

Jae-Myoung Kim

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REMARK ON REGULARITY CRITERION FOR WEAK SOLUTIONS
TO THE SHEAR THINNING FLUIDS

JAE-MYOUNG KIM, Andong

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Abstract. J. Q. Yang (2019) established a regularity criterion for the 3D shear thinning fluids in the whole space \mathbb{R}^3 via two velocity components. The goal of this short note is to extend this result in viewpoint of Lorentz space.

Keywords: shear thinning fluids; regularity criterion

MSC 2020: 76D05, 35Q35

1. INTRODUCTION

In this paper, we study the shear thinning fluids:

$$(1.1) \quad \begin{cases} u_t - \nabla \cdot (|Du|^{p-2} Du) + (u \cdot \nabla)u + \nabla \pi = f, \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } Q_T := \mathbb{R}^3 \times (0, T),$$

where $u = (u_1, u_2, u_3): \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}^3$ is the flow velocity vector, $\pi: \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}$ is the pressure. Also, $Du = \frac{1}{2}(\nabla u + \nabla u^T)$. We consider the initial value problem of (1.1), which requires initial conditions

$$(1.2) \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3.$$

We assume that the initial data $u_0(x) \in L^2(\mathbb{R}^3)$ hold the incompressibility, i.e., $\operatorname{div} u_0(x) = 0$.

The non-Newtonian flows is a fluid such that the relation between the shear stress and the shear strain rate is non-linear. Typical examples are as follows: honey, blood, paint, melted butter and corn starch (see e.g. [2], [5], [14]).

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We report shortly some known results related to the existence of solutions and regularity issues. The existence of weak solutions was firstly shown for $\frac{11}{5} \leq q$ in [9]–[11], and later, the result was improved up to $\frac{6}{5} < q$ in [7]. Here, weak solutions are meant to solve the equations in the sense of distributions and satisfy

$$\sup_{0 \leq t \leq T} \|u(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^T \|\nabla u(t)\|_{L^q(\mathbb{R}^3)}^q dt \leq C \|u_0\|_{L^2(\mathbb{R}^3)}^2.$$

Málek et al. proved in [13] that a strong solution exists globally in time in periodic domains for $q \geq \frac{11}{5}$ in \mathbb{R}^3 (see [17] for the whole space case). Here by strong solutions we mean solutions satisfying the following energy estimate:

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^1(\mathbb{R}^3)}^2 + \int_0^T \int_{\mathbb{R}^3} |Du|^{q-2} |\nabla Du|^2 \leq C \|u_0\|_{H^1(\mathbb{R}^3)}^2.$$

Also, they established local existence of strong solution in time for $q > \frac{5}{3}$ in three dimensional periodic domains (refer to [4] for shear thinning case, $\frac{7}{5} < q < 2$).

When $\frac{8}{5} < p \leq 2$, in particular, for the regularity issue for (1.1), Bae et al. (see [3]) obtained the following regularity of weak solutions of (1.1)–(1.2) in the class

$$(1.3) \quad u \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \quad \frac{3}{\alpha} + \frac{5p-6}{2\beta} \leq \frac{5p-8}{2}, \quad \alpha > \frac{6}{5p-8}.$$

On the other hand, Yang in [19] improved regularity condition (1.3) by only the two-component velocity field. More precisely, for $\tilde{u} = (u_1, u_2, 0)$ he proved the regularity of weak solutions in the class

$$(1.4) \quad \tilde{u} \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \quad \frac{3}{\alpha} + \frac{5p-6}{2\beta} \leq \frac{5p-8}{2}, \quad \alpha > \frac{6}{5p-8}.$$

For $\alpha = \infty$, Ahmad et al. (see [1]) improved Yang’s result (1.4). More precisely, for $\tilde{u} = (u_1, u_2, 0)$ he proved the regularity of weak solutions in the class

$$\tilde{u} \in L^{(5p-6)/(5p-8)}(0, T; \text{BMO}(\mathbb{R}^3)).$$

In this direction, our main result is read as follows:

Theorem 1.1. *Suppose that u is a weak solution to (1.1)–(1.2) with $\frac{8}{5} < p \leq 2$. Then there exists a constant $\delta > 0$ such that u is a strong solution on $(0, T]$ provided that $u \in L^{\beta, \infty}(0, T; L^{\alpha, \infty}(\mathbb{R}^3))$ and*

$$\|u\|_{L^{\beta, \infty}(0, T; L^{\alpha, \infty}(\mathbb{R}^3))} \leq \delta \quad \text{with} \quad \frac{3}{\alpha} + \frac{5p-6}{2\beta} \leq \frac{5p-8}{2}, \quad \alpha > \frac{6}{5p-8}.$$

Theorem 1.2. *Suppose that u is a weak solution to (1.1)–(1.2) with $\frac{8}{5} < p \leq 2$. Then there exists a constant $\delta > 0$ such that u is a strong solution on $(0, T]$ provided that $\tilde{u} \in L^{\beta, \infty}(0, T; L^{\alpha, \infty}(\mathbb{R}^3))$ and*

$$\|\tilde{u}\|_{L^{\beta, \infty}(0, T; L^{\alpha, \infty}(\mathbb{R}^3))} \leq \delta \quad \text{with} \quad \frac{3}{\alpha} + \frac{5p-6}{2\beta} \leq \frac{5p-8}{2}, \quad \alpha > \frac{6}{5p-8}.$$

Remark 1.3. For a half space \mathbb{R}_+^3 with the slip boundary condition instead of \mathbb{R}^3 , our results are also established due to Sobolev embedding (see e.g. [8], pages 215–216).

2. PRELIMINARIES

In this subsection we introduce the notation. For $1 \leq q \leq \infty$, we denote by $W^{k, q}(\Omega)$ the usual Sobolev spaces, namely $W^{k, q}(\Omega) = \{f \in L^q(\Omega) : D^\alpha f \in L^q(\Omega), 0 \leq |\alpha| \leq k\}$. The set of q th power Lebesgue integrable functions on Ω is denoted by $L^q(\Omega)$ and $L_{\text{loc}}^q(\Omega)$ indicates the set of locally q th power Lebesgue integrable functions defined on Ω . Let $\mathcal{O} \subset \Omega$ and $J \subset I$, we denote $\|f\|_{L_{x,t}^{p,q}(\mathcal{O} \times J)} = \| \|f\|_{L^p(\mathcal{O})} \|_{L^q(J)}$. For vector fields u, v we write $(u_i v_j)_{i,j=1,2,3}$ as $u \otimes v$. We denote by $A : B = a_{ij} b_{ij}$ the 3×3 matrices $A = (a_{ij})$, $B = (b_{ij})$. The letter C is used to represent a generic constant, which may change from line to line. Before the proof, we will list some lemmas needed for the proof. First of all, we recall Korn's inequality in [17], Lemma 2.7.

Lemma 2.1. *Let $1 < q < \infty$. Assume that u is in $W^{1, q}(\mathbb{R}^3)$. Then*

$$\|\nabla u\|_{L^q(\mathbb{R}^3)} \leq C \|Du\|_{L^q(\mathbb{R}^3)},$$

where C is a positive constant depending on q .

We recall a key estimate for the stress tensor in [3] or [19].

Lemma 2.2. *Let $1 < p < 2$. Suppose that $u \in W^{2, s}(\mathbb{R}^3)$ for $1 < s < 2$. Then*

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla^2 u|^s dx &\leq C \left(\int_{\mathbb{R}^3} \sum_{i,j=1}^3 \nabla(|D(u)|^{p-2} D_{i,j}(u)) \cdot \nabla D_{i,j}(u) dx \right)^{s/2} \\ &\times \left(\int_{\mathbb{R}^3} |\nabla u|^{(2-p)s/(2-s)} dx \right)^{(2-s)/2}. \end{aligned}$$

For $s = p$, in particular, we have

$$\int_{\mathbb{R}^3} |\nabla^2 u|^p dx \leq \varepsilon \int_{\mathbb{R}^3} \sum_{i,j=1}^3 \nabla(|D(u)|^{p-2} D_{i,j}(u)) \cdot \nabla D_{i,j}(u) dx + C \int_{\mathbb{R}^3} |\nabla u|^p dx.$$

