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CONDITIONAL DISTRIBUTIVITY OF OVERLAP FUNCTIONS OVER UNINORMS WITH CONTINUOUS UNDERLYING OPERATORS

Hui Liu and Wenle Li

The investigations of conditional distributivity are encouraged by distributive logical connectives and their generalizations used in fuzzy set theory and were brought into focus by Klement in the closing session of Linzs 2000. This paper is mainly devoted to characterizing all pairs (O, F) of aggregation functions that are satisfying conditional distributivity laws, where O is an overlap function, and F is a continuous t-conorm or a uninorm with continuous underlying operators.

Keywords: aggregation function, overlap function, uninorm, conditional distributivity

Classification: 08A72, 94A08

1. INTRODUCTION

Aggregation functions are being intensively studied because of their essential roles in many fields and disciplines from mathematics and natural sciences to economics and social sciences. While a high level of attention in this area is concerned on the characterizations of pairs of aggregation functions which satisfy the distributive laws. This topic was derived from [18], and investigations have covered t-norms and t-conorms [12, 20], quasi-arithmetic means [32], pseudo-arithmetical operations [29], uninorms and nullnorms [10, 19, 39], semi-t-operators and uninorms [31, 40, 41], 2-uninorms [30], Mayor's aggregation operators [5]. Afterwards researchers investigated the problem of distributivity on the restricted domain since this particular approach produced a larger variety of solutions. This type of distributivity is known as the conditional distributivity or the restricted distributivity [6, 8, 11–13]. The significance of this considered topic follows not only from the theoretical point of view, but also from its applicability in the integration theory [35–37] and the utility theory [3, 7].

Overlap function [15], as a not necessarily associative binary aggregation function, was introduced by Bustince et al. in 2009 for the purpose of dealing with the frequent overlapping problem of classification in image processing, and the main use of overlap function is to measure the overlapping degree between the two functions standing for object and background, respectively. Some other interesting applications of overlap

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functions include but not exclusively limited to fuzzy preference modeling [16], decision making [27] and fuzzy community detection problems [4]. The work on the distributivity law for overlap functions was started by Qiao in 2019, who proposed the distributivity equation of uninorm over overlap function in [23], and studied the equation when the uninorm U belonged to one of the usual classes $\mathcal{U}_{\min}, \mathcal{U}_{\max}$ [21], the family of idempotent uninorms, representable uninorms. Afterwards Liu and Zhao gave characterizations of the distributivity for continuous t-norms and uninorms continuous in $(0,1)^2$ with respect to overlap functions [17]. In the meantime, Zhu et al. considered the distributivity of overlap functions with neutral element 1 over uninorms [25]. Later on, Zhang and Qin discussed the distributivity between overlap functions and uni-nullnorms [34], null-uninorms [38] or 2-uninorms [33].

This paper is to extend research towards distributivity equations on the restricted domain for overlap functions over uninorms. Because of the heated discussions on the structure of uninorms with continuous underlying operators have been going on for many years, our concern is on the conditional distributivity equation:

$$O(x, U(y,z)) = U(O(x,y), O(x,z)), \quad x, y, z \in [0,1],\ U(y,z) < 1,$$

where O is an overlap function, and U is a uninorm with continuous underlying operators. The results presented in this paper will provide characterizations of some new pairs of aggregation functions that fulfill conditional distributivity. Also, the given results of conditional distributivity can not be obtained from the classical distributivity, which illustrates the strength and usefulness of the conditional case. Motivation for this line of investigation lies in possibility of obtaining new pairs of aggregation functions that further on can be applied in the utility theory for modeling some specific problems.

The paper is organized as follows. Section 2 contains preliminary notions concerning overlap functions, t-conorms and uninorms. In the following, two directions on conditional distributivity for overlap functions are investigated. The first one is conditional distributivity over continuous t-conorms(see Section 3), while the second one is over the uninorms with continuous underlying t-norms and t-conorms(see Section 4). The concluding remarks are given in Section 5.

2. PRELIMINARIES

In this section, we will present some main definitions and results which will be used throughout the paper. For a further reading about such concepts, we recommend [1, 9, 12–14, 21, 22, 28].

Definition 2.1. (Klement et al. [12]) A bivariate function $T : [0,1]^2 \to [0,1]$ (or $S : [0,1]^2 \to [0,1]$) is called a t-norm(or t-conorm) if it is commutative, associative, increasing and 1(or 0) is the neutral element.

The following are three basic t-norms, named T_M, T_P and T_L, respectively:

(i) the minimum t-norm: $T_M(x,y) = \min(x,y), \quad x, y \in [0,1]$;

(ii) the product t-norm: $T_P(x,y) = xy, \quad x, y \in [0,1]$;

(iii) the Łukasiewicz t-norm: $T_L(x,y) = \max(x+y-1, 0), \quad x, y \in [0,1]$.

Dually, the three basic t-conorms S_M, S_P and S_L can be given by, respectively:

(i) the maximum t-conorm: $S_M(x,y) = \max(x,y), \quad x,y \in [0,1]$;
(ii) the probabilistic sum t-conorm: $S_P(x,y) = x+y-xy, \quad x,y \in [0,1]$;
(iii) the Łukasiewicz t-conorm: $S_L(x,y) = \min(x+y,1), \quad x,y \in [0,1]$.

Definition 2.2. (Klement et al. [12], Fodor et al. [21]) A bivariate function $U : [0,1]^2 \to [0,1]$ is called a uninorm if it is commutative, associative, increasing and $e \in [0,1]$ is a neutral element.

Write $\mathcal{U}(e)$ as the family of uninorms with neutral element e. Note that t-norms and t-conorms are special cases of uninorms, i. e., the uninorm U becomes a t-norm when $e = 1$; the uninorm U becomes a t-conorm when $e = 0$. With any uninorm U with neutral element $e \in]0,1[$, we can associate two binary operations T_U, $S_U : [0,1]^2 \to [0,1]$ defined by

$$T_U(x,y) = \frac{U(ex,ey)}{e},$$

and

$$S_U(x,y) = \frac{U(e+(1-e)x, e+(1-e)y) - e}{1-e},$$

then T_U is a t-norm and S_U is a t-conorm. In other words, on $[0,e]^2$ any uninorm U is determined by a t-norm T_U, and on $[e,1]^2$ any uninorm U is determined by a t-conorm S_U; T_U is called the underlying t-norm, and S_U is called the underlying t-conorm. U is always written as $U = \langle T, e, S\rangle$ and the set of all uninorms with continuous underlying operators is denoted by $\mathcal{COU}$. Now, let us denote the remaining part of the unit square by $A(e)$, i. e.,

$$A(e) = [0,1]^2 \setminus ([0,e]^2 \bigcup [e,1]^2).$$

On the set $A(e)$, any uninorm U is bounded by the minimum and maximum of its arguments, i. e., for any $(x,y) \in A(e)$, it holds that

$$\min(x,y) \le U(x,y) \le \max(x,y).$$

Definition 2.3. (Baets [1]) Let U be a uninorm, and $a \in [0,1]$. If $U(a,a) = a$, then a is said to be an idempotent element of U. Denote the set of idempotent elements of U by $Id(U)$. Moreover, if $Id(U) = [0,1]$, then U is said to be idempotent.

Definition 2.4. (Baets et al. [2]) Let $g : [0,1] \to [0,1]$ be a decreasing function and let G be the graph of g, that is

$$G = \{(x, g(x)) | x \in [0,1]\}.$$

For any discontinuity point s of g, let us denote by s^- and s^+ the corresponding lateral limits, that are $s^- = \lim_{x \to s^-} g(x)$ and $s^+ = \lim_{x \to s^+} g(x)$. Then, we define the completed graph of g, as the set

$$F_g = G \bigcup \{(0,y) | y > g(0)\} \bigcup \{(1,y) | y < g(1)\} \bigcup D_g,$$

where $D_g = \{(s,y) | s \textit{ is the discontinuity point of } g, \; s^- < y < s^+\}$.

Definition 2.5. (Baets et al. [2]) A decreasing function $g : [0,1] \to [0,1]$ is called Id-symmetrical if, for all $(x,y) \in [0,1]^2$, it holds that

$$(x,y) \in F_g \Leftrightarrow (y,x) \in F_g.$$

Theorem 2.6. (Ruiz et al. [9]) Let $U \in \mathcal{U}(e)$ with $e \in (0,1)$. Then U is idempotent if and only if there exists an Id-symmetrical decreasing function $g : [0,1] \to [0,1]$ with fixed point e such that

$$U(x,y) = \begin{cases} \min(x,y), & y < g(x) \text{ or } (y = g(x) \text{ and } x < g(g(x))), \\ \max(x,y), & y > g(x) \text{ or } (y = g(x) \text{ and } x > g(g(x))), \\ x \text{ or } y, & \text{otherwise}, \end{cases} \tag{1}$$

and U is commutative on the set $\{(x,g(x))|x = g(g(x))\}$. Such function g is usually called the associated function of U.

We denote the idempotent uninorm U with neutral element e and associated function g by $U = \langle e,g\rangle_{ide}$, and the set of all idempotent uninorms is denoted by $\mathcal{U}_{ide}$ throughout our paper.

Here are two special idempotent uninorms, named $U_e^{\min}$ and $U_e^{\max}$:

$$U_e^{\min}(x,y) = \begin{cases} \max(x,y) & (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise}, \end{cases}$$

$$U_e^{\max}(x,y) = \begin{cases} \min(x,y) & (x,y) \in [0,e]^2, \\ \max(x,y) & \text{otherwise}. \end{cases}$$

Definition 2.7. (Martín et al. [22]) A uninorm $U : [0,1]^2 \to [0,1]$ is called locally internal if it satisfies $U(x,y) \in \{x,y\}$ *for all* $x,y \in [0,1]$.

Any idempotent uninorm is locally internal.

Theorem 2.8. (Baets et al. [2], Martín et al. [22]) Let $U \in \mathcal{U}(e)$ be locally internal on $A(e)$ with $e \in (0,1)$, then there exists an Id-symmetrical decreasing function $g : [0,1] \to [0,1]$ with $g(e) = e$ such that for all $(x,y) \in A(e)$, U has the form of Eq.(1).

The uninorm, which is locally internal on $A(e)$, with associated function g is denoted by $U = \langle e,g\rangle_{loc}$, and write the set of uninorms locally internal on $A(e)$ as $\mathcal{U}_{loc}$.

Proposition 2.9. (Li et al. [13]) Let $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$, then the following statements hold:

(1) If $T_U = (\langle a_k, b_k, T_k\rangle)_{k\in K}$ and $S_U = S_M$, then $U(x,y) \in \{x,y\}$ for all $(x,y) \in A(e)$, i. e., U is locally internal in $A(e)$.

