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ON LOCALES WHOSE COUNTABLY COMPACT SUBLOCALES HAVE COMPACT CLOSURE

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Abstract. Among completely regular locales, we characterize those that have the feature described in the title. They are, of course, localic analogues of what are called cl-isocompact spaces. They have been considered in T. Dube, I. Naidoo, C. N. Ncube (2014), so here we give new characterizations that do not appear in this reference.

Keywords: frame; locale; isocompact; cl-isocompact; fully cl-isocompact

MSC 2020: 06D22, 54B10, 54D20, 54D30

1. Introduction

In [6], the authors define isocompact locales in a manner that extends conservatively the topological notion with the same name that was introduced by Bacon, see [1]. To recall, Bacon defined a topological space X to be isocompact if every closed countably compact subset of X is compact. The localic definition adopted in [6] is a direct translation of the one of Bacon, modulo replacing "subset" with "sublocale". The localic definition is a conservative extension of its spatial counterpart, in the sense that a topological space is isocompact if and only if the frame of its open subsets is isocompact.

Requiring the closure of every countably compact complemented sublocale to be compact defines what are called cl-isocompact locales. They are conservative extensions of Sakai's cl-isocompact spaces (see [16]). These are spaces the closures of whose countably compact subsets are compact. A formally stronger condition requires the closure of every countably compact sublocale to be compact. Locales with this feature are called fully cl-isocompact.

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All the three variants of isocompactness mentioned above were introduced and studied in [6]. This paper should then be viewed as a continuation of [6] in that it supplements the results in that paper by providing other characterizations of isocompact, cl-isocompact, and fully cl-isocompact locales. The characterizations given here include extensions of some characterizations in [7]. Apart from that, we also improve some results in [6]. One such improvement is Theorem 6.4 in which we show that arbitrary products of fully cl-isocompact locales are fully cl-isocompact. In [6] this was proved only for binary (and hence finite) products with a proof which cannot be mimicked for arbitrary products.

Here is an overview of the paper. The Preliminaries (in Section 2) serve mainly the purpose of fixing non-standard (and some standard) notation. We have thus elected to be brief and recall only a few notions. Our thesis is that the theory of frames and locales has by now come of age, and, furthermore, the readers of this article are likely to be au fait with the rudiments of point-free topology.

In Section 3 we assemble some tools that we require mainly for the results in Section 4. The characterizations of isocompact and fully cl-isocompact frames are in Section 4. Section 5 is about some applications of the results in the preceding section. Let us elaborate somewhat. A localic image of an isocompact (or fully cl-isocompact) frame need not be isocompact (or fully cl-isocompact). We give an example of such failure and then apply the characterizations in Section 4 to show that, subject to some conditions, the image of an isocompact (or fully cl-isocompact frame) also has the same property. Interestingly, the conditions for the weaker notion are strictly less stringent than those for the stronger notion.

The last section deals with localic products of these types of locales, but treated in the category Frm, so that we actually deal with coproducts of frames.

2. Preliminaries

Our references for frames and locales are [12] and [13]. As in these references, we do will not work strictly within either Frm or Loc, but rather we will avail ourselves tools from both categories even within the same proof in some instances. Throughout this section, and, in fact, throughout the paper, L denotes a frame.

2.1. Frames and their homomorphisms. The asterisk will be used as a subscript to denote the right adjoint of a frame homomorphism, and as a superscript to denote the pseudocomplement of an element. All frames are assumed to be completely regular. We shall view the Stone-Čech compactification of L as the frame of *strongly regular* ideals of $\operatorname{Coz} L$, the cozero part of L. To recall, these are the lattice ideals $J \subseteq \operatorname{Coz} L$ such that for every $u \in J$ there exists some $v \in J$ with $u \prec\!\prec v$. We denote by $j_L\colon \beta L \to L$ the dense onto frame homomorphism that takes a strongly regular ideal of $\operatorname{Coz} L$ to its join.

The only algebraic structures that will be considered below are frames, so the word "homomorphism" will throughout be understood to mean a frame homomorphism. For any $a \in L$, we write $\kappa_a\colon L \to {\uparrow}a$ for the homomorphism given by $x \mapsto a \vee x$. The right adjoint of κ_a is the inclusion ${\uparrow}a \rightarrowtail L$. For any homomorphism $h\colon L \to M$ there is the factorization

$$L \xrightarrow{\kappa_{h_*0}} {\uparrow}h_*(0) \xrightarrow{\widehat{h}} M,$$

where $\widehat{h}$ maps as h. The homomorphism $\widehat{h}$ is dense. Since the homomorphism $\kappa_{h_*(0)}$ is onto, the factorization is sometimes called the *dense-onto factorization.* We will use that name.

2.2. Sublocales. We denote by $\mathcal{S}(L)$ the coframe of sublocales of L. The *supplement* of a sublocale S is the sublocale

$$S^{\#} = \bigvee\{T \in \mathcal{S}(L)\colon\ S \cap T = \mathsf{O}\},$$

where O denotes the *void* sublocale, namely, $\mathsf{O} = \{1\}$. If a sublocale S of L has a complement in $\mathcal{S}(L)$, it is said to be *complemented.* This is the case precisely when $S \cap S^{\#} = \mathsf{O}$. If S is complemented, then its complement is exactly $S^{\#}$.

The closed (or open) sublocale of L associated to $x \in L$ is denoted by $\mathfrak{c}_L(x)$ (or $\mathfrak{o}_L(x)$). Since $\mathfrak{c}_L(x) = {\uparrow}x$, we shall use either notation, as convenient. We write $\operatorname{Cld}(L)$ (or $\operatorname{Opn}(L)$) for the lattice (ordered by inclusion) of all closed (or open) sublocales of L.

