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POSITIVE UNKNOWN INPUTS FUNCTIONAL OBSERVERS NEW DESIGN FOR POSITIVE LINEAR SYSTEMS

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This paper deals with the problem of designing positive functional observers for positive linear systems subject to unknown inputs. The order of the designed observer is equal to the dimension of the functional to be estimated. The designed functional observer is always nonnegative at any time and converges asymptotically to the real functional state vector. In fact, we propose a new positive reduced order observer for positive linear systems affected by unknown inputs. The proposed procedure is based on the positivity of an augmented system composed of dynamics of both considered system and proposed observer and also, on the unbiasedness of the estimation error by the resolution of Sylvester equation. Then existence conditions of such observers are formulated in terms of linear programming (LP) problem, where we use the Perron–Frobenius theorem applied to Metzler matrices. An algorithm that summarizes the different steps of the proposed positive functional observer design is given. Finally, numerical example and simulation results are given to illustrate the effectiveness of the proposed design method.

Keywords: positive systems, functional observers, unknown inputs, linear systems, LP problem

Classification: 93C05, 93C28, 93B53

Notations: We shall use throughout the paper the following notations:

- Let $\Re$ be the set of real numbers; $\Re_+^n$ denotes the nonnegative orthant of the n-dimensional real space $\Re^n$ and $\Re^{m\times n}$ is the set of $m \times n$ matrices for which all entries belong to $\Re$.
- For a matrix $A \in \Re^{m\times n}$, a_{ij} denotes the element located at the ith row $(i \leq m)$ and jth column $(j \leq n)$.
- A matrix A is said to be nonnegative, denoted by $A \succ 0$, if $\forall (i,j),\ a_{ij} \geq 0$. It is said to be positive, if $\forall (i,j),\ a_{ij} \geq 0, \exists (i,j),\ a_{ij} > 0$. Note that definitions of nonnegative and positive matrices are equivalent, except when a nonnegative matrix is identically zero which is the degenerate case and is of no interest. So, we consider that these two definitions are equivalent in general cases.

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- A matrix A is said to be negative, denoted by $A \prec 0$, if $\forall(i,j),\ a_{ij} \leq 0$.
- $A > 0$ (respectively, $A < 0$) means that the matrix A is positive definite (respectively, negative definite), $A \geq 0$ (respectively, $A \leq 0$) means that the matrix A is positive semidefinite (respectively, negative semidefinite)
- For a real matrix A, A^T denotes the transpose.
- A^- denotes any generalized inverse of matrix A, i. e. verifies $AA^-A = A$.
- $\lambda_i(A)$ designs the ith eigenvalue of matrix A.
- diag (v) denotes the diagonal matrix formed from the vector v.
- Ones (m_z) denotes an $m_z \times 1$ vector of ones $: = \begin{pmatrix} 1, \dots, 1 \end{pmatrix}^T$.
- I and 0 are the identity matrix and the zero matrix of appropriate dimensions.
- s denotes the Laplace variable, it is a complex number.
- $v \succ 0$ denotes a vector v such that for all its coordinates it holds, $\forall i,\ v_i \geq 0$.

1. INTRODUCTION

Real systems in many area as biomedicine [5], biology [23], physiology [15], epidemiology [31], industrial engineering [3]. . . are positive. Positive systems, belongs to an important class of dynamical systems whose states are nonnegative for any nonnegative initial condition and any nonnegative input. In view of these widespread applications, it is necessary to investigate the study and synthesis problems for positive linear systems (see [14, 19], . . .). Note that, positive systems differ from linear standard systems by the existence of positivity constraints. In fact, for example, if a system is controllable, the poles of the system can be placed arbitrarily, whereas for positive one this feature may not be true owing the positivity constraints on systems matrices.

On the other hand, in practice, many control processes require the availability of some components of the system state vector for the purpose of monitoring for example. This problem, has motivated a great deal of work to the observers design for linear systems [6, 25, 30]. . . when the states are not measurable or measured. Mainly, problem of functional observer design is equivalent to find an observer, that estimates a linear combination of the states of a system using the input and output measurements. Such estimator has the same order as the linear combination to be estimated. Note that, it has been the object of numerous studies for non positive systems where the aim is only to minimise the estimation error (make the estimation error converge to zero) [6], [7], [9]. . . . For positive systems, in addition to minimizing this error, positive observers must also guarantee the nonnegativity of the state estimates [8, 10, 12]. This makes the positive observer design significantly more challenging [22], [2], [27]. . . . For positive systems with unknown inputs (see [28]. . .) a design method via LMIs is provided. In [21] and [29], unknown input observers for discrete-time positive systems, state and unknown input observers for positive systems were reported. Despite the recent advances on observers design for positive systems, to the best of our knowledge, little attention has been paid for the positive functional observer design for positive linear systems subject to unknown inputs [28], which motivates the present work. This is an important problem finding its way into multiple engineering applications such as in fault detection…and it is an important research topic since these kind of observes are very important in practice as they possess real physical meaning which motivates the present work. Notice that, in this paper we means by unknown inputs inputs that are totally unknown not even the borders of their energy, which is of practical interest.