(2) If $T_U = T_M$ and $S_U = (\langle c_j, d_j, S_j\rangle)_{j\in J}$, then $U(x,y) \in \{x,y\}$ for all $(x,y) \in A(e)$, i. e., U is locally internal in $A(e)$.

Definition 2.10. (Bustince et al. [14]) A bivariate function $O : [0,1]^2 \to [0,1]$ is called an overlap function if, for all $x, y \in [0,1]$, the following conditions hold:

($O1$) O is commutative;
($O2$) $O(x,y) = 0$ iff $xy = 0$;
($O3$) $O(x,y) = 1$ iff $xy = 1$;
($O4$) O is increasing;
($O5$) O is continuous.

In some contexts, the positivity of boundary conditions in Definition 2.10 is not an indispensable property, especially in behavior classification problems. The papers [24,26] introduced the concept of 0-overlap function, 1-overlap function and 2-dimensional general overlap function, which differed in the boundary conditions, allowing the functions to have zero or one divisors. The bivariate function $O : [0,1]^2 \to [0,1]$ is called a 0-overlap function if only condition ($O2$) in Definition 2.10 is downgraded to

($O2$)$'$ if $xy = 0$ then $O(x,y) = 0$.

O is called a 1-overlap function if only ($O3$) in Definition 2.10 is replaced by

($O3$)$'$ if $xy = 1$ then $O(x,y) = 1$.

O is called 2-dimensional general overlap function if ($O2$) and ($O3$) in Definition 2.10 is reduced to ($O2$)$'$ and ($O3$)$'$.

Here are some conclusions related to Cauchy function equation used in solving equations.

Definition 2.11. (Kuczma [28]) The function $f : \mathbb{R} \to \mathbb{R}$ is called the additive function if, for any $x, y \in \mathbb{R}$, f satisfies the following Cauchy equation:

$$f(x+y) = f(x) + f(y). \tag{2}$$

Theorem 2.12. (Kuczma [28]) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If f satisfies Eq. (2), then there exists $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{R}$.

3. CONDITIONAL DISTRIBUTIVITY OF OVERLAP FUNCTIONS OVER CONTINUOUS T-CONORMS

Since it always requires more sophisticated approaches while solving conditional distributivity equations, the first concern of this paper is to solve the conditional distributivity of overlap functions over continuous t-conorms. As is well-known, the distributivity of overlap function over t-conorm is so strong a condition that it reduces the t-conorm to be S_M. Thus, it seems to be a reasonable direction of research to restrict the domain of the distributivity law to obtain some new solutions that are non-idempotent.

Definition 3.1. An overlap function O is conditional distributive over a t-conorm S if, for all $x, y, z \in [0,1]$,

$$O(x, S(y,z)) = S(O(x,y), O(x,z)), \quad \text{whenever } S(y,z) < 1. \tag{3}$$

In order to simplify the researches on the conditional distributivity of overlap functions over continuous t-conorms, we firstly discuss the relationship between conditional distributivity and the $\varphi-$conjugate.

Proposition 3.2. Let O be an overlap function, S be a t-conorm, and φ be an automorphism. Then O is conditionally distributive over S if and only if O_φ is conditionally distributive over S_φ.

Proof. According to the nature of automorphism, we only need to prove one direction. Let's assume O is conditionally distributive over S. For any $x, y, z \in [0, 1]$, if $S_\varphi(y, z) < 1$, then $S(\varphi(y), \varphi(z)) < 1$, so one has

$$\begin{aligned} O_\varphi(x, S_\varphi(y, z)) =& \varphi^{-1}(O(\varphi(x), \varphi(S_\varphi(y, z)))) \\ =& \varphi^{-1}(O(\varphi(x), S(\varphi(y), \varphi(z)))) \\ =& \varphi^{-1}(S(O(\varphi(x), \varphi(y)), O(\varphi(x), \varphi(z)))) \\ =& S_\varphi(\varphi^{-1}(O(\varphi(x), \varphi(y))), \varphi^{-1}(O(\varphi(x), \varphi(z)))) \\ =& S_\varphi(O_\varphi(x, y), O_\varphi(x, z)). \end{aligned}$$

Thus, O_φ is conditional distributive over S_φ. □

Next, the conditional distributivity of overlap functions over continuous t-conorms will be discussed in detail.

Lemma 3.3. Let O be an overlap function, S be a continuous t-conorm, $u \in (0, 1)$ be an idempotent element of S. If O is conditionally distributive over S, then $[0, O(1, u)] \subseteq Id(S)$.

Proof. For all $x \in [0, 1]$, since O is conditionally distributive over S, it holds that

$$O(x, u) = O(x, S(u, u)) = S(O(x, u), O(x, u)).$$

That is, $O(x, u)$ is an idempotent element of S for each $x \in [0, 1]$. Thus, it follows from (O2) and (O5) that $[0, O(1, u)] \subseteq Id(S)$. □

Lemma 3.3 illustrates that the maximal non-trivial idempotent element of a continuous t-conorm plays important roles in investigating our conditional distributivity problem. Therefore, we firstly discuss the solutions of Eq.(3) when the continuous t-conorm has no non-trivial idempotent element, i. e., it is a continuous Archimedean t-conorm.

Theorem 3.4. Let O be an overlap function, S be a continuous Archimedean t-conorm. Then O is conditionally distributive over S if and only if S is a nilpotent t-conorm, and if $S = (S_L)_\varphi$ for an automorphism φ then $O = (T_P)_\varphi$.

Proof. ($\Rightarrow$) Suppose S is a strict t-conorm, then by Proposition 3.2, it is enough to consider the case $S = S_P$. Since O is conditionally distributive over S_P, it holds that $O(x, S_P(y, y)) = S_P(O(x, y), O(x, y)) = 2O(x, y) - O(x, y)^2$ for all $x \in (0, 1)$ and $y \in [0, 1)$. Thus, it follows from the continuity of O and S_P that $O(x, 1) = \lim_{y \to 1} O(x, S_P(y, y)) = \lim_{y \to 1} (2O(x, y) - O(x, y)^2) = 2O(x, 1) - O(x, 1)^2$, and then it implies that $O(x, 1) = 1$ or $O(x, 1) = 0$. Thus, by $(O2)$ and $(O3)$, there holds $x = 1$ or $x = 0$, an obvious contradiction with the value of x. Consequently, S is a nilpotent t-conorm. Similarly, our priority is to consider $S = S_L$. For any $x, y, z \in [0, 1]$, if $S_L(y, z) < 1$, i. e., $y + z < 1$, then it follows from conditional distributivity that

$$S_L(O(x, y), O(x, z)) = O(x, S_L(y, z)) = O(x, y + z) < 1.$$

Furthermore, it holds that

$$O(x, y + z) = O(x, y) + O(x, z). \tag{4}$$

Fixed $x \in [0, 1]$, define a function $f_x : [0, 1) \to \mathbb{R}$ by $f_x(y) = O(x, y)$, then it holds that $f_x(0) = 0$, and Eq.(4) can be rewritten as $f_x(y + z) = f_x(y) + f_x(z)$. Obviously, f_x is continuous, and satisfies Eq.(2). Therefore, by Theorem 2.12, there exists $c_x \in \mathbb{R}$ such that $f_x(y) = c_x y$ for all $y \in [0, 1)$. In addition, because of continuity,

$$f_x(y) = c_x y, \; y \in [0, 1]. \tag{5}$$

Specially, if $x = 1$, it holds that $f_1(1) = O(1, 1) = 1$. Thus, it follows from Eq.(5) that $c_1 = 1$. Furthermore, for each $y \in [0, 1]$, we can obtain $O(1, y) = f_1(y) = y$. And then, $c_x = f_x(1) = x$ for all $x \in [0, 1]$. Thus, O is actually T_P. Moreover, associated with Proposition 3.2, if there exists automorphism φ such that $S = (S_L)_\varphi$, then $O = (T_P)_\varphi$. Obviously, O is a strict t-norm.

($\Leftarrow$) By Proposition 3.2, we only need to prove that T_P is conditionally distributive over S_L. For any $x, y, z \in [0, 1]$, if $S_L(y, z) < 1$, we have $y + z < 1$ and $xy + xz < 1$, and then,

$$T_P(x, S_L(y, z)) = x(y + z) = xy + xz = S_L(T_P(x, y), T_P(x, z)).$$

Thus, T_P is conditionally distributive over S_L. □

Next, we will discuss the conditional distributivity of an overlap function over the continuous t-conorm with a maximum non-trivial idempotent element.

Proposition 3.5. Let O be an overlap function, S be a continuous t-conorm, $a \in (0, 1)$ be the maximum non-trivial idempotent element of S. If O is conditionally distributive over S, then $O(1, a) = a$.

Proof. As we all know, $S |_{[a,1]^2}$ is isomorphic to a continuous Archimedean t-conorm. And Lemma 3.3 shows that $[0, O(1, a)] \subseteq Id(S)$, so $O(1, a) \leq a$. Assume $O(1, a) < a$, and let $H = \{t \in [0, 1] \mid O(1, t) = a\}$, then $H \neq \emptyset$. Take $\alpha = \wedge H$, then $O(1, \alpha) = a$ and $a < \alpha < 1$. Since $S |_{[a,1]^2}$ is isomorphic to a continuous Archimedean t-conorm, then $\alpha = S(\beta, \beta) > \beta$ holds for some $\beta \in (a, 1)$. Moreover, due to the conditional distributivity, we have $a = O(1, \alpha) = O(1, S(\beta, \beta)) = S(O(1, \beta), O(1, \beta))$. Then it follows from the structure of S that $O(1, \beta) = a$, it means $\beta \in H$ and $\alpha \leq \beta$, an obvious contradiction. Consequently, $O(1, a) = a$. □

Proposition 3.6. Let O be an overlap function, S be a continuous t-conorm, $a \in (0,1)$ be the maximum non-trivial idempotent element of S. If O is conditionally distributive over S, then $O(x,a) = O(x,1)$ holds for all $x \in [0,a]$.