For a sublocale A of L, we denote by $\nu_A\colon L \to A$ the associated frame surjection, and recall that

$$\nu_A(x) = \bigwedge\{a \in A\colon\ a \geqslant x\}.$$

With this notation, we then have

$$\operatorname{Cld}(A) = \{A \cap F\colon\ F \in \operatorname{Cld}(L)\} = \{\mathfrak{c}_A(\nu_A(x))\colon\ x \in L\},$$
$$\operatorname{Opn}(A) = \{A \cap U\colon\ U \in \operatorname{Opn}(L)\} = \{\mathfrak{o}_A(\nu_A(x))\colon\ x \in L\}.$$

Let us recall the identities

$$\bigcap_{i\in I} \mathfrak{c}_L(a_i) = \mathfrak{c}_L\Big(\bigvee_{i\in I} a_i\Big), \quad \mathfrak{c}_L(a) \vee \mathfrak{c}_L(b) = \mathfrak{c}_L(a \wedge b),$$
$$\bigvee_{i\in I} \mathfrak{o}_L(a_i) = \mathfrak{o}_L\Big(\bigvee_{i\in I} a_i\Big), \quad \mathfrak{o}_L(a) \cap \mathfrak{o}_L(b) = \mathfrak{o}_L(a \wedge b).$$

For an arbitrary subset S of L, not necessarily a sublocale, we write

$$\mathfrak{c}_L[S] = \{\mathfrak{c}_L(x)\colon\ x \in S\} \quad \text{and} \quad \mathfrak{o}_L[S] = \{\mathfrak{o}_L(x)\colon\ x \in S\}.$$

2.3. Localic maps. A localic map $f\colon L \to M$ gives rise to two mappings

$$f[\text{-}]\colon \mathcal{S}(L) \to \mathcal{S}(M) \quad \text{and} \quad f_{-1}[\text{-}]\colon \mathcal{S}(M) \to \mathcal{S}(L)$$

defined by

$$f[A] = \{f(x)\colon x \in A\} \quad \text{and} \quad f_{-1}[B] = \bigvee \{S \in \mathcal{S}(L)\colon S \subseteq f^{-1}[B]\},$$

where $f^{-1}[\text{-}]$ is the set-theoretic inverse-image mapping. For any $S \in \mathcal{S}(L)$ and $T \in \mathcal{S}(M)$, we always have the equivalence

$$f[S] \subseteq T \Leftrightarrow S \subseteq f_{-1}[T].$$

The mapping $f[\text{-}]$ preserves joins. The mapping $f_{-1}[\text{-}]$ preserves meets (recall that they are intersections) and finite joins. Also, $f_{-1}[\text{-}]$ preserves arbitrary joins of open sublocales, that is, if $\{V_i\colon i \in I\}$ is a collection of open sublocales of M, then

$$f_{-1}\Big[\bigvee_{i\in I} V_i\Big] = \bigvee_{i\in I} f_{-1}[V_i].$$

If $f[U]$ is an open sublocale of M for every open sublocale U of L, then f is said to be an *open map*. If $f[A]$ is a closed sublocale of M for every closed sublocale A of L, then f is said to be a *closed map*. For the latter, we will recall some useful characterizations where they will be needed. A *closed homomorphism* is one whose right adjoint is a closed map.

2.4. Covers and coverings. A *cover* of L is a set $C \subseteq L$ such that $\bigvee C = 1$. To avoid ambiguity, we say a collection $\mathscr{A}$ of sublocales of L is a *covering* of L if $\bigvee\{A\colon A \in \mathscr{A}\} = L$, where the join is calculated in $\mathcal{S}(L)$. This terminology is not standard. A cover consists of elements of L, whereas a covering consists of sublocales of L. If every sublocale in a covering $\mathscr{A}$ of L is open, then $\mathscr{A}$ is an *open covering* of L. There is a bijection between covers and open coverings, given by

$$C \mapsto \mathfrak{o}_L[C] \quad \text{and} \quad \mathscr{A} \mapsto \mathfrak{o}_L^{-1}[\mathscr{A}] = \{x \in L\colon \mathfrak{o}_L(x) \in \mathscr{A}\}.$$

3. Assembling some tools

As the heading suggests, in this section we assemble some tools that will be used in the next section, where the main theorem is proved. Let us recall the following terminology from [3]. If A is a sublattice of L, an ideal J in A is said to be *σ-proper* if $\bigvee S \neq 1$ for any countable $S \subseteq J$, and *completely proper* if $\bigvee J \neq 1$.

On the other hand, as in spaces, let us say a filter $\mathcal{F}$ in a sublattice of $\mathrm{Cld}(L)$ is *fixed* in the case when $\bigcap\{F\colon F \in \mathcal{F}\} \neq \mathsf{O}$, and has the *countable intersection property* if $\bigcap\{C\colon C \in \mathcal{C}\} \neq \mathsf{O}$ for every countable $\mathcal{C} \subseteq \mathcal{F}$.

Next, for a lattice M, let us write $\mathrm{Sub}(M)$ for the set of sublattices of M. Note that if L is a frame, then the map $\mathfrak{c}_L[\text{-}]\colon \mathrm{Sub}(L) \to \mathrm{Sub}(\mathrm{Cld}(L))$ is a bijection. Let us record the following results which are easy to verify using the properties of the map $\mathfrak{c}_L\colon L \to \mathcal{S}(L)$.

Lemma 3.1. *Let L be a frame.*

(1) *If A is a sublattice of L and J is an ideal in A, then $\mathfrak{c}_L[J]$ is a filter in the lattice $\mathfrak{c}_L[A]$. Furthermore:*
   (a) *J is σ-proper if and only if $\mathfrak{c}_L[J]$ has the countable intersection property.*
   (b) *J is completely proper if and only if $\mathfrak{c}_L[J]$ is fixed.*

(2) *If $\mathcal{A}$ is a sublattice of $\mathrm{Cld}(L)$ and $\mathcal{F}$ is a filter in $\mathcal{A}$, then $\mathfrak{c}_L^{-1}[\mathcal{F}]$ is an ideal in the lattice $\mathfrak{c}_L^{-1}[\mathcal{A}]$. Furthermore:*
   (a) *$\mathcal{F}$ has the countable intersection property if and only if $\mathfrak{c}_L^{-1}[\mathcal{F}]$ is σ-proper.*
   (b) *$\mathcal{F}$ is fixed if and only if $\mathfrak{c}_L^{-1}[\mathcal{F}]$ is completely proper.*

The following lemma leads to a result (Corollary 3.3 below) that will be used in the proof of the main theorem in this section.

Lemma 3.2. *Let L be a frame and A be a sublattice of L. Then the following statements are equivalent.*

(1) *Every σ-proper ideal of A is completely proper.*
(2) *Every cover of L by elements of A admits a countable subcover.*
(3) *Every covering of L by sublocales belonging to $\mathfrak{o}_L[A]$ admits a countable subcovering.*

P r o o f. The equivalence of statements (2) and (3) follows easily from the fact that $\mathfrak{o}_L(a) = L$ if and only if $a = 1$.