Note that [13] addresses the problem of positive observer design for positive time-delay systems subject to unknown inputs, where the designed observer is of full order one. On the contrary, here we focus on the purpose of functional observer design for standard linear systems. Our aim is to estimate only a functional or a linear combination of the state that can be useful for control purposes. Therefore, in this paper, we consider a new problem of designing functional positive observer for positive linear standard systems subject to unknown inputs. In fact, we propose a new positive reduced order observer (its order is equal to the dimension of the vector to be estimated) for positive linear systems subject to unknown inputs. The proposed approach is based on the positivity of an augmented system composed of dynamics of both considered system and proposed observer and also, on the unbiasedness of the estimation error by the resolution of Sylvester equation. Then existence conditions of such observers are formulated in terms of linear programming (LP) problem, where we use the Perron–Frobenius theorem applied to Metzler matrices. The different steps of the proposed approach are summarized. Numerical example and simulation results are finally given to illustrate the effectiveness of the proposed design method.

2. PROBLEM STATEMENT

Let us consider the following linear multivariable continuous system described by

$$\dot{x}(t) = Ax(t) + Bu(t) + Dd(t) \tag{1a}$$

$$z(t) = Kx(t) \tag{1b}$$

$$y(t) = Cx(t) \tag{1c}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^p$ is the control input vector, $d(t) \in \Re^q$ represents the unknown inputs vector, $y(t) \in \Re^m$ is the output vector and $z(t) \in \Re^{m_z}$ is the the functional to be estimated. A, B, D, K and C are known constant matrices of appropriate dimensions.

Further, it is assumed through the paper that:

Assumption 2.1. 1) rank $K = m_z$, $m_z \leq n$.

2) rank $C = m$, $m \leq n$.

In the following section, we give necessary conditions that ensure the positivity of the proposed linear system (1).

3. BASIC RESULTS ON POSITIVE LINEAR STANDARD SYSTEMS

Definition 3.1. (Shafai et al. [28]) A linear system is said to be positive if its state and output are both nonnegative ($x(t) \in \Re^n_+$, $y(t) \in \Re^m_+$ $\forall t \geq 0$) for any nonnegative input and nonnegative initial state.

Definition 3.2. (Luenberger [26]) A square real matrix M is called a Metzler matrix if its off-diagonal elements are nonnegative, i. e. $m_{ij} \succ 0$, $i \neq j$.

Now, we present two lemmas which will be used in the sequel of the paper.

Lemma 3.3. (Shafai et al. [28]) System (1) is positive if and only if A is a Metzler matrix and $B \in \Re^{n\times p}_+$, $D \in \Re^{n\times q}_+$, $K \in \Re^{m_z\times n}_+$, $C \in \Re^{m\times n}_+$ are nonnegative matrices: ($B \succ 0$, $D \succ 0$, $K \succ 0$ and $C \succ 0$).

Lemma 3.4. A matrix G is a generalized inverse of positive matrix $\tilde{A}$ if it satisfies $\tilde{A}G\tilde{A} = \tilde{A}$. Let $\tilde{A}^-$ be one generalized inverse of $\tilde{A}$ i. e. $\tilde{A}\tilde{A}^-\tilde{A} = \tilde{A}$, then

$$G = \tilde{A}^- + \tilde{V}(I - \tilde{A}\tilde{A}^-) + (I - \tilde{A}^-\tilde{A})\tilde{W} \tag{2}$$

where $\tilde{V}$ and $\tilde{W}$ are arbitrary matrices.

