k \in]0, e[_{L_n}$, then we have $h(i) + h(j) \leq 0$. Otherwise, suppose $h(i) + h(j) > 0$, then $k = U(i, j) = h^{(-1)}(h(i) + h(j)) = \max\{r \in L_n \mid h(r) \leq h(i) + h(j)\}$. Thus, $0 = h(e) \geq h(k+1) > h(i) + h(j)$ since $k \in]0, e[_{L_n}$, which is a contradiction. From the fact $h(i) + h(j) \leq 0$, it follows that $k = U(i, j) = \min\{r \in L_n \mid h(r) \geq h(i) + h(j)\}$. Therefore, we have $h(k-1) < h(i) + h(j) \leq h(k)$, that is, $a_{k-1} < a_i + a_j \leq a_k$.

If $a_{k-1} < a_i + a_j \leq a_k$ when $k \in]0, e[_{L_n}$, then $h(k-1) < h(i) + h(j) \leq h(k) < h(e) = 0$. Thus, it is obvious that $U(i, j) = \min\{r \in L_n \mid h(r) \geq h(i) + h(j)\} = k$ from Theorem 3.1.

Similarly, we can obtain the conclusion that $U(i, j) = k \in]e, n[_{L_n}$ if and only if $a_k \leq a_i + a_j < a_{k+1}$.

(ii) If $h(i) + h(j) \leq 0$, then $e = U(i, j) = h^{(-1)}(h(i) + h(j)) = \min\{r \in L_n \mid h(r) \geq h(i) + h(j)\}$ if and only if $h(e-1) < h(i) + h(j) \leq h(e)$. If $h(i) + h(j) > 0$, then $e = U(i, j) = h^{(-1)}(h(i) + h(j)) = \max\{r' \in L_n \mid h(r') \leq h(i) + h(j)\}$ if and only if $h(e) \leq h(i) + h(j) < h(e+1)$. So, $U(i, j) = e$ if and only if $h(e-1) < h(i) + h(j) < h(e+1)$. That is, $U(i, j) = e$ if and only if $a_{e-1} < a_i + a_j < a_{e+1}$.

(iii) Similar to the proof of (i), we know $h(i)+h(j) \leq 0$ if $U(i,j) = 0$ and $h(i)+h(j) > 0$ if $U(i,j) = n$. So it is easily to obtain that $U(i,j) = 0$ if and only if $a_0 \geq a_i + a_j$ and $U(i,j) = n$ if and only if $a_n \leq a_i + a_j$ from Theorem 3.1. □

Proposition 3.9. Let $e \in L_n$, $h \in \mathcal{H}$ with $h(e) = 0$. If $U = \langle h \rangle \in \mathcal{F}_2$, then the following statements are true.

(i) When $\max(i,j) > e$:

- $U(i,j) = k \in]0,n[_{L_n}$ if and only if $a_k \leq a_i + a_j < a_{k+1}$;
- $U(i,j) = 0$ if and only if $a_0 \leq a_i + a_j < a_1$;
- $U(i,j) = n$ if and only if $a_n \leq a_i + a_j$.

(ii) When $\max(i,j) \leq e$:

- $U(i,j) = k \in]0,n[_{L_n}$ if and only if $a_{k-1} < a_i + a_j \leq a_k$;
- $U(i,j) = 0$ if and only if $a_0 \geq a_i + a_j$;
- $U(i,j) = n$ if and only if $a_{n-1} < a_i + a_j \leq a_n$.

Proof. (i) When $\max(i,j) > e$. It is clear that $k = U(i,j) = \max\{r \in L_n \mid h(r) \leq h(i) + h(j)\}$ if and only if $h(k) \leq h(i) + h(j) < h(k+1)$. That is, $U(i,j) = k$ if and only if $a_k \leq a_i + a_j < a_{k+1}$.

(ii) When $\max(i,j) \leq e$. It is evident that $k = U(i,j) = \min\{r \in L_n \mid h(r) \geq h(i) + h(j)\}$ if and only if $h(k) \geq h(i) + h(j) > h(k-1)$. That is $U(i,j) = k$ if and only if $a_k \geq a_i + a_j > a_{k-1}$.

The other situations can be obtained in a similar way. □

Proposition 3.10. Let $e \in L_n$, $h \in \mathcal{H}$ with $h(e) = 0$. If $U = \langle h \rangle \in \mathcal{F}_3$, then the following statements are true.

(i) When $\min(i,j) \geq e$:

- $U(i,j) = k \in]0,n[_{L_n}$ if and only if $a_k \leq a_i + a_j < a_{k+1}$;
- $U(i,j) = 0$ if and only if $a_0 \leq a_i + a_j < a_1$;
- $U(i,j) = n$ if and only if $a_n \leq a_i + a_j$.