(1) $\Rightarrow$ (2): Suppose, by way of contradiction, that L has a cover C consisting of elements of A which has no countable subcover. Then the set

$$J = \Big\{a \in A\colon\ a \leqslant \bigvee S \text{ for some countable } S \subseteq C\Big\}$$

is an ideal of A because if for $i \in \{1,2\}$, $a_i \leqslant S_i$ for a countable $S_i \subseteq C$, then $S_1 \cup S_2$ is a countable subset of C with $a_1 \vee a_2 \leqslant \bigvee(S_1 \cup S_2)$. Since a countable union of countable sets is countable, reasoning as in the previous sentence, one sees that J is σ-proper because any countable subset of C has join unequal to 1, by supposition. Therefore $\bigvee J \neq 1$, which then implies that $\bigvee C \neq 1$ because $C \subseteq J$. This is a contradiction.

(2) $\Rightarrow$ (1): Let J be a σ-proper ideal of A. Therefore $\bigvee J \neq 1$, otherwise J would be a cover of L consisting of elements of A but admitting no countable subcover. $\square$

As remarked earlier, if $\mathcal{A}$ is a sublattice of $\mathrm{Cld}(L)$, then $\mathcal{A} = \mathfrak{c}_L[A]$ for a sublattice A of L. Furthermore, if $\mathcal{F}$ is a filter in $\mathcal{A}$, then the set $\mathfrak{c}_L^{-1}[\mathcal{F}]$ is an ideal in A such that $\mathcal{F}$ has the countable intersection property if and only if $\mathfrak{c}_L^{-1}[\mathcal{F}]$ is σ-proper. We therefore have the following corollary.

Corollary 3.3. *Let L be a frame and let $\mathfrak{L}$ be a sublattice of $\mathrm{Cld}(L)$. Then the following statements are equivalent.*

(a) *Every filter in $\mathfrak{L}$ with the countable intersection property is fixed.*
(b) *Every covering of L by sublocales which are complements of members of $\mathfrak{L}$ has a countable subcovering.*

4. Isocompactness and its stronger variants

We start by recalling that a frame is *countably compact* if every countable cover has a finite subcover. Equivalently, every countable open covering has a finite subcovering. We must remark that, unlike compact regular frames, countably compact regular frames need not be spatial (see [5]).

The following characterization of countably compact sublocales in terms of open coverings of the ambient frame will be useful. Of course, to say a sublocale is countably compact means that it is countably compact as a frame in its own right. Call a sequence $(U_n)_{n\in\mathbb{N}}$ of sublocales of L *increasing* in the case when $U_n \subseteq U_{n+1}$ for every n. Using the fact (proved by Isbell in [10]) that families of open sublocales are *distributive*, meaning that if $\{U_i\colon i \in I\}$ is a family of open sublocales of L, then, for any sublocale A of L,

$$A \cap \bigvee_{i\in I} U_i = \bigvee_{i\in I} (A \cap U_i),$$

it is easy to prove the following.

Lemma 4.1. *The following statements are equivalent for a sublocale A of L.*

(1) *A is countably compact.*
(2) *If $(U_n)_{n\in\mathbb{N}}$ is a sequence of open sublocales of L with $A \subseteq \bigvee_{n\in\mathbb{N}} U_n$, then there are finitely many indices $n_1, \ldots, n_k$ such that $A \subseteq U_{n_1} \vee \ldots \vee U_{n_k}$.*
(3) *If $(U_n)_{n\in\mathbb{N}}$ is an increasing sequence of open sublocales of L with $A \subseteq \bigvee_{n\in\mathbb{N}} U_n$, then $A \subseteq U_{n_0}$ for some $n_0 \in \mathbb{N}$.*

Now here are the definitions of the frames that will form the main subject of study in this article. The definitions come from [6].

Definition 4.2. A frame L is said to be *isocompact* if every closed countably compact sublocale of L is compact, *closure-isocompact* (abbreviated cl-isocompact) if the closure of every countably compact complemented sublocale of L is compact, *fully closure-isocompact* (abbreviated fully cl-isocompact) if the closure of every countably compact sublocale of L is compact.

As observed in [6], the implications

$$\text{fully cl-isocompact} \Rightarrow \text{cl-isocompact} \Rightarrow \text{isocompact}$$

hold, with the latter non-reversible. We do not know if the first is reversible.

Before we plough ahead with the main purpose of this article, let us take this opportunity to record some observations which are not recorded in [6]. We start with one which casts some light on the question of reversibility of the first implication above. We record it as a proposition which suggests that to settle the question, it suffices to determine whether in cl-isocompact frames countably compact sublocales have countably compact closures.

Proposition 4.3. *The following statements are equivalent for a frame L.*

(1) *L is fully* cl-*isocompact.*
(2) *L is* cl-*isocompact and for every countably compact sublocale A of L, there is a complemented countably compact sublocale A' of L with $\overline{A} = \overline{A'}$.*
(3) *L is* cl-*isocompact and every countably compact sublocale of L has a countably compact closure.*

Proof. (1) $\Rightarrow$ (2): Let A be a countably compact sublocale of L. Since L is fully cl-isocompact, it is cl-isocompact, and so $\overline{A}$ is compact, and hence countably compact. Therefore $\overline{A}$ is a complemented countably compact sublocale of L, whose closure coincides with that of A.

(2) $\Rightarrow$ (3): Let A be a countably compact sublocale of L. By hypothesis, there is a complemented countably compact sublocale A' of L with $\overline{A'} = \overline{A}$. Since L is cl-isocompact, by hypothesis, $\overline{A'}$ is compact, and hence countably compact. Therefore $\overline{A}$ is countably compact.

(3) $\Rightarrow$ (1): Let A be a countably compact sublocale of L. By hypothesis, $\overline{A}$ is countably compact, and so $\overline{A}$ is a complemented countably compact sublocale of the cl-isocompact frame L, which implies that $\overline{A}$ is compact. Therefore L is fully cl-isocompact. □

The next observation adds to a list of examples of fully cl-isocompact frames. Among the examples of fully cl-isocompact frames mentioned in [6] are paracompact frames. Here we point out that every realcompact frame is fully cl-isocompact. The definition of realcompact frames can be found in [3]. See also [4] for some properties of these frames. For the definition and properties of pseudocompact frames that we shall refer to [2]. On more than one occasion below we shall use the fact that a frame which has a dense pseudocompact sublocale is itself pseudocompact, see [6], Lemma 4.3.