Proof. The case of $x = 0$ is obvious. Specially, $S = (\langle a, 1, S^* \rangle)$, where S^* is a continuous Archimedean t-conorm. So for any $\epsilon \in (0, 1-a)$, there exists $t_\epsilon \in (a,1)$ such that $1 - \epsilon = S(t_\epsilon, t_\epsilon) > t_\epsilon$. Thus, for $x \in (0,a]$, since O is conditionally distributive over S, it follows from Proposition 3.5 and the structure of S that

$$O(x, 1-\epsilon) = O(x, S(t_\epsilon, t_\epsilon)) = S(O(x,t_\epsilon), O(x,t_\epsilon)) = O(x, t_\epsilon).$$

Take $H_{x,\epsilon} = \{y \in (a,1) \mid O(x,y) = O(x, 1-\epsilon)\}$, then $H_{x,\epsilon} \neq \emptyset$. Let $m_{x,\epsilon} = \bigwedge H_{x,\epsilon}$, then by the continuity of O, one has $O(x, m_{x,\epsilon}) = O(x, 1-\epsilon)$ and $a \leq m_{x,\epsilon} < 1$. We assert $m_{x,\epsilon} = a$. In fact, if $m_{x,\epsilon} > a$, then there exists $n_{x,\epsilon} \in (a,1)$ such that $m_{x,\epsilon} = S(n_{x,\epsilon}, n_{x,\epsilon}) > n_{x,\epsilon}$. Thus, it follows from the conditional distributivity that $O(x, m_{x,\epsilon}) = O(x, S(n_{x,\epsilon}, n_{x,\epsilon})) = S(O(x, n_{x,\epsilon}), O(x, n_{x,\epsilon})) = O(x, n_{x,\epsilon})$, then one has $n_{x,\epsilon} \in H_{x,\epsilon}$, a contradiction. Thus, $m_{x,\epsilon} = a$ and $O(x, 1-\epsilon) = O(x,a)$. Hence, from the randomness of ϵ and continuity of O, it can be concluded $O(x,a) = O(x,1)$ for all $x \in [0,a]$. □

Proposition 3.7. Let O be an overlap function, S be a continuous t-conorm, and $a \in (0,1)$ be the maximum non-trivial idempotent element of S. If O is conditionally distributive over S, then for any $x, y \in [a,1]$, $O(x,y) = a$ if and only if $x = a$ or $y = a$.

Proof. ($\Leftarrow$) It is acquirable from Proposition 3.5 and Proposition 3.6.
($\Rightarrow$) Suppose $O(x_0, y_0) = a$ holds for some $x_0, y_0 \in (a,1]$. Let $b = \bigvee\{x \in (a,1) | O(x,x) = a\}$, then $b \in (a,1)$ and $O(b,b) = a$. Furthermore, take $H = \{x \in (a,1] \mid O(b,x) = a\}$ and $c = \bigvee H$, then $1 \geq c \geq b > a$ and $O(b,c) = a$.

- If $c = 1$, then $O(1,b) = a$. Take $d = \bigvee\{x \in (a,1) | O(1,x) = a\}$, then $b \leq d < 1$ and $O(1,d) = a$. Since $S \mid_{[a,1]^2}$ is isomorphic to a continuous Archimedean t-conorm, there exists $t \in (a,1)$ such that $t < d < S(t,t) < 1$.Therefore, it follows from the conditional distributivity that

  $$O(1, S(t,t)) = S(O(1,t), O(1,t)) = S(a,a) = a.$$

  Thus, we have $S(t,t) \leq d$, a contradiction.

- If $b \leq c < 1$ and $O(b,c) = a$ hold. Similarly, there exists $u \in (a,1)$ such that $u < c < S(u,u) < 1$. So it holds that $a \leq O(b,u) \leq O(b,c) = a$, i. e., $O(b,u) = a$. Due to the conditional distributivity, we have $O(b, S(u,u)) = S(O(b,u), O(b,u)) = S(a,a) = a$, that is, $S(u,u) \in H$, an obvious contradiction.

Consequently, for any $x, y \in [a,1]$, if $O(x,y) = a$, then $x = a$ or $y = a$. □

Theorem 3.8. Let O be an overlap function, S be a continuous t-conorm, and $a \in (0,1)$ be the maximum non-trivial idempotent element of S. Then O is conditionally distributive over S if and only if S and O have the following forms (see Figure 1):

$$S(x,y) = \begin{cases} a + (1-a)(S_L)_\varphi(\frac{x-a}{1-a}, \frac{y-a}{1-a}), & (x,y) \in [a,1]^2, \\ \max(x,y), & \text{otherwise}, \end{cases}$$

$$O(x,y) = \begin{cases} a + (1-a)(T_P)_\varphi(\frac{x-a}{1-a}, \frac{y-a}{1-a}), & (x,y) \in [a,1]^2, \\ aO_1(\frac{x}{a}, \frac{y}{a}), & (x,y) \in [0,a]^2, \\ aO_1(\frac{\min(x,y)}{a}, 1), & \text{otherwise}, \end{cases}$$

where $(S_L)_\varphi$ is the $\varphi-$conjugation of S_L, $(T_P)_\varphi$ is the $\varphi-$conjugation of T_P, O_1 is a 1-overlap function.

Proof. ($\Rightarrow$) It follows from Theorem 3.4, Proposition 3.6 and Proposition 3.7.

($\Leftarrow$) $\forall\ x,y,z \in [0,1]$, we shall prove Eq. (3) in the following cases:

(i) when $x,y,z \in [0,a]$, $S(y,z) \leq a$, we have

$$O(x,S(y,z)) = O(x,\max(y,z)) = \max(O(x,y),O(x,z)) = S(O(x,y),O(x,z));$$

(ii) when $x,y,z \in [a,1]$, Eq.(3) can be obtained from Theorem 3.4;

(iii) when $x,y \in [0,a]$, $z \in (a,1]$, if $S(y,z) < 1$, then $z \in (a,1)$, and it holds that $S(y,z) = z$, $O(x,y) \leq O(x,z) \leq a$, thus, by the structure of S, one has

$$O(x,S(y,z)) = O(x,z) = \max(O(x,y),O(x,z)) = S(O(x,y),O(x,z));$$

(iv) when $y,z \in [0,a]$, $x \in (a,1]$, by the structure of S and O, we get $S(y,z) = \max(y,z) < 1$, $O(x,y) \leq a$, $O(x,z) \leq a$, thus, it holds that

$$O(x,S(y,z)) = O(x,\max(y,z)) = \max(O(x,y),O(x,z)) = S(O(x,y),O(x,z));$$

(v) when $x,z \in [0,a]$, $y \in (a,1]$, it can be proved similarly to item (iii);

(vi) when $x,y \in (a,1]$, $z \in [0,a)$, if $S(y,z) < 1$, then $y \in (a,1)$, and it follows that $S(y,z) = y$, $O(x,y) > a$, $O(x,z) \leq a$, therefore, by the structure of S, one gets

$$O(x,S(y,z)) = O(x,y) = \max(O(x,y),O(x,z)) = S(O(x,y),O(x,z));$$

(vii) when $y,z \in (a,1]$, $x \in [0,a)$, if $S(y,z) < 1$, then $O(x,S(y,z)) = O(x,y) = O(x,z) = O(x,a)$, thus, it follows from the structure of S that

$$O(x,S(y,z)) = O(x,a) = \max(O(x,y),O(x,z)) = S(O(x,y),O(x,z));$$

(viii) when $x,z \in (a,1]$, $y \in [0,a)$, it can be proved similarly to item (vi).

Consequently, O is conditionally distributive over S. □

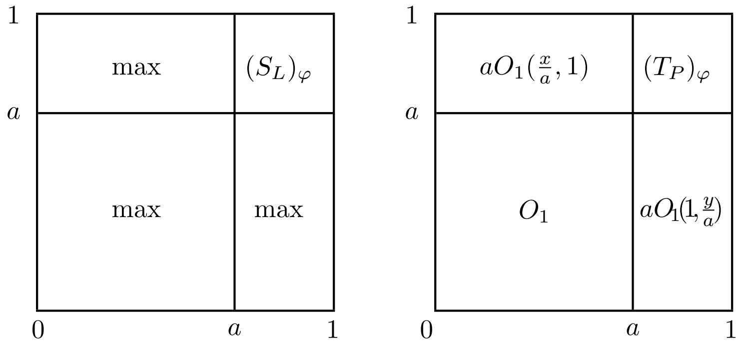

Fig. 1. Structure of S and O from Theorem 3.8.

When every element of [0,1] is an idempotent element of continuous t-conorm S, i. e., $S = S_M$, obviously, any overlap function will be conditionally distributive over S. To sum up, we can give a complete characterization of the conditional distributivity of overlap functions over continuous t-conorms.

Theorem 3.9. Let O be an overlap function, S be a continuous t-conorm. Then O is conditionally distributive over S if and only if one of the following holds:

(1) $S = S_M$.

(2) There exists idempotent element $a \in [0, 1)$ such that S and O have the following forms(see Figure 1):

$$S(x,y) = \begin{cases} a + (1-a)(S_L)_\varphi(\frac{x-a}{1-a}, \frac{y-a}{1-a}), & (x,y) \in [a,1]^2, \\ \max(x,y), & \text{otherwise,} \end{cases}$$

$$O(x,y) = \begin{cases} a + (1-a)(T_P)_\varphi(\frac{x-a}{1-a}, \frac{y-a}{1-a}), & (x,y) \in [a,1]^2, \\ aO_1(\frac{x}{a}, \frac{y}{a}), & (x,y) \in [0,a]^2, \\ aO_1(\frac{\min(x,y)}{a}, 1), & \text{otherwise,} \end{cases}$$

where $(S_L)_\varphi$ is the $\varphi-$conjugation of S_L, $(T_P)_\varphi$ is the $\varphi-$conjugation of T_P, O_1 is a 1-overlap function.

Corollary 3.10. Let O be an overlap function with neutral element 1, S be a continuous t-conorm. Then O is conditionally distributive over S if and only if one of the following holds:

(1) $S = S_M$.

(2) There exists an idempotent element $a \in [0, 1)$ such that $S = (\langle a, 1, (S_L)_\varphi \rangle)$, and $O = (\langle 0, a, O_1 \rangle, \langle a, 1, (T_P)_\varphi \rangle)$, where $(S_L)_\varphi$ is the $\varphi-$conjugation of S_L, $(T_P)_\varphi$ is the $\varphi-$conjugation of T_P, O_1 is an overlap function with neutral element 1.

Example 3.11. Consider the t-conorm S and overlap function O as follows:

$$S(x,y)=\begin{cases}\min(x+y-\frac{1}{2},1), & (x,y)\in[\frac{1}{2},1]^2,\\ \max(x,y), & \text{otherwise,}\end{cases}$$

$$O(x,y)=\begin{cases}2xy-x-y+1, & (x,y)\in[\frac{1}{2},1]^2,\\ \sqrt{x}, & (x,y)\in[0,\frac{1}{4}]\times[\sqrt{x},1],\\ \sqrt{y}, & (y,x)\in[0,\frac{1}{4}]\times[\sqrt{y},1],\\ \frac{1}{2}, & (x,y)\in[\frac{1}{4},\frac{1}{2}]\times[\frac{1}{2},1]\bigcup[\frac{1}{2},1]\times[\frac{1}{4},\frac{1}{2}],\\ \max(x,y), & \text{otherwise,}\end{cases}$$

then one can verify that O is conditionally distributive over the conorm S.