(ii) When $\min(i,j) < e$:

- $U(i,j) = k \in]0,n[_{L_n}$ if and only if $a_{k-1} < a_i + a_j \leq a_k$;
- $U(i,j) = 0$ if and only if $a_0 \geq a_i + a_j$;
- $U(i,j) = n$ if and only if $a_{n-1} < a_i + a_j \leq a_n$.

Proof. The proof is similar to that of Proposition 3.9. □

Next we will discuss the relationship between $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$. Before that, we will use an example to illustrate that there exists some function $h \in \mathcal{H}_0$ with $h(e) = 0$ such that the semi-uninorms $U_1 = \langle h \rangle \in \mathcal{F}_1$, $U_2 = \langle h \rangle \in \mathcal{F}_2$ and $U_3 = \langle h \rangle \in \mathcal{F}_3$ are different from each other.

Example 3.11. Let $h = (-2, -1, 0, 0.5, 3)$. The operations $U_1 = \langle h \rangle \in \mathcal{F}_1$, $U_2 = \langle h \rangle \in \mathcal{F}_2$ and $U_3 = \langle h \rangle \in \mathcal{F}_3$ shown in Table 2 are three different commutative semi-uninorms with the neutral element $e = 2$ on L_4.

(a) $U_1 = \langle h \rangle \in \mathcal{F}_1$

U_1	0	1	2	3	4
0	0	0	**0**	1	3
1	0	0	**1**	2	3
2	**0**	**1**	**2**	**3**	**4**
3	1	2	**3**	3	4
4	3	3	**4**	4	4

(b) $U_2 = \langle h \rangle \in \mathcal{F}_2$

U_2	0	1	2	3	4
0	0	0	**0**	0	3
1	0	0	**1**	1	3
2	**0**	**1**	**2**	**3**	**4**
3	0	1	**3**	3	4
4	3	3	**4**	4	4

(c) $U_3 = \langle h \rangle \in \mathcal{F}_3$

U_3	0	1	2	3	4
0	0	0	**0**	1	4
1	0	0	**1**	2	4
2	**0**	**1**	**2**	**3**	**4**
3	1	2	**3**	3	4
4	4	4	**4**	4	4

Tab. 2. Three different commutative semi-uninorms U_1, U_2 and U_3 on L_4.

Theorem 3.12. Let $e \in L_n$ and $h \in \mathcal{H}$ with $h(e) = 0$. If $U_1 = \langle h \rangle \in \mathcal{F}_1$ and $U_2 = \langle h \rangle \in \mathcal{F}_2$, then $U_1(i, j) = U_2(i, j)$ for all $i, j \in L_n$ if and only if h satisfies one of the following two conditions:

(i) $h(0) + h(e + 1) > 0$;

(ii) $h(i) + h(j) \in \mathrm{Ran}(h)$ when $(i, j) \in A(e)$ and $h(i) + h(j) \leq 0$.

Proof. If $U_1 = U_2$, then we have $U_1(i, j) = U_2(i, j)$ for any $(i, j) \in A(e)$. Suppose that there exists $i_0 \in [0, e[_{L_n}$ and $j_0 \in]e, n]_{L_n}$ such that $h(i_0) + h(j_0) \leq 0$ and $h(i_0) + h(j_0) \notin \mathrm{Ran}(h)$. Then, there is an element $k \in]0, e]_{L_n}$ such that $h(i_0)+h(j_0) \in]h(k-1), h(k)[_{L_n}$. Thus, $U_1(i_0, j_0) = h^{(-1)}(h(i_0) + h(j_0)) = \min\{r \in L_n \mid h(r) \geq h(i_0) + h(j_0)\} = k$ and $U_2(i_0, j_0) = \max\{r \in L_n \mid h(r) \leq h(i_0) + h(j_0)\} = k - 1$. That is, $U_1(i_0, j_0) = k \neq k - 1 = U_2(i_0, j_0)$, which is a contradiction. Therefore, there are only two possibilities for $h(i) + h(j)$ when $(i, j) \in A(e)$. One is that $h(i) + h(j) > 0$, which is equivalent to $h(0) + h(e + 1) > 0$. The other is that $h(i) + h(j) \in \mathrm{Ran}(h)$ if $h(i) + h(j) \leq 0$.