Example 4.4. Every realcompact frame is fully cl-isocompact. To see this, let A be a countably compact sublocale of a realcompact frame L. Then $\overline{A}$ is pseudocompact because it has a dense countably compact (and hence pseudocompact) sublocale. As shown in [4], Lemma 4.8, closed sublocales of a realcompact frame are realcompact. Therefore $\overline{A}$ is both realcompact and pseudocompact, so, by [3], Corollary to Proposition 4, $\overline{A}$ is compact, showing that L is fully cl-isocompact.

Our final observation is about fully cl-isocompact and cl-isocompact frames that have no points. In this regard, recall that a nontrivial (meaning one in which $0 \neq 1$) compact frame has at least one point (see [12], Lemma III 1.9).

Observation 4.5. A frame with no points is fully cl-isocompact (or cl-isocompact) if and only if its void sublocale is the only one that is countably compact (or complemented and countably compact). Indeed, if O is the only sublocale of L that is countably compact, then L is fully cl-isocompact. On the other hand, suppose L has no points and is fully cl-isocompact. Suppose, on the contrary, that A is a non-void countably compact sublocale of L. Then $\overline{A}$ is a nontrivial compact sublocale of L, and hence has a point. But points of a sublocale are points of the frame that reside inside the sublocale. This yields a contradiction. The other assertion is verified similarly.

Now, reverting to the main theme, we are aiming for characterizations of fully cl-isocompact frames motivated by [7], Theorems 1.9 and 1.10. Towards that end, let us define a collection $\mathcal{K}(L) \subseteq \mathcal{S}(L)$ by setting

$$\mathcal{K}(L) = \{L\} \cup \{A \in \mathcal{S}(L)\colon\ A = \overline{A_1} \cap \ldots \cap \overline{A_n} \text{ for a countably compact } A_i \in \mathcal{S}(L)\}.$$

Observe that $\mathcal{K}(L)$ is a sublattice of $\mathrm{Cld}(L)$. Indeed, O and L belong to $\mathcal{K}(L)$, and the meet of any two members is clearly also a member. Regarding the join of two members, say, $A = \overline{A_1} \cap \ldots \cap \overline{A_n}$ and $B = \overline{B_1} \cap \ldots \cap \overline{B_m}$, with A_i and B_j countably compact,

$$A \vee B = \bigcap_{i,j} (\overline{A_i} \vee \overline{B_j}) = \bigcap_{i,j} \overline{A_i \vee B_j},$$

which belongs to $\mathcal{K}(L)$ because, as follows easily from Lemma 4.1, the join of finitely many countably compact sublocales is countably compact.

R e m a r k 4.6. This remark is due to the referee. When L is fully cl-isocompact, then $\mathcal{K}(L)$ is simply the system of compact sublocales of L, plus L itself.

As commented in the previous section, there is a sublattice of L, which we denote by $\mathrm{K}(L)$, such that $\mathcal{K}(L) = \mathfrak{c}_L[\mathrm{K}(L)]$. In fact, $\mathrm{K}(L) = \mathfrak{c}_L^{-1}[\mathcal{K}(L)]$, so explicitly,

$$\mathrm{K}(L) = \{0\} \cup \Big\{ a \in L\colon\ a = \bigwedge A_1 \vee \ldots \vee \bigwedge A_n \text{ for some countably compact } A_i \in \mathcal{S}(L) \Big\}.$$

We need a lemma to prove the main theorem below which characterizes fully cl-isocompact frames. Recall that a frame homomorphism $h\colon M \to L$ is *coz-codense* if whenever $h(c) = 1$, with $c \in \operatorname{Coz} M$, then $c = 1$. It is *coz-onto* if for every $d \in \operatorname{Coz} L$ there exists some $c \in \operatorname{Coz} M$ such that $h(c) = d$. Banaschewski and Gilmour showed in [2], Corollary 5 that the homomorphism $j_L\colon \beta L \to L$ is coz-onto. Walters-Wayland showed in [17], Proposition 7.5 that L is pseudocompact if and only if every dense onto homomorphism $M \to L$, with M compact, is coz-codense.

Lemma 4.7. *If L is pseudocompact and $h\colon M \to L$ is dense onto, then h is coz-codense.*

P r o o f. Let u be a cozero element of M with $h(u) = 1$. The composite

$$\beta M \xrightarrow{j_M} M \xrightarrow{h} L$$

is a dense homomorphism out of a compact frame onto a pseudocompact one. Therefore $h \circ j_M$ is coz-codense, by the result cited from [17]. By the result cited from [2], there is cozero element U of βL with $j_M(U) = u$. Since $h(j_M(U)) = h(u) = 1$, it follows that $U = 1_{\beta L}$, which implies that $u = 1_L$. Therefore h is coz-codense. $\square$

In the upcoming proof we shall use the fact that, as in the case of countable compactness (see Lemma 4.1) a sublocale of L is compact if and only if whenever it is contained in the join of open sublocales of L, then it is contained in the join of finitely many of those open sublocales.

Theorem 4.8. *The following statements are equivalent for a completely regular frame L.*

(1) *L is fully* cl-*isocompact.*
(2) *Every filter in $\mathcal{K}(L)$ with the countable intersection property is fixed.*
(3) *Every σ-proper ideal in $\mathrm{K}(L)$ is completely proper.*
(4) *Every cover of L consisting of elements of $\mathrm{K}(L)$ has a countable subcover.*
(5) *Every covering of L consisting of complements of members of $\mathcal{K}(L)$ has a countable subcovering.*

P r o o f. The equivalence of statements (2) and (5) follows from Corollary 3.3. It is easy to verify that an open sublocale is a complement of a sublocale in $\mathcal{K}(L)$ if and only if it is of the form $\mathfrak{o}_L(a)$ for some $a \in \mathrm{K}(L)$. Therefore the equivalence of statements (3), (4) and (5) follows from Lemma 3.2.