4. CONDITIONAL DISTRIBUTIVITY OF OVERLAP FUNCTIONS OVER UNINORMS WITH CONTINUOUS UNDERLYING OPERATORS

The distributivity law of overlap function over uninorm is too strong a condition that makes the structure of uninorm very simple, the uninorm is actually reduced to T_M, S_M or $U_e^{\min}$. Thus, it is reasonable to restrict the domain of distributivity law to obtain some new solutions. While the uninorms with continuous underlying operators have magic structure, we consider a counterpart of the distributivity of overlap functions over this class of uninorms on the restricted domain. In this section, we shall discuss the conditional distributivity of overlap functions over uninorms with continuous underlying operators, and we suppose that the neutral element e of uninorm belongs to $(0,1)$.

Definition 4.1. An overlap function O is conditionally distributive over a uninorm U with continuous underlying operators if, for all $x,y,z\in[0,1]$,

$$O(x,U(y,z))=U(O(x,y),O(x,z)),\quad \text{whenever } U(y,z)<1. \tag{6}$$

Proposition 4.2. Let O be an overlap function and $U\in\mathcal{COU}$. If O is conditionally distributive over U, then $O(1,e)=e$.

Proof. Since $O(1,0)=0$ and $O(1,1)=1$, then it follows from the continuity of O that there exists $t\in(0,1)$ such that $e=O(1,t)$. Thus, by the conditional distributivity, one has $e=O(1,t)=O(1,U(e,t))=U(O(1,e),O(1,t))=U(O(1,e),e)=O(1,e)$. □

Lemma 4.3. Let O be an overlap function and $U\in\mathcal{COU}$. If O is conditionally distributive over U, and $v\in(0,1)$ is an idempotent element of U, then $[0,O(1,v)]\subseteq Id(U)$.

Proof. The proof is similar to Lemma 3.3. □

From Proposition 4.2 and Lemma 4.3, we can get some results as follows.

Remark 4.4. If overlap function O is conditionally distributive over uninorm U with continuous underlying operators, then $[0,e]\subseteq Id(U)$, which means the underlying t-norm $T_U=T_M$.

Proposition 4.5. Let O be an overlap function and $U \in \mathcal{COU}$. If O is conditionally distributive over U, then $O(x,e) = O(x,1)$ for all $x \in [0,e]$.

Proof. For any $x \in [0,e]$ and $y \in [e,1)$, since O is conditionally distributive over U, it follows from the structrue of U that

$$O(x,y) = O(x,U(e,y)) = U(O(x,e),O(x,y)) = \min(O(x,e),O(x,y)) = O(x,e).$$

Thus, by the continuity of O, $O(x,e) = O(x,1)$ holds for all $x \in [0,e]$. □

Proposition 4.6. Let O be an overlap function and $U \in \mathcal{COU}$. If O is conditionally distributive over U, then $U(0,x) = 0$ for all $x \in (e,1)$.

Proof. Since $T_U = T_M$ and S_U can be written as ordinal sum, it can be known from Proposition 2.9 that U is locally internal uninorm, then $U(0,x) \in \{0,x\}$ for all $x \in (e,1)$. Suppose $U(0,x_0) = x_0$ holds for some $x_0 \in (e,1)$, then by Proposition 4.5 and the monotonicity of O, we have $O(e,x_0) = e$. Since O is conditionally distributive over U, we have

$$e = O(e,x_0) = O(e,U(0,x_0)) = U(O(e,0),O(e,x_0)) = U(0,O(e,x_0)) = U(0,e) = 0,$$

which is an obvious contradiction. Consequently, $U(0,x) = 0$ for all $x \in (e,1)$. □

For the uninorm U with continuous underlying operators, Lemma 4.3 shows that the maximal non-trivial idempotent element of U plays vital roles in investigating the conditional distributivity. With that in mind, we firstly discuss the conditional distributivity of overlap functions over idempotent uninorms.

Theorem 4.7. Let O be an overlap function, $U \in \mathcal{U}_{ide}$. If O is conditionally distributive over U, then U has the following form:

$$U(x,y) = \begin{cases} \max(x,y), & (x,y) \in [e,1]^2, \\ \min(x,y), & (x,y) \in [0,e]^2 \cup ([0,e) \times (e,1)) \cup ((e,1) \times [0,e)), \\ x \text{ or } y, & \text{otherwise.} \end{cases} \tag{7}$$

Proof. Since U is idempotent, then $U = \langle e, g \rangle_{ide}$ for an associated function g. We assert $g(x) = 1$ for all $x \in [0,e)$. In fact, it can be known from Proposition 4.6 that $g(0) = 1$. Assume that there exists $x_0 \in (0,e)$ such that $g(x_0) < 1$, then the following statements hold:

(1) $O(e,x_0) = e$. For any $y \in (g(x_0),1)$, it holds that $x_0 < e \le g(x_0) < y < 1$, then by the structures of U and O, one has $U(y,x_0) = y$ and $O(e,y) = e$. Since O is conditionally distributive over U, one gets

$$e = O(e,y) = O(e,U(y,x_0)) = U(O(e,y),O(e,x_0)) = U(e,O(e,x_0)) = O(e,x_0).$$

(2) $O(1,g(x_0))=e$. In fact, it is obvious when $g(x_0)=e$; when $g(x_0)>e$, it follows from item (1) and Proposition 4.5 that $O(1,x_0)=O(e,x_0)=e$. Thus, for any $z\in(e,g(x_0))$, one has

$$e=O(1,x_0)=O(1,U(z,x_0))=U(O(1,z),O(1,x_0))=U(O(1,z),e)=O(1,z).$$

Therefore, we can get $O(1,g(x_0))=e$ from the continuity of O.

Let $p=\bigwedge\{x\in[0,e)|g(x)<1\}$, then $0\le p<e$. For any $t\in(p,e)$, it holds that $g(t)<1$ and $O(e,t)=e$. Hence, by the continuity of O, we have $O(e,p)=e$. Obviously, $p\neq 0$. And, by Proposition 4.5, we can get $O(1,p)=e$. Take $u\in(0,e)$ such that $g(u)<1$, then $O(1,g(u))=e$, furthermore, there exists $m_u\in(0,p)$ such that $u=O(1,m_u)$. Then $g(m_u)=1$ because of $m_u\in(0,p)$. Moreover, for any $y\in(g(u),1)$, there exists $z_y\in(g(u),1)$ such that $y=O(1,z_y)$ because of the continuity of O. In addition, since O is conditionally distributive over U, we can get from the structure of U that

$$u=O(1,m_u)=O(1,U(z_y,m_u))=U(O(1,z_y),O(1,m_u))=U(y,u)=y. \tag{8}$$

However, $u<e<y$, which contradicts with Eq.(8).

Consequently, $g(x)=1$ for all $x\in[0,e)$. Then by the characterization of associated function, we have

$$g(x)=\begin{cases}1, & x\in[0,e),\\ e, & x\in[e,1).\end{cases}$$

Thus, U has the form of Eq.(7). □

Proposition 4.8. Let O be an overlap function, $U\in\mathcal{U}_{ide}$. If O is conditionally distributive over U, then $O(x,1)=O(x,e)<e$ for all $x\in[0,e)$.

Proof. For $x=0$, it is obvious. Take any $x\in(0,e)$, by Proposition 4.5, we know $O(x,1)=O(x,e)$. Suppose that there exists $x_0\in(0,e)$ such that $O(1,x_0)=O(e,x_0)=e$, then for any $y\in(e,1)$, it follows from Theorem 4.7 and the conditional distributivity that

$$e=O(1,x_0)=O(1,U(y,x_0))=U(O(1,y),O(1,x_0))=U(O(1,y),e)=O(1,y).$$

Furthermore, we can obtain $O(1,1)=e<1$ in that O is continuous, it is a contradiction. □

Theorem 4.9. Let O be an overlap function, U be an idempotent uninorm, and $U=\langle e,g\rangle_{ide}$. Then O is conditionally distributive over U if and only if one of the following statements holds:

(1) U and O have the following form (see Figure 2):

$$U(x,y)=\begin{cases}\max(x,y), & (x,y)\in[e,1]^2,\\ 1, & (x,y)\in((g(1),e)\times\{1\})\cup(\{1\}\times(g(1),e)),\\ \min(x,y), & \text{otherwise},\end{cases} \tag{9}$$

$$O(x,y)=\begin{cases} e+(1-e)O_2(\frac{x-e}{1-e},\frac{y-e}{1-e}), & (x,y)\in[e,1]^2,\\ eO_1(\frac{x}{e},\frac{y}{e}), & (x,y)\in[0,e]^2,\\ eO_1(\frac{\min(x,y)}{e},1), & \text{otherwise},\end{cases} \tag{10}$$

where O_1 is an overlap function, O_2 is a 0-overlap function, O has the following properties:

($\star$) $O(1,g(1))\le g(1)$.

(2) U has the following form:

$$U(x,y)=\begin{cases} \max(x,y), & (x,y)\in[e,1]^2,\\ 1, & (x,y)\in([g(1),e)\times\{1\})\cup(\{1\}\times[g(1),e)),\\ \min(x,y), & \text{otherwise},\end{cases} \tag{11}$$

and O has the form of Eq.(10)(see Figure 3), but O has the following properties:

($\star'$) $O(1,x)<g(1)$ for all $x\in[0,g(1))$.

Proof. ($\Rightarrow$): By Proposition 4.2, Proposition 4.5 and Proposition 4.8, O satisfies Eq.(10), where O_1 is an overlap function, O_2 is a 0-overlap function. And, by Theorem 4.7, the values of g on $[0,1)$ can be obtained, then due to the value of $g(1)$, we shall discuss the cases below.

i) If $g(1)=0$, then two cases should be considered:

- when U is conjunctive, U satisfies Eq.(9), and O obviously satisfies ($\star$);
- when U is disjunctive, U satisfies Eq.(11), and O satisfies ($\star'$).

ii) If $g(1)=e$, then U satisfies Eq.(11), and by Proposition 4.8, O satisfies ($\star'$).

iii) If $g(1)\in(0,e)$, then idempotent uninorm U has the following form:

$$U(x,y)=\begin{cases} 1 \text{ or } g(1), & (x,y)\in\{(1,g(1)),(g(1),1)\},\\ \max(x,y), & (x,y)\in[e,1]^2,\\ 1, & (x,y)\in((g(1),e)\times\{1\})\cup(\{1\}\times(g(1),e)),\\ \min(x,y), & \text{otherwise}.\end{cases}$$

- When $U(1,g(1))=g(1)$, U satisfies Eq.(9). And, since O is conditionally distributive over U, it holds that

  $$O(1,g(1))=O(1,U(1,g(1)))=U(O(1,1),O(1,g(1)))=U(1,O(1,g(1))).$$

  Furthermore, by the structure of U, we get $O(1,g(1))\le g(1)$.
- When $U(1,g(1))=1$, U satisfies Eq.(11). For any $x\in[0,g(1))$, since O is conditionally distributive over U, one has

  $$O(1,x)=O(1,U(1,x))=U(O(1,1),O(1,x))=U(1,O(1,x)),$$

  then it follows from the structure of U that $O(1,x)<g(1)$.