Next, we present some basic facts on Lorentz spaces $L^{p,q}(\Omega)$. For $p, q \in [1, \infty]$ we define

$$L^{p,q}(\mathbb{R}^3) = \{f: f \text{ is a measurable function on } \mathbb{R}^3 \text{ and } \|f\|_{L^{p,q}(\mathbb{R}^3)} < \infty\}$$

with

$$\|f\|_{L^{p,q}(\mathbb{R}^3)} = \begin{cases} \left(p \int_0^\infty \alpha^q |\{x \in \mathbb{R}^3: |f(x)| > \alpha\}|^{q/p} \frac{d\alpha}{\alpha} \right)^{1/q}, & q < \infty, \\ \sup_{\alpha > 0} \alpha |\{x \in \mathbb{R}^3: |f(x)| > \alpha\}|^{1/p}, & q = \infty. \end{cases}$$

And also, followed [18], Lorentz space may be defined by real interpolation methods as

$$L^{p,q}(\mathbb{R}^3) = (L^{p_1}(\mathbb{R}^3), L^{p_2}(\mathbb{R}^3))_{\alpha,q}$$

with

$$\frac{1}{p} = \frac{1-\alpha}{p_1} + \frac{\alpha}{p_2}, \quad 1 \leq p_1 < p < p_2 \leq \infty.$$

We also remind the Hölder inequality in Lorentz spaces (see [15]).

Lemma 2.3. *Assume $1 \leq p_1, p_2 \leq \infty, 1 \leq q_1, q_2 \leq \infty$ and $u \in L^{p_1, q_1}(\mathbb{R}^3), v \in L^{p_2, q_2}(\mathbb{R}^3)$. Then $uv \in L^{p_3, q_3}(\mathbb{R}^3)$ with $1/p_3 = 1/p_1 + 1/p_2$ and $1/q_3 \leq 1/q_1 + 1/q_2$, and the inequality*

$$\|uv\|_{L^{p_3, q_3}(\mathbb{R}^3)} \leq C \|u\|_{L^{p_1, q_1}(\mathbb{R}^3)} \|v\|_{L^{p_2, q_2}(\mathbb{R}^3)}$$

is valid.

We recall the following useful Gronwall lemma required in our proof which was first shown by [6] (see e.g. [16], [12]).

Lemma 2.4. *Let φ be a measurable positive function defined on the interval $[0, T]$. Suppose that there exists $\kappa_0 > 0$ such that for all $0 < \kappa < \kappa_0$ and a.e. $t \in [0, T]$, φ satisfies the inequality*

$$\frac{d}{dt} \varphi \leq \mu \lambda^{1-\kappa} \varphi^{1+2\kappa},$$

where $0 < \lambda \in L^{1,\infty}(0, T)$ and $\mu > 0$ with $\mu \|\lambda\|_{L^{1,\infty}(0, T)} < \frac{1}{2}$. Then φ is bounded on $[0, T]$.

3. REGULARITY CRITERION

Proof of Theorem 1.1. Testing $-\Delta u$ to the fluid equation of (1.1), respectively, and using integration by parts, we have

$$(3.1) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u(\tau)\|_{L_x^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \sum_{i,j=1}^3 \nabla(|D(u)|^{p-2} D_{i,j}(u)) \cdot \nabla D_{i,j}(u) \, dx \\ = \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u \, dx := A. \end{aligned}$$

Now, the first term A is computed as follows. Since $\alpha > 6/(5p-8) > 3p/(4p-6) > p/(p-1)$, we have