In fact it is easy to see that

$$\tilde{A}G\tilde{A} = \tilde{A}\tilde{A}^-\tilde{A} + \tilde{A}\tilde{V}(I - \tilde{A}\tilde{A}^-)\tilde{A} + \tilde{A}(I - \tilde{A}^-\tilde{A})\tilde{W}\tilde{A} = \tilde{A}\tilde{A}^-\tilde{A} = \tilde{A}. \tag{3}$$

Let us now assume that the considered linear system defined in (1) is positive. The functional $z(t) = Kx(t) \in \Re^{m_z}_+$, $m_z \leq n$ is defined as a positive linear function of the state vector, where $K \succ 0$ is any given $m_z \times n$ matrix.

Objective 3.5. Our main purpose in this paper is to design for the positive linear system (1), a reduced order positive linear functional observer of order m_z, ($m_z \leq n$), that generates a positive estimate of functional $z(t)$, $\hat{z}(t) \in \Re^{m_z}_+$ such that the estimation error $e(t) = \hat{z}(t) - z(t)$ converges asymptotically to zero as $t \to \infty$.

Note, that the designed observer uses only the available input and output in order to estimate the linear functional of the state.

4. NEW POSITIVE FUNCTIONAL OBSERVER DESIGN

4.1. Positive functional observer structure

Our aim is to design a new positive reduced order observer with the following structure

$$\dot{\varphi}(t) = N\varphi(t) + Jy(t) + Hu(t) \tag{4a}$$
$$\hat{z}(t) = \varphi(t) + Ey(t) \tag{4b}$$

where $\varphi(t)$ is the observer state, $\hat{z}(t)$ is the estimate of $z(t)$ the functional to be estimated. Observer matrices N, J, H and E are to be designed.

Lemma 4.1. The functional observer defined in (4) is called a positive linear functional observer of system (1) if for any initial condition, $\varphi(0) \in \Re^{m_z}_+$ and all inputs $u(t) \in \Re^p_+, \forall t \geq 0$ then $\hat{z}(t) \in \Re^{m_z}_+$ for all $t \geq 0$ and $\hat{z}(t)$ converges asymptotically to $z(t)$ as $t \to \infty$.

4.2. Existence conditions of positive functional observer

Before providing the first result of the paper, let us compute the estimation error, that is given by:

$$\begin{aligned} e(t) &= \hat{z}(t) - z(t) && \text{(5a)} \\ &= \varphi(t) + (EC - K)x(t). && \text{(5b)} \end{aligned}$$

So, its dynamics can be written as

$$\begin{aligned} \dot{e}(t) &= \dot{\varphi}(t) + (EC - K)\dot{x}(t) && \text{(6a)} \\ &= Ne(t) + (H + ECB - KB)u(t) + (NK - NEC + JC + ECA - KA)x(t) \\ &+ (ECD - KD)d(t). && \text{(6b)} \end{aligned}$$

Furthermore, we propose to compute the following augmented system composed by the system state $x(t)$ and the observer output $\hat{z}(t)$. In fact, it is given by:

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{\hat{z}}(t) \end{pmatrix} &= \begin{pmatrix} A & 0 \\ JC + ECA - NEC & N \end{pmatrix} \begin{pmatrix} x(t) \\ \hat{z}(t) \end{pmatrix} \\ &+ \begin{pmatrix} B \\ H + ECB \end{pmatrix} u(t) + \begin{pmatrix} D \\ ECD \end{pmatrix} d(t). \end{aligned} \tag{7}$$

We are now ready to state the first result of the paper, namely the existence conditions of the proposed functional observer. In fact, the following theorem provides conditions which ensure that the proposed system (4) is a positive linear functional observer of system (1), by providing an output $\hat{z}(t)$ that is always nonnegative and converges asymptotically to the functional $z(t)$.

Theorem 4.2. The observer defined in (4) is an asymptotic positive linear functional observer of system (1) if and only if the following conditions are satisfied:

1) N is Metzler and Hurwitz matrix

2) $JC + ECA - NEC \succ 0$

3) $H + ECB \succ 0$

4) $ECD \succ 0$

5) $NK - NEC + JC + ECA - KA = 0$

6) $H + ECB - KB = 0$

7) $ECD - KD = 0.$

P r o o f . The first part of condition 1) (N is Metzler matrix) with conditions 2), 3) and 4) are obtained by applying lemma 3.3 on augmented system (7). They ensure that the estimate $\hat{z}(t)$, output of the proposed observer (4), be nonnegative all the time.