($\Leftarrow$) Firstly, we discuss the case $g(1)\in(0,e)$.

1) Suppose O and U are given as Eq.(10) and Eq.(9), respectively. For any $x, y, z \in [0,1]$, if $U(y,z) < 1$, then we shall divide our proof into the following cases:

(i) when $x, y, z \in [0,e]$, Eq.(6) holds;

(ii) when $y, z \in [e,1)$ and $x \in [e,1]$, Eq.(6) is easy to get;

(iii) when $x, y \in [0,e]$ and $z \in (e,1)$, it holds that $O(x,y) \leq O(x,z) \leq e$, thus, by the structures of O and U, one has

$$O(x, U(y,z)) = O(x,y) = \min(O(x,y), O(x,z)) = U(O(x,y), O(x,z));$$

(iv) when $x \in [0,e]$, $y \in [0, g(1)]$ and $z = 1$, we have $U(y,1) = y$ and $O(x,y) \leq O(x,z) \leq e$, thus, it holds that

$$O(x, U(y,1)) = O(x,y) = \min(O(x,y), O(x,1)) = U(O(x,y), O(x,1));$$

(v) when $y, z \in [0,e]$ and $x \in (e,1]$, we can obtain $U(y,z) = \min(y,z) < 1$, $O(x,y) \leq e$ and $O(x,z) \leq e$, thus, we have

$$O(x, U(y,z)) = \min(O(x,y), O(x,z)) = U(O(x,y), O(x,z));$$

(vi) when $x, z \in [0,e]$ and $y \in (e,1)$, similarly to case (iii);

(vii) when $x \in [0,e]$, $z \in [0, g(1)]$ and $y = 1$, similarly to case (iv);

(viii) when $x \in (e,1]$, $y \in (e,1)$ and $z \in [0,e)$, we have $U(y,z) = z$, $e \leq O(x,y) < 1$ and $O(x,z) < e$, it follows that

$$O(x, U(y,z)) = O(x,z) = \min(O(x,y), O(x,z)) = U(O(x,y), O(x,z));$$

(ix) when $x \in (e,1]$, $y = 1$ and $z \in [0, g(1)]$, one gets $U(1,z) = z$ and $O(x,1) \geq e$, moreover, $O(x,z) \leq O(1, g(1)) \leq g(1)$ can be obtained by $(\star)$, thus, we have

$$O(x, U(1,z)) = O(x,z) = \min(O(x,1), O(x,z)) = U(O(x,1), O(x,z));$$

(x) when $y, z \in (e,1)$ and $x \in [0,e)$, it holds that $O(x, U(y,z)) = O(x,y) = O(x,z) = O(x,e) < e$, thus, we get

$$O(x, U(y,z)) = O(x,e) = U(O(x,y), O(x,z));$$

(xi) when $x \in (e,1]$, $z \in (e,1)$ and $y \in [0,e)$, similarly to case (viii);

(xii) when $x \in (e,1]$, $z = 1$ and $y \in [0, g(1)]$, similarly to case (ix).

As seen above, in all considered cases we obtain the overlap function O given as Eq.(10) is conditionally distributive over the uninorm U shown in Eq.(9).

2) If O is an overlap function satisfying Eq.(10), and U is a uninorm satisfying Eq.(11), the conditional distributivity can be proved similarly to the case 1).

For the cases $g(1) = 0$ and $g(1) = e$, the conditional distributivity can be easily proved. □

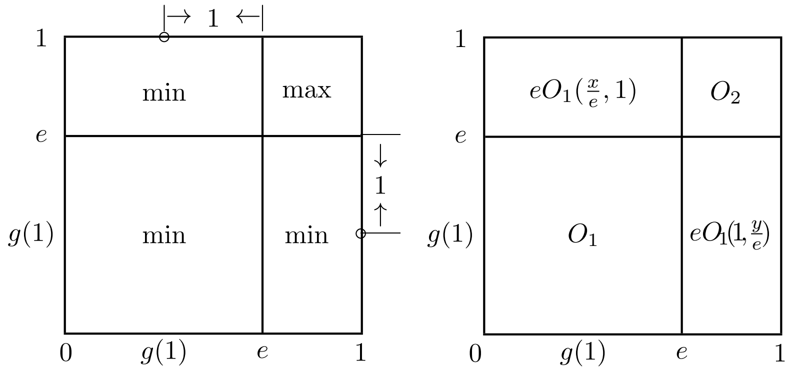

Fig. 2. Structure of U and O from Theorem 4.9(1).

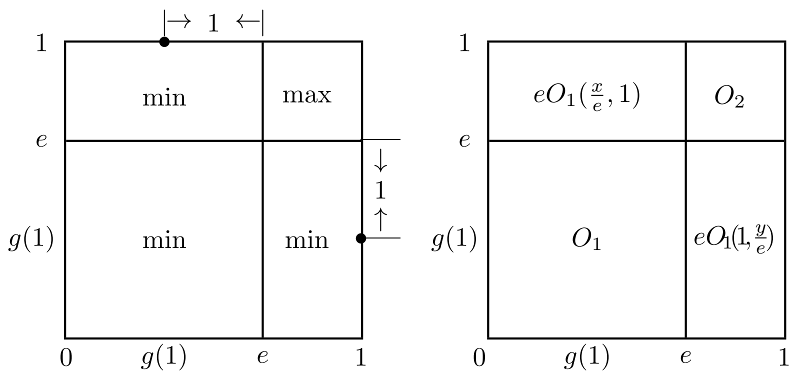

Fig. 3. Structure of U and O from Theorem 4.9(2).

In order to make the difference between the two solutions clearer, let's give an example as follows.

Example 4.10. Consider the idempotent uninorm $U =< \frac{1}{2}, g >$ with associated function

$$g(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}), \\ \frac{1}{2}, & x \in [\frac{1}{2}, 1), \\ \frac{1}{4}, & x = 1. \end{cases}$$

(1) If U is given by

$$U(x, y) = \begin{cases} \max(x, y), & (x, y) \in [\frac{1}{2}, 1]^2 \bigcup ((\frac{1}{4}, \frac{1}{2}) \times \{1\}) \cup (\{1\} \times (\frac{1}{4}, \frac{1}{2})), \\ \min(x, y), & \text{otherwise}, \end{cases}$$

and we define an overlap function O as follows:

$$O(x,y)=\begin{cases}\frac{1}{4}O_1(4x,4y), & (x,y)\in[0,\frac{1}{4}]^2,\\ \max(x+y-\frac{1}{2},\frac{1}{4}), & (x,y)\in[\frac{1}{4},\frac{1}{2}]^2,\\ \frac{1}{2}+\frac{1}{2}O_2(2x-1,2y-1) & (x,y)\in[\frac{1}{2},1]^2,\\ \frac{1}{4}O_1(4x,1), & (x,y)\in[0,\frac{1}{4}]\times[\frac{1}{4},1],\\ \frac{1}{4}O_1(1,4y), & (x,y)\in[\frac{1}{4},1]\times[0,\frac{1}{4}],\\ \min(x,y), & \text{otherwise},\end{cases}$$

where O_1 and O_2 are defined by

$$O_1(x,y)=\begin{cases}\sqrt{2x}, & (x,y)\in[0,\frac{1}{2}]\times[\sqrt{2x},1],\\ \sqrt{2y}, & (y,x)\in[0,\frac{1}{2}]\times[\sqrt{2y},1],\\ \max(x,y), & \text{otherwise},\end{cases}$$

$$O_2(x,y)=\min(x,y)\max(x^2+y^2-1,0),$$

the structure of O and U is shown in Figure 4, then one can verify that O is conditionally distributive over the uninorm U. This fits in with the first solution in Theorem 4.9.

(2) If U is given by

$$U(x,y)=\begin{cases}\max(x,y), & (x,y)\in[\frac{1}{2},1]^2\bigcup([\frac{1}{4},\frac{1}{2})\times\{1\})\cup(\{1\}\times[\frac{1}{4},\frac{1}{2})),\\ \min(x,y), & \text{otherwise},\end{cases}$$

and an overlap function O^* is constructed by

$$O^*(x,y)=\begin{cases}2\min(x\sqrt{y},y\sqrt{x}), & (x,y)\in[0,\frac{1}{4}]^2,\\ \frac{1}{4}+\frac{1}{4}(4x-1)^2(4y-1)^2, & (x,y)\in[\frac{1}{4},\frac{1}{2}]^2,\\ \frac{1}{2}+\min((2x-1)^2,(2y-1)^2)\max(x+y-\frac{3}{2},0), & (x,y)\in[\frac{1}{2},1]^2,\\ \min(x,\frac{\sqrt{x}}{2}), & (x,y)\in[0,\frac{1}{4}]\times[\frac{1}{4},1],\\ \min(y,\frac{\sqrt{y}}{2}), & (x,y)\in[\frac{1}{4},1]\times[0,\frac{1}{4}],\\ \frac{1}{4}+\frac{1}{4}\min((4x-1)^2,(4y-1)^2), & \text{otherwise},\end{cases}$$

their structure is shown in Figure 5, and O_3, O_4 and O_5 are listed as follows:

$$O_3(x,y)=\min(x\sqrt{y},y\sqrt{x}),$$

$$O_4(x,y)=x^2y^2,$$

$$O_5(x,y)=\min(x^2,y^2)\max(x+y-1,0),$$

then O^* is conditionally distributive over the uninorm U. This illustrates the second solution in Theorem 4.9.

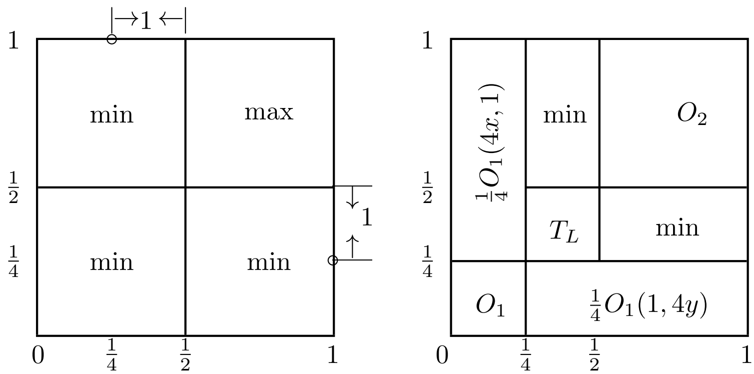

Fig. 4 . Structure of U and O from Example 4.10(1).