$$\begin{aligned} |A| &\leq \|u\|_{L^{\alpha,\infty}} \|\nabla u\|_{L^{p\alpha/(p\alpha-p-\alpha), p/(p-1)}} \|\nabla^2 u\|_{L^{p,6}} \\ &\leq \|u\|_{L^{\alpha,\infty}} \|\nabla u\|_{L^2}^{(8p\alpha-12\alpha-6p)/(5p-6)\alpha} \|\nabla u\|_{L^{3p/(3-p)}}^{3(2p+2\alpha-p\alpha)/(5p-6)\alpha} \|\nabla^2 u\|_{L^p} \\ &\leq \|u\|_{L^{\alpha,\infty}} \|\nabla u\|_{L^2}^{(8p\alpha-12\alpha-6p)/(5p-6)\alpha} \|\nabla^2 u\|_{L^p}^{(2p\alpha+6p)/(5p-6)\alpha} \\ &\leq \|u\|_{L^{\alpha,\infty}}^{\alpha(5p-6)/(5p\alpha-8\alpha-6)} \|\nabla u\|_{L^2}^{(8p\alpha-12\alpha-6p)/(5p\alpha-8\alpha-6)} + \varepsilon \|\nabla^2 u\|_{L^p}^p. \end{aligned}$$

Due to Lemma 2.2, the non-linear term A is estimated by

$$(3.2) \quad \begin{aligned} |A| &\leq \|u\|_{L^{\alpha,\infty}}^{\alpha(5p-6)/(5p\alpha-8\alpha-6)} \|\nabla u\|_{L^2}^{(8p\alpha-12\alpha-6p)/(5p\alpha-8\alpha-6)} \\ &\quad + \varepsilon_1 \varepsilon \int_{\mathbb{R}^3} \sum_{i,j=1}^3 \nabla(|D(u)|^{p-2} D_{i,j}(u)) \cdot \nabla D_{i,j}(u) \, dx + \varepsilon_1 C \int_{\mathbb{R}^3} |\nabla u|^p \, dx. \end{aligned}$$

With the energy estimate and (3.2), (3.1) becomes

$$(3.3) \quad \begin{aligned} \frac{d}{dt} (\|u(t)\|_{L_x^2(\mathbb{R}^3)}^2 + \|\nabla u(t)\|_{L_x^2(\mathbb{R}^3)}^2) \\ + \int_{\mathbb{R}^3} \sum_{i,j=1}^3 \nabla(|D(u)|^{p-2} D_{i,j}(u)) \cdot \nabla D_{i,j}(u) + |\nabla u|^p \, dx \\ \leq \|u\|_{L^{\alpha,\infty}}^{\alpha(5p-6)/(5p\alpha-8\alpha-6)} \|\nabla u\|_{L^2}^{(8p\alpha-12\alpha-6p)/(5p\alpha-8\alpha-6)} \\ \leq \|u\|_{L^{\alpha,\infty}}^{\alpha(5p-6)/(5p\alpha-8\alpha-6)} \\ \quad \times \frac{1}{(1 + \|\nabla u\|_{L^2}^2)^{(4p\alpha-6\alpha-3p)/(5p\alpha-8\alpha-6)-1}} (1 + \|\nabla u\|_{L^2}^2) \end{aligned}$$

Let $Y(t) := 1 + \|u(t)\|_{L_x^2(\mathbb{R}^3)}^2 + \|\nabla u(t)\|_{L_x^2(\mathbb{R}^3)}^2$ and thus (3.3) becomes

$$(3.4) \quad \frac{d}{dt} Y(t) \leq C \|u\|_{L^{\alpha,\infty}}^{\alpha(5p-6)/(5p\alpha-8\alpha-6)} Y(t).$$

We use an argument similar to the one used in the work of Bosia et al.: For $\varepsilon > 0$, let $\beta_\varepsilon = \beta + 2p\varepsilon((5p - 9)\beta - (5p - 6))/(5p - 6)$ and $(\alpha_\varepsilon, \beta_\varepsilon)$ be a Prodi-Serrin pair. We have

$$(3.5) \quad \|u\|_{L^{\alpha_\varepsilon, \infty}(\mathbb{R}^3)}^{\beta_\varepsilon} \leq \|u\|_{L^{\alpha, \infty}(\mathbb{R}^3)}^{\beta(1-\varepsilon)} \|u\|_{L^{6, \infty}}^{2p\varepsilon} \leq C \|u\|_{L^{\alpha, \infty}(\mathbb{R}^3)}^{\beta(1-\varepsilon)} \|\nabla u\|_{L^2(\mathbb{R}^3)}^{2p\varepsilon},$$

where we use

$$\|f\|_{L^{p, q_2}(\mathbb{R}^3)} \leq \left(\frac{q_1}{p}\right)^{1/q_1 - 1/q_2} \|f\|_{L^{p, q_1}(\mathbb{R}^3)}, \quad 1 \leq p \leq \infty, \quad 1 \leq q_1 < q_2 \leq \infty.$$