(1) $\Rightarrow$ (2): Assume that L is fully cl-isocompact, and let $\mathcal{F}$ be a filter in $\mathcal{K}(L)$ with a countable intersection property. Note, from the outset, that $\mathcal{F}$ is a proper filter because it has the countable intersection property. Since L is fully cl-isocompact, each element of $\mathcal{F}$ is a closed sublocale of a compact locale, and is therefore compact. Now, suppose, by way of contradiction, that $\mathcal{F}$ is not fixed. Then

$$L = \mathsf{O}^{\#} = \Big(\bigcap\{F\colon\ F \in \mathcal{F}\}\Big)^{\#} = \bigvee\{F^{\#}\colon\ F \in \mathcal{F}\}.$$

Let H be any element of $\mathcal{F}$. Since the supplement of each sublocale in $\mathcal{F}$ is open, the compactness of H furnishes finitely many elements $F_1, \ldots, F_n$ of $\mathcal{F}$ such that

$$H \subseteq F_1^{\#} \vee \ldots \vee F_n^{\#} = (F_1 \cap \ldots \cap F_n)^{\#},$$

which then implies that $H \cap (F_1 \cap \ldots \cap F_n) = \mathsf{O}$ because a complemented sublocale misses its supplement. This yields a contradiction because both H and $F_1 \cap \ldots \cap F_n$ are members of $\mathcal{F}$, a proper filter.

(3) $\Rightarrow$ (1): Assume that L satisfies the property hypothesized in (3). Observe that if S is a sublocale of L, then $\overline{S}$ also has the property hypothesized in (3). To see this, note first that if a is a nonzero element in $\mathrm{K}(\overline{S})$, then $a \in \mathrm{K}(L)$ because any countably compact sublocale of $\overline{S}$ is also countably compact as a sublocale of L, and furthermore, joins in $\overline{S}$ agree with those in L because $\overline{S}$ is a closed sublocale of L. Now let I be a σ-proper ideal in $\mathrm{K}(\overline{S})$. The set

$$\widehat{I} = \{x \in \mathrm{K}(L)\colon\ x \leqslant u \text{ for some } u \in I\}$$

is easily checked to be a σ-proper ideal in $\mathrm{K}(L)$, containing all nonzero elements of I. Consequently $\widehat{I}$ is completely proper, by hypothesis, and hence I is completely proper.

To prove that L is fully cl-isocompact, let S be a countably compact sublocale of L. We must show that $\overline{S}$ is compact. Since every sublocale is dense in its closure, by what we have demonstrated in the foregoing paragraph we may assume that S is dense in L and then argue that L is compact. Since L has a dense countably compact (and therefore a dense pseudocompact) sublocale, L is pseudocompact. By [3], Corollary to Proposition 4, it suffices to show that L is realcompact. If L

is not realcompact, then there is a σ-proper maximal ideal U of $\operatorname{Coz} L$ such that $\bigvee U = 1$. Define the set $J \subseteq \mathrm{K}(L)$ by

$$J = \Big\{x \in \mathrm{K}(L)\colon\ x \leqslant \bigwedge(\mathfrak{c}_L(u) \cap S) \text{ for some } u \text{ in } U\Big\},$$

and note that J is an ideal in $\mathrm{K}(L)$ containing each $\bigwedge(\mathfrak{c}_L(u) \cap S)$, for $u \in U$, because each $\mathfrak{c}_L(u) \cap S$ is countably compact, being a closed sublocale of the countably compact sublocale S. We show that J is σ-proper. Let $\nu_S\colon L \to S$ be the frame surjection witnessing that S is a sublocale of L, so for any $a \in L$,

$$\nu_S(a) = \bigwedge\{s \in S\colon\ a \leqslant s\} = \bigwedge(\mathfrak{c}_L(a) \cap S).$$

Since S is a dense sublocale of L, ν_S is dense. For any sequence (x_n) in J, choose, for each n, an element $u_n \in U$ such that $x_n \leqslant \bigwedge(\mathfrak{c}_L(u_n) \cap S)$. Since U is σ-proper, $\bigvee_n u_n < 1$. Since ν_S is a dense onto frame homomorphism and S is pseudocompact, ν_S is coz-codense by Lemma 4.7. Since $\bigvee_n u_n \in \operatorname{Coz} L$, we therefore have

$$\bigvee_n x_n \leqslant \bigvee_n \Big(\bigwedge(\mathfrak{c}_L(u_n) \cap S)\Big) = \bigvee_n \nu_S(u_n) \leqslant \nu_S\Big(\bigvee_n u_n\Big) < 1,$$

which shows that J is σ-proper. Therefore, by (3), J is completely proper. But now this is a contradiction because for each $u \in U$,

$$u = \bigwedge \mathfrak{c}_L(u) \leqslant \bigwedge(\mathfrak{c}_L(u) \cap S) \in J,$$

and $\bigvee U = 1$. We conclude therefore that L is realcompact, and hence compact as it is pseudocompact. ☐

The characterization in item (4) in this theorem can be expressed in terms of the subset

$$\mathrm{E}(L) = \Big\{\bigwedge A\colon\ A \text{ is a countably compact sublocale of } L\Big\}$$

of L. Note that:

(a) $\mathrm{E}(L) \subseteq \mathrm{K}(L)$, and

(b) every element of $\mathrm{K}(L)$ is a join of a finite subset of $\mathrm{E}(L)$, including the empty set (so as to cater for 0).

These two properties yield the following corollary.

Corollary 4.9. *A frame L is fully* cl*-isocompact if and only if every cover of L by elements of $\mathrm{E}(L)$ has a countable subcover.*

P r o o f. If L is fully cl-isocompact, then every cover of L by elements of $\mathrm{K}(L)$ has a countable subcover by Theorem 4.8. Since $\mathrm{E}(L)$ is a subset of $\mathrm{K}(L)$, it follows that every cover of L by elements of $\mathrm{E}(L)$ has a countable subcover.

Conversely, suppose that $\bigvee T = 1$ for some $T \subseteq \mathrm{K}(L)$. For each $t \in T$ pick a set $E_t \subseteq \mathrm{E}(L)$ such that $t = \bigvee E_t$, and put $E = \bigcup\limits_{t\in T} E_t$. Clearly, $\bigvee E = 1$, and so, by the present hypothesis, there is a countable $S \subseteq E$ such that $\bigvee S = 1$. If $s \in S$, then $s \in E_t$ for some $t \in T$, so $s \leqslant t$. Thus, each element of S is below an element T, and so T has a countable subset whose join is the top element of L. Therefore L is fully cl-isocompact. □

There are characterizations of cl-isocompact frames similar to those in Theorem 4.8. To obtain them one simply inserts the descriptor "complemented". To be more precise, define the collection $\mathcal{K}^{\mathrm{c}}(L) \subseteq \mathcal{S}(L)$ by decreeing that a sublocale A of L belongs to $\mathcal{K}^{\mathrm{c}}(L)$ if and only if $A = L$ or $A = \overline{A_1} \cap \ldots \cap \overline{A_n}$, for some finitely many complemented countably compact sublocales A_i of L. Then let $\mathrm{K}^{\mathrm{c}}(L)$ be defined analogously to $\mathrm{K}(L)$.