In addition, by considering the expression (6) of the estimation error dynamics, one can conclude that it is unbiased (does not depend explicitly on state $x(t)$ and input $u(t)$) if and only if conditions 5) and 6) are satisfied with the unknown inputs independent observer condition 7). So, the estimation error dynamics $\dot{e}(t) = Ne(t)$ is asymptotically stable if and only if second part of condition 1) (N is Hurwitz) is satisfied.

Finally, it is clear that if conditions of the proposed theorem are satisfied then $\hat{z}(t)$ is always nonnegative and it tends to $z(t)$ asymptotically ($e(t)$ tends to zero asymptotically for any initial value $e(0)$; $\lim_{t\to\infty} e(t) = 0$). □

From previous developments, we can give the existence conditions of the proposed positive unknown inputs functional observer:

Lemma 4.3. The unknown input observer, defined in (4) for positive system (1), exists if:

$$rank(CD) = rank\left[\begin{pmatrix} K \\ C \end{pmatrix} D \right] = q \tag{8}$$

In fact, condition 7) of theorem (4.2) is solvable if (8) is satisfied.

We are now ready to state the second result of the paper, namely the design procedure of the proposed functional observer. In fact, the following subsection is devoted to find the functional observer matrices N, J, H and E such that conditions $1) - 7)$ of Theorem 4.2 are satisfied.

Before continuing, we will give the following remark on the design of positive functional observer with respect to classical functional observer.

Remark 4.4. It is worth noting that the functional observer design for positive systems is significantly more difficult than for systems without nonnegativity constraint. The main feature of the proposed method in this paper, is that the design is reduced to a linear programming problem that makes it interesting and easy to manipulate.

4.3. Positive functional observer synthesis

As first step of the design, let us consider conditions 6) and 7) of theorem 4.2. One can get positive functional observer matrix E from condition 7) of theorem (4.2), if rank condition (8) is verified provided that a nonnegative left inverse of CD exists.

Notice that, a parametrization of all positive generalized inverses of a known and positive matrix $\tilde{A}$ is stated by lemma 3.4. Therefore in order to guarantee the positivity of the generalized inverse of CD, it is sufficient to solve an LP problem (2) to find matrices $\tilde{V}$ and $\tilde{W}$ since CD is known and positive ($C \in \Re_+^{m\times n}$ and $D \in \Re_+^{n\times q}$).

So, observer matrix E can be given by

$$E = KD(CD)^- - Z_1(I - (CD)(CD)^-) \tag{9}$$

where Z_1 is an arbitrary matrix that can be, for simplicity assumed to be zero, which gives that

$$E = KD(CD)^- \tag{10}$$

Then, we can get functional observer matrix H from condition 6) of Theorem 4.2

$$H = KB - KD(CD)^{-}CB \tag{11}$$

Now, to achieve unbiasedness of the proposed observer, the following condition 5) of Theorem 4.2 must hold:

$$NK - NEC + JC + ECA - KA = 0. \tag{12}$$

The equation (12), which has two unknowns, that are observer matrices N and J, can be transformed to

$$[N \quad J]\begin{bmatrix} T \\ C \end{bmatrix} = TA \tag{13}$$

where $T = K - EC$. (E is given by (10)).

For the resolution of (13), let set

$$[N \quad J] = X \tag{14}$$

$$\begin{bmatrix} T \\ C \end{bmatrix} = \Sigma \tag{15}$$

$$TA = \Theta \tag{16}$$

therefore (13) becomes

$$X\Sigma = \Theta. \tag{17}$$

This equation has a solution X if and only if

$$rank \begin{pmatrix} \Sigma \\ \Theta \end{pmatrix} = rank\ \Sigma \tag{18}$$

in this case the general solution for (17), is given by

$$X = \Theta\Sigma^{-} - Z(I_{m_z+m} - \Sigma\Sigma^{-}) \tag{19}$$

where Σ^{-} is a generalized inverse of matrix Σ given by (15) and $Z \in \Re^{m_z\times(m_z+m)}$ is an arbitrary matrix, that will be determined in the sequel. Once matrix X is determined, it is easy to give the expressions of matrices N and J.