Since the pair $(\alpha_\varepsilon, \beta_\varepsilon)$ also meets $2/\alpha_\varepsilon + 3/\beta_\varepsilon = 2$, using (3.5), (3.4) becomes

$$\frac{d}{dt} Y(t) \leq C \|u\|_{L^{\alpha_\varepsilon, \infty}(\mathbb{R}^3)}^{\beta_\varepsilon} Y(t) \leq C \|u\|_{L^{\alpha, \infty}(\mathbb{R}^3)}^{\beta(1-\varepsilon)} Y(t)^{1+2\varepsilon}.$$

Applying Lemma 2.4, we obtain the desired result. \square

Proof of Theorem 1.2. The proof is almost the same as that for Theorem 1.1. Indeed, from the proof of Theorem 1.1, convection term A is decomposed with three parts as follows:

$$\begin{aligned} - \sum_{i,j=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_{kk}^2 u_j \, dx &= \sum_{i=1}^2 \sum_{j,k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j \, dx \\ &\quad + \sum_{j=1}^2 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j \, dx \\ &\quad + \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_3 \partial_k u_3 \, dx \\ &:= A_1 + A_2 + A_3. \end{aligned}$$

For A_1 and A_2 , by integration by parts, we have

$$\begin{aligned} A_1 &= - \sum_{i=1}^2 \sum_{j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_{kk}^2 u_j \, dx - \sum_{i=1}^2 \sum_{j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_k \partial_i u_j \partial_k u_j \, dx, \\ A_2 &= - \sum_{i=1}^2 \sum_{j,k=1}^3 \int_{\mathbb{R}^3} u_j \partial_3 u_j \partial_{kk}^2 u_3 \, dx - \sum_{i=1}^2 \sum_{j,k=1}^3 \int_{\mathbb{R}^3} u_j \partial_3 \partial_k u_j \partial_k u_3 \, dx. \end{aligned}$$

And thus, we know

$$(3.6) \quad |A_1|, |A_2| \leq C \int_{\mathbb{R}^3} |(\tilde{u} \cdot \nabla u) : \nabla^2 u| \, dx.$$

For A_3 , using the divergence free condition, we get

$$\begin{aligned}
 A_3 &= - \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_3 (\partial_1 u_1 + \partial_2 u_2) \partial_k u_3 \, dx \\
 &= - \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_3 \partial_1 u_1 \partial_k u_3 \, dx - \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_3 \partial_2 u_2 \partial_k u_3 \, dx \\
 &= - \sum_{k=1}^3 \int_{\mathbb{R}^3} u_1 \partial_1 \partial_k u_3 \partial_k u_3 \, dx - \sum_{k=1}^3 \int_{\mathbb{R}^3} u_1 \partial_k u_3 \partial_1 \partial_k u_3 \, dx \\
 &\quad - \sum_{k=1}^3 \int_{\mathbb{R}^3} u_2 \partial_2 \partial_k u_3 \partial_k u_3 \, dx - \sum_{k=1}^3 \int_{\mathbb{R}^3} u_2 \partial_k u_3 \partial_2 \partial_k u_3 \, dx
 \end{aligned}$$

and thus, we have

$$(3.7) \quad |A_3| \leq C \int_{\mathbb{R}^3} |(\tilde{u} \cdot \nabla u) : \nabla^2 u| \, dx.$$

Through (3.6) and (3.7), we can see that

$$|A| \leq C \int_{\mathbb{R}^3} |(u \cdot \nabla u) : \nabla^2 u| \, dx.$$

Hence, with the same approach as in Theorem 1.1, we finally obtain the result in Theorem 1.2. The proof is complete. \square

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Author’s address: Jae-Myoung Kim, Department of Mathematics Education, Andong National University, Andong 36729, Republic of Korea, e-mail: cauchy02@naver.com.