Using properties of complemented sublocales, a proof along the lines of the proof of Theorem 4.8 yields the following. To mimic this proof, in showing the implication (3) $\Rightarrow$ (1) you need to observe that if S is a complemented sublocale of L, then S is also complemented as a sublocale of $\overline{S}$, that is, S has a complement in $\mathcal{S}(\overline{S})$.

Theorem 4.10. *The following statements are equivalent for a completely regular frame L.*

(1) *L is* cl-*isocompact.*
(2) *Every filter in $\mathcal{K}^{\mathrm{c}}(L)$ with the countable intersection property is fixed.*
(3) *Every σ-proper ideal in $\mathrm{K}^{\mathrm{c}}(L)$ is completely proper.*
(4) *Every cover of L consisting of elements of $\mathrm{K}^{\mathrm{c}}(L)$ has a countable subcover.*
(5) *Every covering of L consisting of complements of members of $\mathcal{K}^{\mathrm{c}}(L)$ has a countable subcovering.*

Regarding isocompact frames, there are similar characterizations. Since the proofs are also exactly along the same lines, we only set up the tools for stating the characterizations, just as we did for the cl-isocompact case.

Define the sets

$$\mathcal{C}(L) = \{L\} \cup \{S \in \mathcal{S}(L)\colon\ S \text{ is closed and countably compact}\}$$

and

$$\mathrm{C}(L) = \{0\} \cup \{a \in L\colon\ \mathfrak{c}_L(a) \text{ is countably compact}\},$$

and note that they are, respectively, sublattices of $\mathrm{Cld}(L)$ and L, related by the equalities $\mathcal{C}(L) = \mathfrak{c}_L[\mathrm{C}(L)]$ and $\mathrm{C}(L) = \mathfrak{c}_L^{-1}[\mathcal{C}(L)]$.

Theorem 4.11. *The following statements are equivalent for a completely regular frame L.*

(1) *L is isocompact.*
(2) *Every filter in $\mathcal{C}(L)$ with the countable intersection property is fixed.*
(3) *Every σ-proper ideal in $\mathrm{C}(L)$ is completely proper.*
(4) *Every cover of L consisting of elements of $\mathrm{C}(L)$ has a countable subcover.*
(5) *Every covering of L consisting of complements of members of $\mathcal{C}(L)$ has a countable subcovering.*

5. Some applications

A localic image of a fully cl-isocompact frame need not be fully cl-isocompact. In fact, it need not even be isocompact. Here is an example. As mentioned earlier, it is shown in [6], Example 4.6 that every paracompact frame is fully cl-isocompact.

Example 5.1. Let L be a frame which is not isocompact, and consider the *dissolution map* $\mathcal{S}(L)^{\mathrm{op}} \to L$, which is the right adjoint of the frame embedding $a \mapsto \mathfrak{c}_L(a)\colon L \to \mathcal{S}(L)^{\mathrm{op}}$. Since $\mathcal{S}(L)^{\mathrm{op}}$ is paracompact—in fact, ultraparacompact (see [15], Theorem 17)—and paracompact frames are fully cl-isocompact, we have an example because the dissolution map is onto.

We apply Theorem 4.8 to show that the localic image of a fully cl-isocompact frame under an open localic map which pulls back countably compact sublocales to countably compact sublocales is fully cl-isocompact. For this we need to recall from [14], Corollary 5.2 that if $f\colon L \to M$ is an open localic map, then

$$\overline{f_{-1}[T]} = f_{-1}[\overline{T}]$$

for all sublocales T of M.

The following quick lemma will be useful.

Lemma 5.2. *For any localic map $f\colon L \to M$, $f[f_{-1}[T]^{\#}] \subseteq T^{\#}$ for every sublocale T of M.*

Proof. Recall that if $g\colon A \to B$ is a frame homomorphism, then $g(a^*) \leqslant g(a)^*$ for every $a \in A$, and so

$$a^* \leqslant g_*(g(a^*)) \leqslant g_*(g(a)^*).$$

Applying this to the frame homomorphism $f_{-1}[\text{-}]\colon \mathcal{S}(M)^{\mathrm{op}} \to \mathcal{S}(L)^{\mathrm{op}}$ and keeping in mind that the partial order is now $\supseteq$ yields the result. □

In the proof that follows we invoke condition (5) in Theorem 4.8.

Corollary 5.3. *Suppose that $f\colon L \to M$ is an onto open localic map which pulls back countably compact sublocales to countably compact sublocales. Then M is fully* cl-*isocompact if L is fully* cl-*isocompact.*

Proof. We start by showing that f pulls back sublocales in $\mathcal{K}(M)$ to sublocales in $\mathcal{K}(L)$. So let $A \in \mathcal{K}(M)$, and find countably compact sublocales $A_1, \ldots, A_n$ of M such that

$$A = \overline{A_1} \cap \ldots \cap \overline{A_n}.$$

Since f is an open map, we have the equalities

$$f_{-1}[A] = f_{-1}[\overline{A_1}] \cap \ldots \cap f_{-1}[\overline{A_n}] = \overline{f_{-1}[A_1]} \cap \ldots \cap \overline{f_{-1}[A_n]},$$

whence we deduce that $f_{-1}[A]$ belongs to $\mathcal{K}(L)$. Now, suppose that $\{A_i\colon i \in I\}$ is a collection of members of $\mathcal{K}(M)$ such that $\{A_i^{\#}\colon i \in I\}$ is a covering of M. Then

$$M = \bigvee_{i\in I} A_i^{\#} = \Big(\bigcap_{i\in I} A_i\Big)^{\#}.$$

Since each A_i is a closed sublocale, $\Big(\bigcap\limits_{i\in I} A_i\Big)^{\#}$ is the complement of the closed sublocale $\bigcap\limits_{i\in I} A_i$, and so, since $f_{-1}[\text{-}]$ preserves complements, we have the equalities

$$L = f_{-1}[M] = f_{-1}\Big[\Big(\bigcap_{i\in I} A_i\Big)^{\#}\Big] = \Big(f_{-1}\Big[\bigcap_{i\in I} A_i\Big]\Big)^{\#} = \Big(\bigcap_{i\in I} f_{-1}[A_i]\Big)^{\#} = \bigvee_{i\in I} f_{-1}[A_i]^{\#}.$$