In fact,

$$N = X\begin{pmatrix} I_{m_z} \\ 0_{m\times m_z} \end{pmatrix} = A_{11} - ZB_{11} \tag{20}$$

where

$$A_{11} = \Theta\Sigma^{-}\begin{pmatrix} I_{m_z} \\ 0_{m\times m_z} \end{pmatrix} \tag{21}$$

$$B_{11} = (I_{m_z+m} - \Sigma\Sigma^{-})\begin{pmatrix} I_{m_z} \\ 0_{m\times m_z} \end{pmatrix} \tag{22}$$

and

$$J = X \begin{pmatrix} 0_{m_z \times m} \\ I_m \end{pmatrix} = A_{22} - ZB_{22} \tag{23}$$

with

$$A_{22} = \Theta\Sigma^{-} \begin{pmatrix} 0_{m_z \times m} \\ I_m \end{pmatrix} \tag{24}$$

$$B_{22} = (I_{m_z+m} - \Sigma\Sigma^{-}) \begin{pmatrix} 0_{m_z \times m} \\ I_m \end{pmatrix}. \tag{25}$$

Hence functional observer matrices N and J are determined if and only if the matrix Z is known.
Now, we propose a method to compute this matrix Z such that conditions 1)- 4) of Theorem 4.2 are satisfied; Note that equations (17), (11) and (10) correspond to conditions 5), 6) and 7) of Theorem 4.2.

For that, we recall the following result that will be useful in the sequel.

Lemma 4.5. (Arrow [1]) The Perron–Frobenius theorem applied to matrices $S \in \mathtt{M} \cap \mathtt{H}$, where $\mathtt{M}$ denotes the set of Metzler matrices and $\mathtt{H}$ denotes the set of Hurwitz matrices, states that: for all $S \in \mathtt{M} \cap \mathtt{H}$, there exists a nonnegative vector v i.e. $v \succ 0$, such that $Sv \prec 0$.

Now, let us note that conditions 3) and 4) of Theorem 4.2 are all time verified due to the nonnegativity of matrices B, C, D, K and $(CD)^{-}$. In addition, from condition 1) of Theorem 4.2 and lemma 4.5, there exists a strictly positive $v \in \Re^{m_z}$ that verifies following inequalities:

$$v \succ 0 \tag{26}$$

$$Nv \prec 0 \tag{27}$$

$$N \; diag(v) \succ 0 \quad for \quad i \neq j. \tag{28}$$

Note that we can write $v =$diag$(v)\times$ Ones(m_z). As such, we can rewrite condition 1) and condition 2) of Theorem 4.2 (that are useful for the positivity of the $\hat{z}(t)$) in terms of this vector v, where we define $V_1 = \text{diag}(v)$, as follows:

- N is Metzler and Hurwitz matrix:

$$\begin{cases} v \succ 0 \\ N^T V_1 Ones(m_z) \prec 0 \\ N^T V_1 \succ 0 \quad for \quad 1 \leq i \neq j \leq m_z. \end{cases} \tag{29}$$

- From condition 2) of Theorem 4.2, we can write the following equivalent condition

$$(JC + ECA - NEC)^T V_1 \succ 0. \tag{30}$$

At this stage, using equivalent conditions (29)-(30), we're ready to state the main result of the paper that permits to obtain for observer matrices N and J such that conditions 1) and 2) of Theorem (4.2) are satisfied. Note that based on linear programming problem, one can get the gain matrix Z which parameterizes the positive functional observer matrices N and J (see (20) and (23)).

So, we give the following theorem which permits to ensure the existence of the unknown input functional positive observer and to obtain the observer's matrices.

Theorem 4.6. The positive unknown inputs functional observer (4) for positive linear system (1) exists if the following conditions holds:

1. Rank conditions (8) and (18) are satisfied.
2. The positivity test of a one generalized inverse $(CD)^-$ of CD as mentioned by lemma (3.4) is satisfied.
3. Verify that the following linear programming (LP) problem in variables $\mathbf{V_1} \in \Re^{m_z \times m_z}$, where $\mathbf{V_1} = diag(v)$ for a strictly positive vector v and $\mathbf{Y} \in \Re^{(m_z+m) \times m_z}$ is feasible:

$$\begin{cases} v \succ 0 \\ A_{11}^T v - B_{11}^T \mathbf{Y}\ Ones(m_z) \prec 0 \\ A_{11}^T \mathbf{V_1} - B_{11}^T \mathbf{Y} \succ 0\ for \quad 1 \leq i \neq j \leq m_z \\ (A_{22}C + E_1 A - A_{11}E_1)^T \mathbf{V_1} + (B_{11}E_1 - B_{22}C)^T \mathbf{Y} \succ 0 \end{cases} \tag{31}$$

where $E_1 = EC$.

If this proposed (LP) problem is feasible, variables $\mathbf{V_1}$ and $\mathbf{Y}$ are consequently known. Then, the gain Z is given by $Z = \mathbf{V_1}^{-1}\mathbf{Y}^T$.