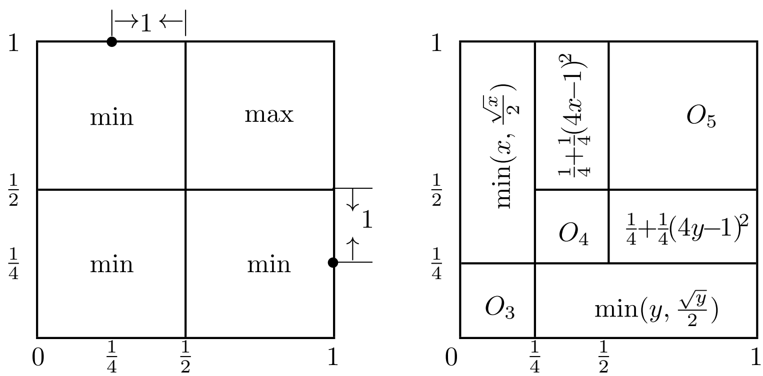

Fig. 5. Structure of U and O from Example 4.10(2).

Above all, we have completely characterized the conditional distributivity of overlap functions over idempotent uninorms. The maximum idempotent element, but not 1, is crucial for us to study the conditional distributivity of overlap function over uninorm with continuous underlying operators. As we all know, the neutral element e is an idempotent element, so by Remark 4.4, we can directly get $[0, e] \subseteq Id(U)$, namely $T_U = T_M$, therefore the maximum non-trivial idempotent element of uninorm will fall in $[e, 1)$.

Proposition 4.11. Let O be an overlap function, $U \in \mathcal{COU}$, and $a \in [e, 1)$ is the maximum non-trivial idempotent element of U. If O is conditionally distributive over U, then $O(1, a) = a$.

P r o o f. If $a = e$, it follows directly from Proposition 4.2. If $a \in (e, 1)$, since S_U is continuous, and a is the maximum non-trivial idempotent element of U, we have that $U\mid_{[a,1]^2}$ is isomorphic to a continuous Archimedean t-conorm. Additionally, by Lemma 4.3, we get $[0, O(1, a)] \subseteq Id(U)$, so it holds that $O(1, a) \leq a$. Assume $O(1, a) < a$, then there exists $t \in (a, 1)$ such that $a = O(1, t)$ in that O is continuous. Let $B =$

$\{x \in (a,1) \mid O(1,x) = a\}$, then $B \neq \emptyset$. Take $m = \bigwedge B$, then $O(1,m) = a$ and $a < m < 1$ hold. For $m \in (a,1)$, since $U \mid_{[a,1]^2}$ is a continuous Archimedean t-conorm, then $m = U(n,n) > n$ for some $n \in (a,1)$. Thus, we can get from the conditional distributivity that $a = O(1,m) = O(1,U(n,n)) = U(O(1,n),O(1,n))$. Furthermore, by the structure of U, we have $O(1,n) = a$, i. e., $n \in B$ and $n \geq m$, which is a contradiction. Thus, $O(1,a) = a$. □

Remark 4.12. If overlap function O is conditionally distributive over U with continuous underlying operators, and $a \in [e,1)$ is the maximum non-trivial idempotent element of U, then the underlying operators of U are $T_U = T_M$ and $S_U = (\langle \frac{a-e}{1-e}, 1, S^* \rangle)$, where S^* is a continuous Archimedean t-conorm.

Proposition 4.13. Let O be an overlap function, $U \in \mathcal{COU}$, and $a \in [e,1)$ is the maximum non-trivial idempotent element of U. If O is conditionally distributive over U, then $O(x,a) = O(x,1)$ for all $x \in [0,a]$.

Proof. If $a = e$, it is given by Proposition 4.5. If $a \in (e,1)$, $O(0,a) = O(0,1) = 0$ is obvious. For any $y \in (a,1)$, since $U \mid_{[a,1]^2}$ is a continuous Archimedean t-conorm, there exists $t_y \in (a,1)$ such that $y = U(t_y,t_y) > t_y$. For any $x \in (0,a]$, it follows from the conditional distributivity that

$$O(x,y) = O(x,U(t_y,t_y)) = U(O(x,t_y),O(x,t_y)) = O(x,t_y).$$

Let $B_{x,y} = \{u \in (a,1) \mid O(x,u) = O(x,y)\}$, then $B_{x,y} \neq \emptyset$, take $m_{x,y} = \bigwedge B_{x,y}$, we have $O(x,m_{x,y}) = O(x,y)$ and $a \leq m_{x,y} < 1$. Next, we shall prove $m_{x,y} = a$. Suppose that $m_{x,y} > a$, then there exists $n_{x,y} \in (a,1)$ such that $m_{x,y} = U(n_{x,y},n_{x,y}) > n_{x,y}$. So by the conditional distributivity, one gets $O(x,m_{x,y}) = O(x,U(n_{x,y},n_{x,y})) = U(O(x,n_{x,y}),O(x,n_{x,y})) = O(x,n_{x,y})$, i. e., $n_{x,y} \in B_{x,y}$. However $n_{x,y} < m_{x,y}$, which is a contradiction with the definition of $m_{x,y}$. Thus $m_{x,y} = a$, i. e., $O(x,y) = O(x,a)$. Moreover, it follows from the continuity of O that $O(x,a) = O(x,1)$ for all $x \in [0,a]$.

□

Proposition 4.14. Let O be an overlap function, $U \in \mathcal{COU}$, and $a \in [e,1)$ is the maximum non-trivial idempotent element of U. If O is conditionally distributive over U, then for any $x,y \in [a,1]$, $O(x,y) = a$ if and only if $x = a$ or $y = a$.

Proof. ($\Leftarrow$) It can be obtained by Proposition 4.11 and Proposition 4.13.

($\Rightarrow$) Suppose that there exist $x_0, y_0 \in (a,1]$ such that $O(x_0,y_0) = a$, then there must be $z_0 \in (a,1)$ such that $O(z_0,z_0) = a$. Let $b = \bigvee\{x \in (a,1) | O(x,x) = a\}$, then $b \in (a,1)$ and $O(b,b) = a$. Because $U \mid_{[a,1]^2}$ is a continuous Archimedean t-conorm, we have $t < b < U(t,t) < 1$ for some $t \in (a,1)$, hence $a \leq O(b,t) \leq O(b,b) = a$, i. e., $O(b,t) = a$. Therefore, by the conditional distributivity, we have

$$O(b,U(t,t)) = U(O(b,t),O(b,t)) = U(a,a) = a.$$

Let $B = \{x \in (a,1] \mid O(b,x) = a\}$, then $U(t,t) \in B$. Take $c = \bigvee B$, it holds that $c > b$ and $O(b,c) = a$. For c, there exists $v \in (a,1)$ such that $v < c < U(v,v) < 1$, then

$O(b,v)=a$. Owing to the conditional distributivity, we have $O(b,U(v,v))=U(a,a)=a$. That is, $U(v,v)\in B$, but $U(v,v)>c$, it contradicts with the definition of c. Thus, for any $x,y\in[a,1]$, if $O(x,y)=a$, then $x=a$ or $y=a$. □

Theorem 4.15. Let O be an overlap function, $U\in\mathcal{COU}$, and $a\in[e,1)$ is the maximum non-trivial idempotent element of U. If O is conditionally distributive over U, then U has the following form:

$$U(x,y)=\begin{cases} a+(1-a)(S_L)_\varphi(\frac{x-a}{1-a},\frac{y-a}{1-a}), & (x,y)\in[a,1]^2,\\ \max(x,y), & (x,y)\in[e,a]^2\cup([e,a]\times(a,1])\cup((a,1]\times[e,a]),\\ \min(x,y), & (x,y)\in[0,e]^2\cup([0,e)\times(e,1))\cup((e,1)\times[0,e)),\\ x \text{ or } y, & \text{otherwise.}\end{cases} \tag{12}$$

Proof. We have known $T_U=T_M$ and S_U can be represented by ordinal sum, so by Proposition 2.9, U is a locally internal uninorm, that is, there exists an associated function g such that $U=\langle e,g\rangle_{loc}$. Similar to the proof of Theorem 4.7, we can get $g(x)=1$ for all $x\in[0,e)$. Thus, according to the property of g, we have

$$g(x)=\begin{cases}1, & x\in[0,e),\\ e, & x\in[e,1).\end{cases}$$

Then by the conditional distributivity, we know $O|_{[a,1]^2}$ is conditionally distributive over $U|_{[a,1]^2}$. Therefore, combined with Proposition 4.14 and Theorem 3.4, uninorm U has the form of Eq.(12). □

Proposition 4.16. Let O be an overlap function, $U\in\mathcal{COU}$, and $a\in[e,1)$ is the maximum non-trivial idempotent element of U. If O is conditionally distributive over U, then $O(x,1)=O(x,e)<e$ for all $x\in[0,e)$.

Proof. The proof is similar to Proposition 4.8. □

Theorem 4.17. Let O be an overlap function, $U\in\mathcal{COU}$, and $a\in[e,1)$ is the maximum non-trivial idempotent element of U. Then O is conditionally distributive over U if and only if one of the following conditions holds:

(1) U and O have the following forms(see Figure 6):

$$U(x,y)=\begin{cases} a+(1-a)(S_L)_\varphi(\frac{x-a}{1-a},\frac{y-a}{1-a}), & (x,y)\in[a,1]^2,\\ \max(x,y), & (x,y)\in[e,a]^2\cup([e,a]\times(a,1])\cup((a,1]\times[e,a]),\\ 1, & (x,y)\in((g(1),e)\times\{1\})\cup(\{1\}\times(g(1),e)),\\ \min(x,y), & \text{otherwise,}\end{cases} \tag{13}$$

$$O(x,y)=\begin{cases} a+(1-a)(T_P)_{\varphi}(\frac{x-a}{1-a},\frac{y-a}{1-a}), & (x,y)\in[a,1]^2,\\ e+(a-e)O_2(\frac{x-e}{a-e},\frac{y-e}{a-e}), & (x,y)\in[e,a]^2,\\ eO_1(\frac{x}{e},\frac{y}{e}), & (x,y)\in[0,e]^2,\\ e+(a-e)O_2(\frac{\min(x,y)-e}{a-e},1), & (x,y)\in([e,a]\times(a,1])\cup((a,1]\times[e,a]),\\ eO_1(\frac{\min(x,y)}{e},1), & (x,y)\in([0,e)\times(e,1])\cup((e,1]\times[0,e)),\end{cases} \tag{14}$$

where g is the associated function of U, $(S_L)_{\varphi}$ is the $\varphi-$conjugate of S_L, $(T_P)_{\varphi}$ is the $\varphi-$conjugate of T_P, O_1 is an overlap function, O_2 is a 2-dimentional general overlap function, and O also satisfies the following condition:

$(\star)$ $O(1,g(1))\leq g(1)$.