Thus, $\{f_{-1}[A_i]^{\#}\colon i \in I\}$ is a covering of L by complements of members of $\mathcal{K}(L)$, and so there is a countable $K \subseteq I$ such that $L = \bigvee\limits_{k\in K} f_{-1}[A_k]^{\#}$. Consequently, since f is onto, and taking into account Lemma 5.2, we see that

$$M = f[L] = f\Big[\bigvee_{k\in K} f_{-1}[A_k]^{\#}\Big] = \bigvee_{k\in K} f[f_{-1}[A_k]^{\#}] \subseteq \bigvee_{k\in K} A_k^{\#},$$

whence we deduce that the covering $\{A_i^{\#}\colon i \in I\}$ of M admits a countable subcovering, and therefore M is fully cl-isocompact. □

The referee has added the following result, together with the proof. It is with his/her (implied) consent that we include it. The reader should note that the condition on the localic map is weakened in that we do not require openness. This should perhaps not be a surprise because cl-isocompactness is formally weaker than full cl-isocompactness.

Proposition 5.4. *Let $f\colon L \to M$ be an onto localic map which pulls back countably compact sublocales to countably compact sublocales. If L is* cl-*isocompact, then so is M.*

P r o o f. Let S be a complemented countably compact sublocale of M. Then $f_{-1}[S]$ is a complemented countably compact sublocale of L. Since L is cl-isocompact, $\overline{f_{-1}[S]}$ is compact. Then $f[\overline{f_{-1}[S]}]$ is compact too. Now, because S is complemented and surjections are stable under pullback along complemented inclusions, one has $S = f[f_{-1}[S]] \subseteq f[\overline{f_{-1}[S]}]$. Taking closures and using that $f[\overline{f_{-1}[S]}]$ is closed (because it is compact), it follows that $\overline{S} \subseteq f[\overline{f_{-1}[S]}]$. Hence, $\overline{S}$ is compact (as a closed sublocale of the compact $f[\overline{f_{-1}[S]}]$). □

Finally, we consider the (even weaker) isocompact case. In this instance it is condition (c) in Theorem 4.11 that we shall use.

Corollary 5.5. *The localic image of an isocompact frame under a localic map which pulls back closed countably compact sublocales to countably compact sublocales is isocompact.*

P r o o f. Let L be isocompact and $f\colon L \to M$ be an onto localic map such that $f_{-1}[A]$ is countably compact whenever A is a closed countably compact sublocale of M. Denote by $h\colon M \to L$ the left adjoint of f. We show that $h[\mathrm{C}(M)] \subseteq \mathrm{C}(L)$. Let $a \in \mathrm{C}(M)$. Then $\mathfrak{c}_M(a)$ is countably compact, so, by hypothesis, $f_{-1}[\mathfrak{c}_M(a)]$ is countably compact, that is, $\mathfrak{c}_L(h(a))$ is countably compact, and hence $h(a) \in \mathrm{C}(L)$. Now suppose that B is a cover of M consisting of elements of $\mathrm{C}(M)$. Then $h[B]$ is a cover of L consisting of elements of $\mathrm{C}(L)$, hence, there is a countable subset S of B such that $h[S]$ is a cover of L because L is isocompact. Since h is one-one (as f is onto), S is a cover of M, and therefore M is isocompact. □

R e m a r k 5.6. Localic maps which pull back closed countably compact sublocales to countably compact sublocales include the *proper* ones. Recall that a localic map is called proper if it is closed and it preserves directed joins. In fact, as was shown in the course of the proof of [6], Proposition 4.9, proper maps pull back arbitrary complemented countably compact sublocales to countably compact sublocales.

R e m a r k 5.7. (a) In the course of the proof of Corollary 5.5 we saw that if $f\colon L \to M$ is a localic map which pulls back closed countably compact sublocales to countably compact sublocales, then $h[\mathrm{C}(M)] \subseteq \mathrm{C}(L)$ for the left adjoint h of f. In fact, the two conditions are equivalent, as one checks easily.

(b) In a similar vein, one shows that $f_{-1}[A] \in \mathcal{K}(L)$ for every $A \in \mathcal{K}(M)$ if and only if $h[\mathrm{K}(M)] \subseteq \mathrm{K}(L)$.

6. A bit on coproducts

In this section we prove that an arbitrary coproduct of fully cl-isocompact frames is fully cl-isocompact. As mentioned in Introduction, this improves [6], Proposition 4.11, where it is shown that the coproduct of two fully cl-isocompact frames is fully cl-isocompact. We refer to [13], Chapter IV for the construction and properties of coproducts. As in our reference, given a collection $\{L_i\colon\ i \in J\}$ of frames, we write

$$\iota_\alpha\colon\ L_\alpha \to \bigoplus_{i\in J} L_i \quad \text{and} \quad p_\alpha\colon\ \bigoplus_{i\in J} L_i \to L_\alpha$$

for the αth coproduct injection and projection, respectively. Recall that p_α is the right adjoint of ι_α.

We recall from [11], 1.6.3 that if J is an index set and, for each $i \in J$, $h_i\colon\ L_i \to M_i$ is a closed homomorphism, then the induced homomorphism $\bigoplus\limits_J h_i\colon\ \bigoplus\limits_J L_i \to \bigoplus\limits_J M_i$ is closed. Next, let us observe the following about localic images of countably compact sublocales. Recall that if $f\colon\ L \to M$ is a localic map, then $f_{-1}[\text{-}]$ preserves joins of open sublocales.

Lemma 6.1. *If $f\colon\ L \to M$ is a localic map and S is a countably compact sublocale of L, then $f[S]$ is countably compact.*

P r o o f. Let $(U_n)_{n\in\mathbb{N}}$ be an increasing sequence of open sublocales of M such that $f[S] \subseteq \bigvee\limits_n U_n$. Then

$$S \subseteq f_{-1}\Big[\bigvee_n U_n\Big] = \bigvee_n f_{-1}[U_n].$$

Since $(f_{-1}[U_n])_{n\in\mathbb{N}}$ is an increasing sequence of open sublocales of L, there is an index k such that $S \subseteq f_{-1}[U_k]$. Therefore $f[S] \subseteq U_k$, and so $f[S]$ is countably compact. $\square$

The following lemma is possibly folklore, but we furnish a proof because we do not have reference. Note that any dense closed frame homomorphism $g\colon\ A \to B$ is one-one because for any $a \in A$,

$$g_*(g(a) \vee 0) = a \vee g_*(0) = a,$$

so if $g(a_1) = g(a_2)$, then $a_1 = a_2$.