Proof.

1. Let's consider (29) with (20) and using the fact that $v = \mathbf{V_1}\ Ones(m_z)$, yields to the first three inequalities of (31) where $\mathbf{Y} = Z^T \mathbf{V_1}$.
2. Finally, by considering inequality (30), using (20) and (23), the last inequality holds with $\mathbf{Y} = Z^T \mathbf{V_1}$ and $E_1 = EC$.

   This completes the proof of the proposed theorem.

□

In the following section, we intend to summarise the different steps that must be achieved to design the proposed positive functional observer.

5. POSITIVE FUNCTIONAL OBSERVER DESIGN STEPS SUMMARY

1. Check existence condition of the observer (8).
2. Verify the existence of a one nonnegative generalized inverse $(CD)^-$ of (CD), using lemma 3.4.
3. Get observer matrices E and H from (10) and (11).
4. Compute Σ and Θ from (15) and (16).
5. Verify rank condition (18).
6. Compute A_{11}, B_{11}, A_{22} and B_{22} from relations (21), (22), (24) and (25).
7. If the Linear programming (LP) problem is feasible, get matrices V_1 and Y.
8. Compute matrix gain Z by $Z = V_1^{-1} Y^T$.
9. Get filter matrices N and J from (20) and (23).

Note, if rank conditions fails and/or the linear programming problem isn't feasible, the observer does not exist and we much augment the dimension of the filter to hope to have a functional filter. And if the algorithm runs well until the end, then the proposed positive functional observer description (4) for positive linear system (1) is obtained.

The following section is devoted to demonstrate the effectiveness of the proposed approach on a numerical example.

6. NUMERICAL RESULTS

Consider the system presented in section 2, where

$$A = \begin{bmatrix} -15 & 2 & 3 & 1 \\ 2 & -9 & 1 & 3 \\ 1 & 2 & -14 & 1 \\ 3 & 1 & 1 & -10 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \\ 1 \end{bmatrix}, K = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

With the computational approach presented in section 5 which is based on LP and after verifying rank conditions (8) and (18), we have obtained the following results:

1. $E = \begin{bmatrix} 1.2 & 0.6 \\ 4 & 2 \end{bmatrix}, H = \begin{bmatrix} 1.6 \\ 3 \end{bmatrix}.$

2. $\Sigma = \begin{bmatrix} 5.3 & 6.3 \\ 4.7 & 4.2 \\ 0.1 & 0.1 \end{bmatrix}$ and $\Theta = \begin{bmatrix} 4.94 & -7.15 \\ -3.94 & -0.1 \end{bmatrix}.$

   Then, we have found that the LP is feasible. One such feasible solution to the LP problem (31) provides:

3. $N = \begin{pmatrix} -1.3333 & 0.6667 \\ 0.3333 & -0.6667 \end{pmatrix}$ and $J = \begin{pmatrix} 76.4889 & 62.1539 \\ 83.0434 & 38.1760 \end{pmatrix}$.

So, the proposed design of the positive observer for positive linear system subject to unknown inputs is obtained.

Simulation results are illustrated by Figures $3, 4, 5, 6, 7$ and 8, where we present in Figures 1 and 2 the behavior of the used known and unknwon inputs $u(t)$ and $d(t)$. Note that Figures $3-4$ demonstrates the positivity of the functional state components $z_1(t)$ and $z_2(t)$, Figures $5-6$ shows the responses of the estimated functional state components $\hat{z}_1(t)$ and $\hat{z}_2(t)$ and in Figures $7-8$ we draw the estimation error components. It is clear that the estimates are nonnegative and the designed functional observer estimated the functional state $z(t)$ as expected. Finally, simulation results show the behavior of the proposed positive functional filter for positive linear systems and so, the effectiveness of our approach.

Remark 6.1. We can consider a matrix A that is not necessarily stable for system (1) and the proposed method permits to design the functional observer as expected. In fact, for example for same matrices B, C, D, D_1, C and K where,

$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 9 & 1 & 3 \\ 1 & 2 & 14 & 1 \\ 3 & 1 & 1 & 0 \end{bmatrix}$ and using the same computational approach, we obtain the following functional observer matrices: $E = \begin{bmatrix} 1.2 & 0.6 \\ 4 & 2 \end{bmatrix}$, $H = \begin{bmatrix} 1.6 \\ 3 \end{bmatrix}$,

$N = \begin{pmatrix} -1.3333 & 0.6666 \\ 0.3333 & -0.6667 \end{pmatrix}$ and $J = \begin{pmatrix} 60.4216 & 54.4866 \\ 20.6115 & 3.7441 \end{pmatrix}$.