(2) U has the following form(see Figure 7):

$$U(x,y)=\begin{cases} a+(1-a)(S_L)_{\varphi}(\frac{x-a}{1-a},\frac{y-a}{1-a}), & (x,y)\in[a,1]^2,\\ \max(x,y), & (x,y)\in[e,a]^2\cup([e,a]\times(a,1])\cup((a,1]\times[e,a]),\\ 1, & (x,y)\in([g(1),e)\times\{1\})\cup(\{1\}\times[g(1),e)),\\ \min(x,y), & \text{otherwise},\end{cases} \tag{15}$$

where g is the associated function of U, $(S_L)_{\varphi}$ is the $\varphi-$conjugate of S_L, $(T_P)_{\varphi}$ is the $\varphi-$conjugate of T_P, and O has the form of Eq.(14), but O satisfies the following condition:

$(\star')$ $\forall\, x\in[0,g(1))$, $O(1,x)<g(1)$ holds.

Proof. Since the proof for $a=e$ is similar and simpler to that for $a\in(e,1)$, only the proof for $a\in(e,1)$ is presented here.

($\Rightarrow$) Obviously, O satisfies Eq.(14), where $(T_P)_{\varphi}$ is the $\varphi-$conjugate of T_P, O_1 is an overlap function, $O_2:[0,1]^2\to[0,1]$ satisfies $(O1)$, $(O4)$, $(O5)$, $O_2(1,1)=1$, and $O_2(0,x)=O_2(x,0)=0$ for all $x\in[0,1]$, i.e., O_2 is a 2-dimentional general overlap function. And by Theorem 4.15, $g\mid_{[0,1)}$ can be determined, then according to the value of $g(1)$, we shall discuss the following cases.

i) If $g(1)=0$, then we investigate two cases:

- when U is conjunctive, U satisfies Eq.(13), and O satisfies $(\star)$;
- when U is disjunctive, U satisfies Eq.(15), and O satisfies $(\star')$.

ii) If $g(1)=e$, then U satisfies Eq.(15), and by Proposition 4.17, O satisfies $(\star')$.

iii) If $g(1)\in(0,e)$, then U has the following form:

$$U(x,y)=\begin{cases} a+(1-a)(S_L)_{\varphi}(\frac{x-a}{1-a},\frac{y-a}{1-a}), & (x,y)\in[a,1]^2,\\ \max(x,y), & (x,y)\in[e,a]^2\cup([e,a]\times(a,1])\cup((a,1]\times[e,a]),\\ 1, & (x,y)\in((g(1),e)\times\{1\})\cup(\{1\}\times(g(1),e)),\\ 1 \text{ or } g(1), & (x,y)\in\{(1,g(1)),(g(1),1)\},\\ \min(x,y), & \text{otherwise},\end{cases}$$

(a) when $U(1, g(1)) = g(1)$, U satisfies Eq.(13), and since O is conditionally distributive over U, it holds that

$$O(1, g(1)) = O(1, U(1, g(1))) = U(1, O(1, g(1))),$$

furthermore, by the structure of U, one gets $O(1, g(1)) \leq g(1)$;

(b) when $U(1, g(1)) = 1$, U satisfies Eq.(15), for any $x \in [0, g(1))$, since O is conditionally distributive over U, one has $O(1, x) = O(1, U(1, x)) = U(1, O(1, x))$, then it follows from the structure of U that $O(1, x) < g(1)$.

($\Leftarrow$) Considering $g(1) \in (0, e)$ at first.

1) Suppose O and U are given as Eq.(14) and Eq.(13), respectively. For any $x, y, z \in [0, 1]$, if $U(y, z) < 1$, then we shall divide our proof into the following parts:

(i) when $(x, y, z) \in [0, e]^3 \bigcup [e, a]^3$, Eq.(6) holds;

(ii) when $x, y, z \in [a, 1]$, by Theorem 3.4, Eq.(6) holds;

(iii) when $x \in [0, e]$, $y \in [0, e)$ and $z \in (e, 1)$, it holds that $O(x, y) \leq O(x, z) \leq e$, thus, by the structures of U and O, we have

$$O(x, U(y, z)) = O(x, y) = \min(O(x, y), O(x, z)) = U(O(x, y), O(x, z));$$

(iv) when $x \in [0, e]$, $y = e$ and $z \in (e, 1)$, it holds that $O(x, y) = O(x, z) \leq e$, thus, there holds

$$O(x, U(y, z)) = O(x, z) = \min(O(x, y), O(x, z)) = U(O(x, y), O(x, z));$$

(v) when $x \in [0, e]$, $y \in [0, g(1)]$ and $z = 1$, we have $U(y, 1) = y$ and $O(x, y) \leq O(x, 1) \leq e$, then it follows that

$$O(x, U(y, 1)) = O(x, y) = U(O(x, y), O(x, 1));$$

(vi) when $y, z \in [0, e]$ and $x \in (e, 1]$, then it holds that $U(y, z) = \min(y, z) < 1$, $O(x, y) \leq e$, $O(x, z) \leq e$, thus, we have

$$O(x, U(y, z)) = \min(O(x, y), O(x, z)) = U(O(x, y), O(x, z));$$

(vii) when $x, z \in [0, e]$ and $y \in (e, 1)$, similarly to item (iii) and (iv);

(viii) when $x \in [0, e]$, $z \in [0, g(1)]$ and $y = 1$, similarly to item (v);

(ix) when $x \in (e, 1]$, $y \in (e, 1)$ and $z \in [0, e)$, we have $U(y, z) = z$, $e \leq O(x, y) < 1$ and $O(x, z) < e$, thus, it follows from the structures of U that

$$O(x, U(y, z)) = O(x, z) = \min(O(x, y), O(x, z)) = U(O(x, y), O(x, z));$$

(x) when $x \in (e, 1]$, $y = 1$ and $z \in [0, g(1)]$, we have $U(1, z) = z$ and $O(x, 1) \geq e$, moreover, we can get $O(x, z) \leq O(1, g(1)) \leq g(1)$ by condition ($\star$), thus, by the structure of U, we have

$$O(x, U(1, z)) = O(x, z) = \min(O(x, 1), O(x, z)) = U(O(x, 1), O(x, z));$$

(xi) when $y, z \in (e, 1)$ and $x \in [0, e)$, we have $O(x, U(y, z)) = O(x, y) = O(x, z) = O(x, e) < e$, thus, it holds that

$$O(x, U(y, z)) = O(x, e) = U(O(x, y), O(x, z));$$

(xii) when $x \in (e, 1]$, $z \in (e, 1)$ and $y \in [0, e)$, similarly to item (ix);

(xiii) when $x \in (e, 1]$, $z = 1$ and $y \in [0, g(1)]$, similarly to item (x);

(xiv) when $x, y \in [e, a]$ and $z \in (a, 1)$, we have $U(y, z) = z$, and $O(x, y), O(x, z) \in [e, a]$, thus, it follows from the structure of U that

$$O(x, U(y, z)) = O(x, z) = \max(O(x, y), O(x, z)) = U(O(x, y), O(x, z));$$

(xv) when $y, z \in [e, a]$ and $x \in (a, 1]$, it holds that $U(y, z) = \max(y, z)$, and $O(x, y), O(x, z) \in [e, a]$, hence, we get

$$O(x, U(y, z)) = \max(O(x, y), O(x, z)) = U(O(x, y), O(x, z));$$

(xvi) when $x, z \in [e, a]$ and $y \in (a, 1)$, the proof is similar to item (xiv);

(xvii) when $x \in (a, 1]$, $y \in (a, 1)$ and $z \in [e, a]$, we have $U(y, z) = y$, $O(x, y) \in [a, 1)$ and $O(x, z) \in [e, a]$, thereby, it holds that

$$O(x, U(y, z)) = O(x, y) = \max(O(x, y), O(x, z)) = U(O(x, y), O(x, z));$$

(xviii) when $y, z \in (a, 1)$ and $x \in [e, a]$, we have $U(y, z) \in (a, 1)$ and $O(x, U(y, z)) = O(x, y) = O(x, z) \in [e, a]$, sequentially, we get

$$O(x, U(y, z)) = O(x, y) = U(O(x, y), O(x, z));$$

(xix) when $x \in (a, 1]$, $z \in (a, 1)$ and $y \in [e, a]$, similarly to item (xvii).

As a result, the overlap function O given as Eq.(14) is conditionally distributive over the uninorm U shown in Eq.(13).

2) If O is an overlap function satisfying Eq.(14), and U is a uninorm satisfying Eq.(15), the conditional distributivity can be proved similarly to the case 1).

For the cases $g(1) = 0$ and $g(1) = e$, the conditional distributivity can be easily proved. □

Example 4.18. **(1)** Take $\varphi(x) = x^2$, and consider the uninorm U and overlap function O as follows (see Figure 8):

$$U(x, y) = \begin{cases} \frac{3}{4} + \sqrt{\min((x - \frac{3}{4})^2 + (y - \frac{3}{4})^2, \frac{1}{16})}, & (x, y) \in [\frac{3}{4}, 1]^2, \\ \max(x, y), & (x, y) \in ([\frac{1}{2}, \frac{3}{4}] \times [\frac{1}{2}, 1]) \cup ((\frac{3}{4}, 1] \times [\frac{1}{2}, \frac{3}{4}]), \\ 1, & (x, y) \in ((\frac{1}{4}, \frac{1}{2}) \times \{1\}) \cup (\{1\} \times (\frac{1}{4}, \frac{1}{2})), \\ \min(x, y), & \text{otherwise,} \end{cases}$$

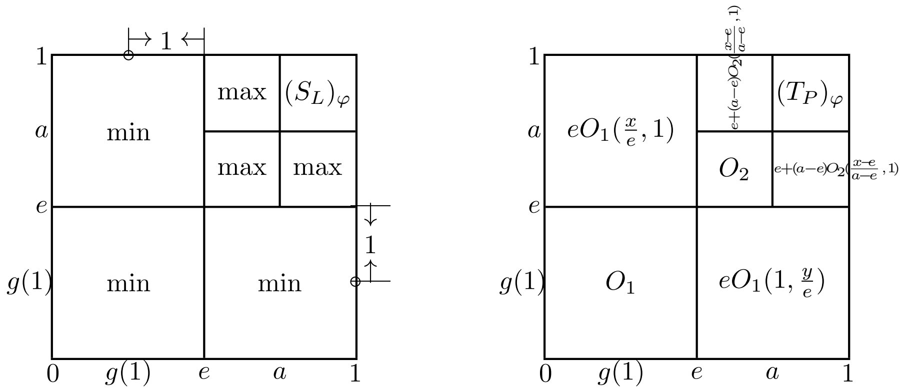

Fig. 6. Structure of U and O from Theorem 4.17(1).