If $h(0)+h(e+1) > 0$, then $h(i)+h(j) > h(e)+h(e+1) > 0$ for any $(i, j) \in A(e)$. Thus, we have $U_1(i, j) = h^{(-1)}(h(i) + h(j)) = \max\{r \in L_n \mid h(r) \leq h(i) + h(j)\} = U_2(i, j)$ for any $(i, j) \in A(e)$. If $h(i) + h(j) \in \mathrm{Ran}(h)$ when $(i, j) \in A(e)$ and $h(i) + h(j) \leq 0$, then $h(U_1(i, j)) = h \circ h^{(-1)}(h(i) + h(j)) = h(i) + h(j) = \max\{h(r) \in \mathrm{Ran}(h) \mid h(r) \leq h(i) + h(j)\} = h \circ \max\{r \in L_n \mid h(r) \leq h(i) + h(j)\} = h(U_2(i, j))$. Thus, we have $U_1(i, j) = U_2(i, j)$ for $(i, j) \in A(e)$ and $h(i) + h(j) \leq 0$ because h is a strictly increasing function. From Theorem 3.4, it follows that $U_1(i, j) = U_2(i, j)$ for any $i, j \in [0, e]_{L_n} \cup [e, n]_{L_n}$. In summary, $U_1 = U_2$ holds. □

Theorem 3.13. Let $e \in L_n$ and $h \in \mathcal{H}$ with $h(e) = 0$. If $U_1 = \langle h \rangle \in \mathcal{F}_1$ and $U_3 = \langle h \rangle \in \mathcal{F}_3$, then $U_1(i, j) = U_3(i, j)$ for all $i, j \in L_n$ if and only if h satisfies one of the following two conditions:

(i) $h(e-1)+h(n) \leq 0$;

(ii) $h(i)+h(j) \in \mathrm{Ran}(h)$ when $(i,j) \in A(e)$ and $h(i)+h(j) > 0$.

Proof. The proof is similar to that of Theorem 3.12. □

Theorem 3.14. Let $e \in L_n$ and $h \in \mathcal{H}$ with $h(e) = 0$. If $U_2 = \langle h \rangle \in \mathcal{F}_2$ and $U_3 = \langle h \rangle \in \mathcal{F}_3$, then $U_2(i,j) = U_3(i,j)$ for all $i,j \in L_n$ if and only if $h(i)+h(j) \in \mathrm{Ran}(h)$ for any $(i,j) \in A(e)$.

Proof. Suppose that there exists $(i_0, j_0) \in A(e)$ such that $h(i_0)+h(j_0) \notin \mathrm{Ran}(h)$. Then, there is an element $k' \in]0,n]_{L_n}$ such that $h(i_0)+h(j_0) \in]h(k'-1), h(k')[_{L_n}$. Thus, $U_2(i_0,j_0) = \max\{r \in L_n \mid h(r) \leq h(i_0)+h(j_0)\} = k'-1$ and $U_3(i_0,j_0) = \min\{r \in L_n \mid h(r) \geq h(i_0)+h(j_0)\} = k'$. That is, $U_2(i_0,j_0) = k'-1 \neq k' = U_3(i_0,j_0)$, which is a contradiction. Therefore, we have $h(i)+h(j) \in \mathrm{Ran}(h)$ for any $(i,j) \in A(e)$. Conversely, $h(U_2(i,j)) = h \circ \max\{r \in L_n \mid h(r) \leq h(i)+h(j)\} = \max\{h(r) \in \mathrm{Ran}(h) \mid h(r) \leq h(i)+h(j)\} = h(i)+h(j) = \min\{h(r) \in \mathrm{Ran}(h) \mid h(r) \geq h(i)+h(j)\} = h \circ \min\{r \in L_n \mid h(r) \geq h(i)+h(j)\} = h(U_3(i,j))$ for any $(i,j) \in A(e)$. So we have $U_2(i,j) = U_3(i,j)$ for any $(i,j) \in A(e)$ since h is a strictly increasing function. From Theorem 3.4, it follows that $U_2(i,j) = U_3(i,j)$ for any $i,j \in [0,e]_{L_n} \cup [e,n]_{L_n}$. In a word, we obtain that $U_2 = U_3$ holds. □

By combining Theorems 3.12, 3.13 and 3.14, we can obtain the relationship among the families $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$.

Theorem 3.15. Let $e \in L_n$ and $h \in \mathcal{H}$ with $h(e) = 0$. If $U_1 = \langle h \rangle \in \mathcal{F}_1$, $U_2 = \langle h \rangle \in \mathcal{F}_2$ and $U_3 = \langle h \rangle \in \mathcal{F}_3$, then $U_1(i,j) = U_2(i,j) = U_3(i,j)$ for all $i,j \in L_n$ if and only if $h(i)+h(j) \in \mathrm{Ran}(h)$ for any $(i,j) \in A(e)$.

4. CONCLUSION

In this work, we have studied semi-uninorms on finite chains generated by strictly increasing functions with the usual addition. Firstly, we introduced a family of commutative semi-uninorms generated by a strictly increasing function h, its pseudo-inverse $h^{(-1)}$ and the usual addition. Then, we recalled other two families of additively generated commutative semi-uninorms proposed by Han and Liu. Furthermore, we not only studied the structures and properties of semi-uninorms in each family but also presented the relationship among these three families. We also characterized uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ that are generated by additive generators.

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