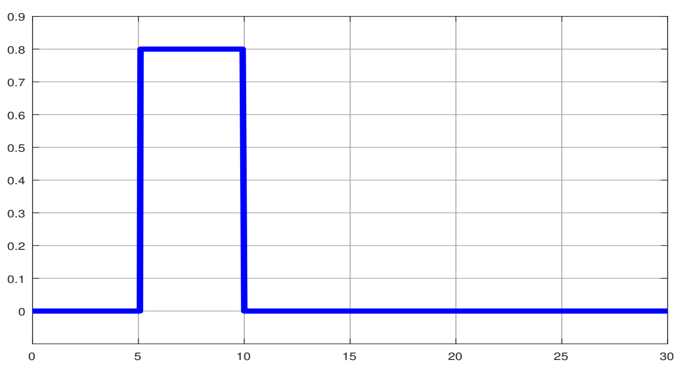

Fig. 1. Known input $u(t)$ behavior.

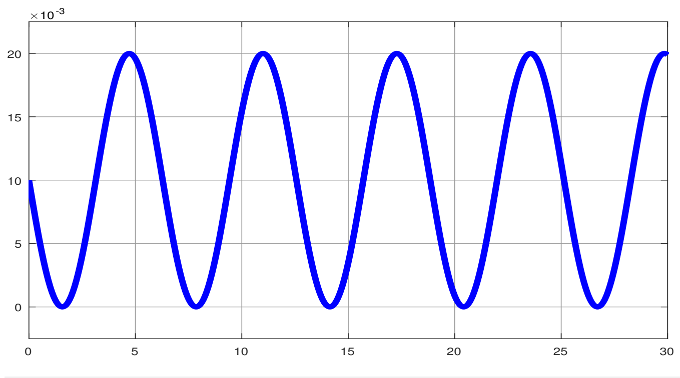

Fig. 2. Unknown inputs $d(t)$ behavior.

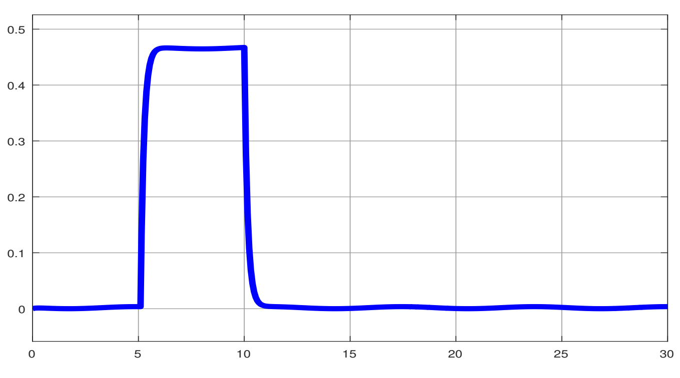

Fig. 3. First component $z_1(t)$ of the functional to be estimated.

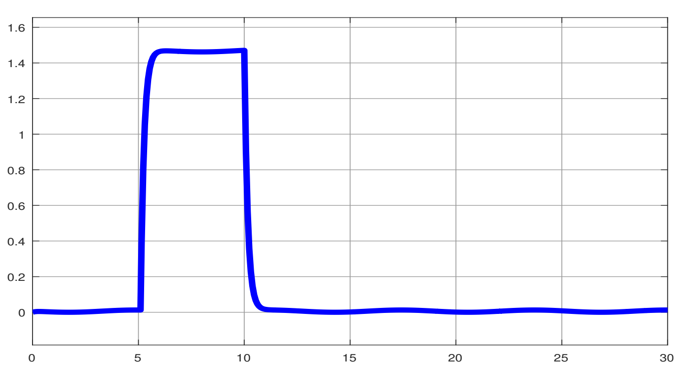

Fig. 4. Second component $z_2(t)$ of the functional to be estimated.