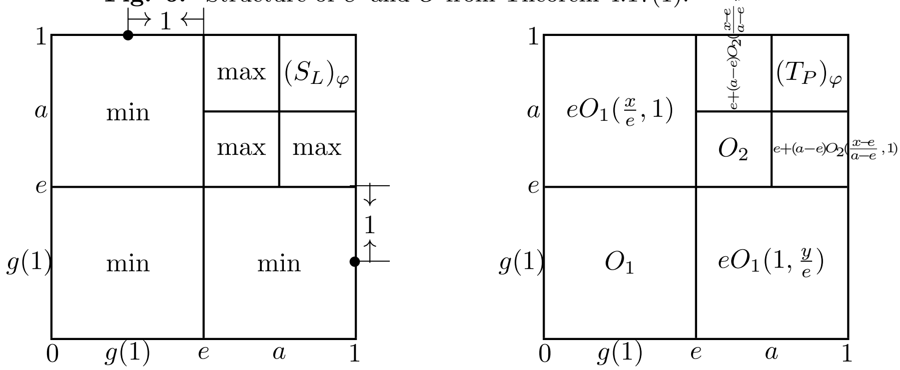

Fig. 7 Structure of U and O from Theorem 4.17(2)

$$O(x,y)=\begin{cases} 4xy-3x-3y+3, & (x,y)\in[\frac{3}{4},1]^2,\\ \frac{5}{8}+\frac{1}{8}O_1(8x-5,8y-5), & (x,y)\in[\frac{5}{8},\frac{3}{4}]^2,\\ \max(x+y-\frac{5}{8},\frac{1}{2}), & (x,y)\in[\frac{1}{2},\frac{5}{8}]^2,\\ \frac{1}{4}+\sqrt{\max((x-\frac{1}{4})^2+(y-\frac{1}{4})^2-\frac{1}{16},0)}, & (x,y)\in[\frac{1}{4},\frac{1}{2}]^2,\\ \frac{1}{4}O_1(4x,4y), & (x,y)\in[0,\frac{1}{4}]^2,\\ \frac{5}{8}+\frac{1}{8}O_1(8\min(x,y)-5,1), & (x,y)\in([\frac{5}{8},\frac{3}{4})\times(\frac{3}{4},1])\cup((\frac{3}{4},1]\times[\frac{5}{8},\frac{3}{4})),\\ \frac{1}{4}O_1(4\min(x,y),1), & (x,y)\in([0,\frac{1}{4})\times(\frac{1}{4},1])\cup((\frac{1}{4},1]\times[0,\frac{1}{4})),\\ \min(x,y), & \text{otherwise,} \end{cases}$$

where O_1 is defined by

$$O_1(x,y)=\begin{cases} 2x, & (x,y)\in[0,\frac{1}{2}]\times[2x,1],\\ 2y, & (y,x)\in[0,\frac{1}{2}]\times[2y,1],\\ \max(x,y), & \text{otherwise,} \end{cases}$$

it can be verified that O is conditionally distributive over the uninorm U, which corresponds to the first case of Theorem 4.17.

(2) If U^* is defined by

$$U^*(x,y)=\begin{cases} \frac{3}{4}+\sqrt{\min((x-\frac{3}{4})^2+(y-\frac{3}{4})^2,\frac{1}{16})}, & (x,y)\in[\frac{3}{4},1]^2,\\ \max(x,y), & (x,y)\in([\frac{1}{2},\frac{3}{4}]\times[\frac{1}{2},1])\cup((\frac{3}{4},1]\times[\frac{1}{2},\frac{3}{4}]),\\ 1, & (x,y)\in([\frac{1}{4},\frac{1}{2})\times\{1\})\cup(\{1\}\times[\frac{1}{4},\frac{1}{2})),\\ \min(x,y), & \text{otherwise,} \end{cases}$$

and an overlap function O^* is defined by

$$O^*(x,y)=\begin{cases} 4xy-3x-3y+3, & (x,y)\in[\frac{3}{4},1]^2,\\ \frac{5}{8}+\frac{1}{8}O_2(8x-5,8y-5), & (x,y)\in[\frac{5}{8},\frac{3}{4}]^2,\\ \max(x+y-\frac{5}{8},\frac{1}{2}), & (x,y)\in[\frac{1}{2},\frac{5}{8}]^2,\\ \frac{1}{4}+\sqrt{\max((x-\frac{1}{4})^2+(y-\frac{1}{4})^2-\frac{1}{16},0)}, & (x,y)\in[\frac{1}{4},\frac{1}{2}]^2,\\ 8xy(x+y), & (x,y)\in[0,\frac{1}{4}]^2,\\ \frac{5}{8}+\frac{1}{8}O_2(8\min(x,y)-5,1), & (x,y)\in([\frac{5}{8},\frac{3}{4})\times(\frac{3}{4},1])\cup((\frac{3}{4},1]\times[\frac{5}{8},\frac{3}{4})),\\ 2x^2+\frac{x}{2}, & (x,y)\in[0,\frac{1}{4})\times(\frac{1}{4},1],\\ 2y^2+\frac{y}{2},, & (x,y)\in(\frac{1}{4},1]\times[0,\frac{1}{4}),\\ \min(x,y), & \text{otherwise,} \end{cases}$$

where O_2 is defined by

$$O_2(x,y)=\begin{cases} \sqrt{2x}, & (x,y)\in[0,\frac{1}{2}]\times[\sqrt{2x},1],\\ \sqrt{2y}, & (y,x)\in[0,\frac{1}{2}]\times[\sqrt{2y},1],\\ \max(x,y), & \text{otherwise,} \end{cases}$$

then O^* is conditionally distributive over the uninorm U^*, it accords with the second case of Theorem 4.17. The specific structure of O^* and U^* is shown in Figure 9, and the corresponding O_3 is given as $O_3(x,y)=xy\frac{x+y}{2}$.

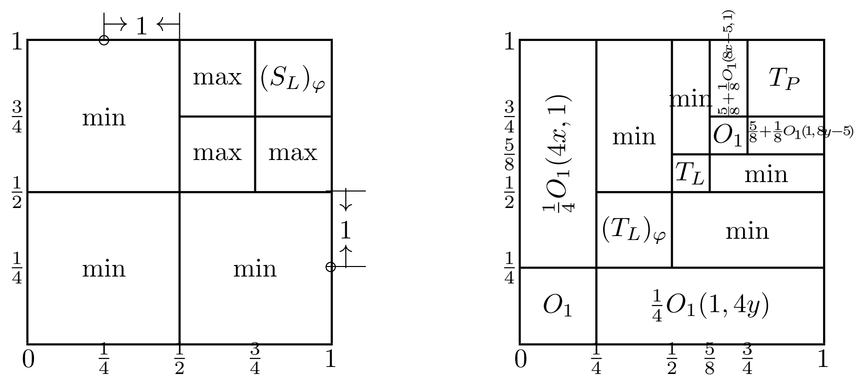

Fig. 8. Structure of U and O from Example 4.18(1).

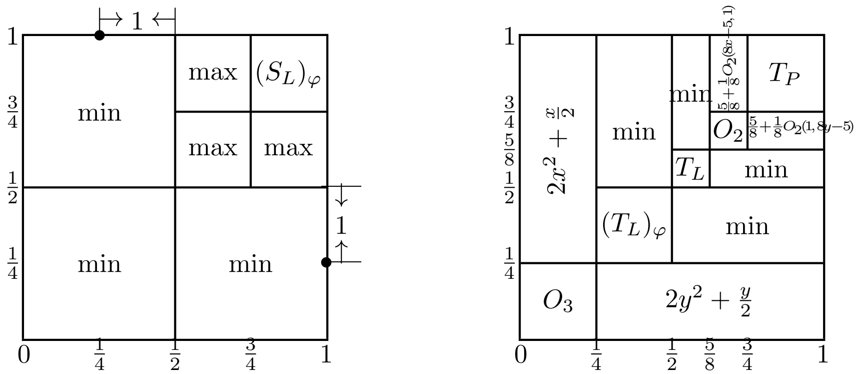

Fig. 9. Structure of U and O from Example 4.18(2).

So far, the full characterization of overlap functions over uninorms with continuous underlying operators can be summarized.

Theorem 4.19. Let O be an overlap function, $U \in \mathcal{COU}$. Then O is conditionally distributive over U if and only if one of the following cases hold:

(1) U is an idempotent uninorm, and U satisfies Eq.(9), O satisfies Eq.(10)(see Figure 2), where g is the associated function of U, O_1 is an overlap function, O_2 is a 0-overlap function, and O satisfies $(\star)$.

(2) U is an idempotent uninorm, and U satisfies Eq.(11), O satisfies Eq.(10)(see Figure 3), where g is the associated function of U, O_1 is an overlap function, O_2 is a 0-overlap function, but O satisfies $(\star')$.

(3) There exists $a \in [e, 1)$ such that U satisfies Eq.(13), O satisfies Eq.(14)(see Figure 6), where g is the associated function of U, $(S_L)_\varphi$ is the $\varphi-$conjugate of S_L, $(T_P)_\varphi$ is the $\varphi-$conjugate of T_P, O_1 is an overlap function, O_2 is a 2-dimentional general overlap function, and O also satisfies $(\star)$.

(4) There exists $a \in [e, 1)$ such that U satisfies Eq.(15), O satisfies Eq.(14)(see Figure 7), where g is the associated function of U, $(S_L)_\varphi$ is the $\varphi-$conjugate of S_L, $(T_P)_\varphi$ is the $\varphi-$conjugate of T_P, O_1 is an overlap function, O_2 is a 2-dimentional general overlap function, and O also satisfies $(\star')$.

5. CONCLUSIONS

Conditional distributivity of overlap functions over t-conorms and uninorms are considered throughout this paper. The distributivity for overlap functions with respect to uninorms is rather a strong condition, since it simplifies the structure of the inner operators considerably, and the uninorms are being actually reduced to idempotent uninorms. In Section 3, we have given a characterization of all pairs (O, S) satisfying distributivity law on the restricted domain, where O is an overlap function and S is a continuous t-conorm. And then, the conditional distributivity of overlap functions over uninorms with continuous underlying operators is fully characterized based on Section 3. It turns out that the conditional distributivity produces a larger variety of solutions. Researches in Section 3 and 4 are continuation investigations of conditional distributivity for overlap function. In the forthcoming work, the distributivity and conditional distributivity for some other classes of aggregation functions will be considered, and our further research will focus on the obtained structures and possible applications to utility theory.

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