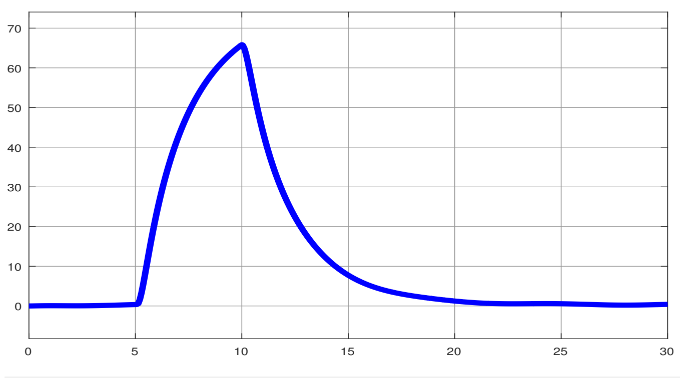

Fig. 5. First component $\hat{z}_1(t)$ of the estimated functional $\hat{z}(t)$.

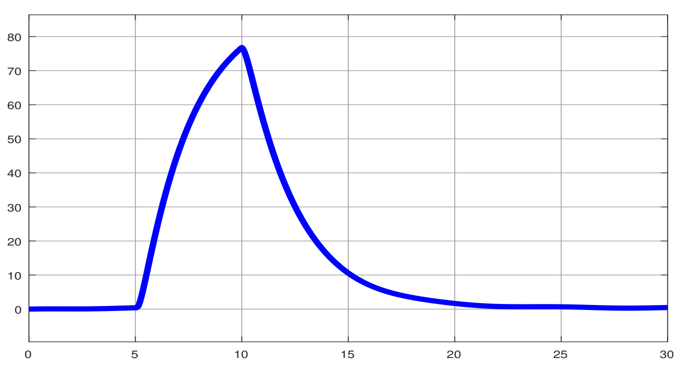

Fig. 6. Second component $\hat{z}_2(t)$ of the estimated functional $\hat{z}(t)$.

7. CONCLUSION

In this paper, we have presented new results for designing positive functional observers for linear positive standard systems subject to unknown inputs. The observer is of reduced order, it is equal to the dimension of the functional to be estimated. It is always nonnegative at any time and converges asymptotically to the real functional state vector. The proposed approach is based on the nonnegativity of an augmented system consisted of the dynamics of both considered system and proposed observer and it is based also, on the unbiasedness of the estimation error by the resolution of Sylvester equation. Then existence conditions of such observers are formulated in terms of linear programming (LP) problem, where we use the Perron–Frobenius theorem applied to Metzler matrices. The proposed new approach for positive functional observer design is summarized by an algorithm that gives different steps useful for this new design. Numerical example and

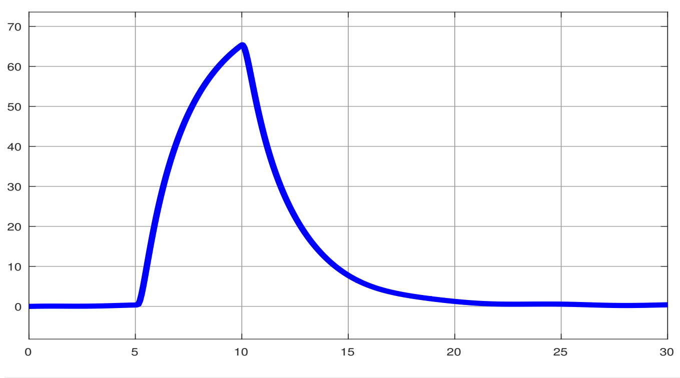

Fig. 7. The first component $e_1(t)$ of the estimation error $e(t)$.

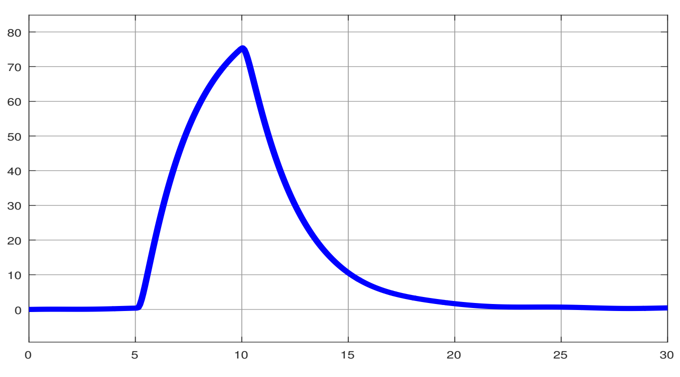

Fig. 8. The second component $e_2(t)$ of the estimation error $e(t)$.

simulation results have been given to illustrate the effectiveness of the proposed design method.

Availability of data and materials: Not Applicable.

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