Lemma 6.2. *If $h\colon\ L \to M$ is a closed onto frame homomorphism, then $M \cong \mathfrak{c}_L(h_*(0))$.*

P r o o f. In the dense-onto factorization

$$L \xrightarrow{\kappa_{h_*0}} \mathfrak{c}_L(h_*(0)) \xrightarrow{\widehat{h}} M$$

it is easy to show that $\widehat{h}$ is also a closed map because from the equality $h = \widehat{h} \circ \kappa_{h_*(0)}$ we deduce that $\widehat{h}_*$ maps as h_* because the right adjoint of $\kappa_{h_*(0)}$ is the inclusion map $\mathfrak{c}_L(h_*(0)) \rightarrowtail L$. But any dense closed homomorphism is one-one, so $\widehat{h}$ is one-one. Since h is onto, $\widehat{h}$ is also onto, and hence $\widehat{h}$ is an isomorphism. □

We need yet another lemma. Recall that if A is a sublocale of L, then the right adjoint of the frame surjection $\upsilon_A\colon L \to A$ is the inclusion map $A \rightarrowtail L$. Hence, in the dense-onto factorization, the frame $\mathfrak{c}_L((\nu_A)_*(0_A))$ is exactly $\mathfrak{c}_L(0_A)$.

Lemma 6.3. *Let $\{L_i\colon i \in J\}$ be a family of frames and let A be a sublocale of $\bigoplus_J L_i$. For each i, put $a_i = p_i(0_A)$. For brevity, write*

$$\kappa = \bigoplus_J \kappa_{a_i}\colon \bigoplus_J L_i \to \bigoplus_J \mathfrak{c}_{L_i}(a_i) \quad \textit{and} \quad M = \bigoplus_J \mathfrak{c}_{L_i}(a_i).$$

Then $\kappa_*(0_M) \leqslant 0_A$.

P r o o f. Let $L = \bigoplus_J L_i$ and consider any basic element $\oplus_i x_i \in L$ that is below $\kappa_*(0_M)$. Then

$$0_M = \kappa(\oplus_i x_i) = \oplus_i \kappa_i(x_i) = \oplus_i (a_i \vee x_i).$$

This implies that there is an index $k \in J$ such that

$$a_k \vee x_k = 0_{\mathfrak{c}_{L_k}(a_k)} = a_k,$$

whence $x_k \leqslant a_k$. Since $a_k = p_k(0_A)$, applying the frame homomorphism $\iota_k\colon L_k \to \bigoplus_J L_i$, we obtain

$$\oplus_i x_i = \bigwedge_i \iota_i(x_i) \leqslant \iota_k(x_k) \leqslant \iota_k(a_k) = \iota_k(p_k(0_A)) \leqslant 0_A,$$

which proves the lemma because the basic elements generate the coproduct. □

Now here is the result that we have been building towards.

Theorem 6.4. *The coproduct of fully* cl-*isocompact frames is fully* cl-*isocompact.*

P r o o f. Let $\{L_i\colon i \in J\}$ be a family of fully cl-isocompact frames. Let A be a countably compact sublocale of $\bigoplus_J L_i$. We must show that $\overline{A}$ is compact. For any $i \in J$, the sublocale $p_i[A]$ of L_i is countably compact, by Lemma 6.1. Therefore $\overline{p_i[A]}$ is compact because L_i is fully cl-isocompact. Put $a_i = \bigwedge p_i[A]$, so $\overline{p_i[A]} = \mathfrak{c}_{L_i}(a_i)$. Note that $a_i = p_i(0_A)$. Since coproducts of compact frames are compact, $\bigoplus_J \mathfrak{c}_{L_i}(a_i)$ is compact. Write $L = \bigoplus_J L_i$, $M = \bigoplus_J \mathfrak{c}_{L_i}(a_i)$, and $\kappa\colon L \to M$ for the homomorphism induced by the homomorphisms $\kappa_{a_i}\colon L_i \to \mathfrak{c}_{L_i}(a_i)$. By Lemma 6.2, $M \cong \mathfrak{c}_L(\kappa_*(0_M))$, and so $\mathfrak{c}_L(\kappa_*(0_M))$ is a compact sublocale of L. Since $\kappa_*(0_M) \leqslant 0_A$, by Lemma 6.3, it follows that $\mathfrak{c}_L(0_A)$ is compact. Since $\mathfrak{c}_L(0_A) = \overline{A}$, we have shown that $\bigoplus_J L_i$ is fully cl-isocompact. □

For further use, let us extract some observation that has come to the fore in the process of the foregoing proof. Let $h_i\colon L_i \to M_i$ be a frame homomorphism for each $i \in J$, and let A be a sublocale of $\bigoplus_J L_i$. Put $L = \bigoplus_J L_i$. For each $i \in J$, put $a_i = p_i(0_A)$ and $M = \bigoplus_J \mathfrak{c}_{L_i}(a_i)$. Then, for the homomorphism $\kappa = \oplus_i \kappa_{a_i}$, there is a frame isomorphism $\tau\colon \bigoplus_J \mathfrak{c}_{L_i}(a_i) \to \mathfrak{c}_L(\kappa_*(0_M))$. Since $\overline{A} = \mathfrak{c}_L(0_A)$ and since $\kappa_*(0_M) \leqslant 0_M$, we have the composite

$$L \xrightarrow{\kappa} M \xrightarrow{\tau} \mathfrak{c}_L(\kappa_*(0_M)) \xrightarrow{\alpha} \overline{A} \xrightarrow{\gamma} A,$$

where α maps as κ_{0_M} and γ is the left adjoint of the localic embedding $A \rightarrowtail \overline{A}$. Since each κ_{a_i} is onto, κ is onto, and hence every homomorphism in this composite is onto. From this we deduce that if A is a sublocale (a closed sublocale) of $\bigoplus_J L_i$, then A is isomorphic to a sublocale (a closed sublocale) of $\bigoplus_J \overline{p_i[A]}$. We will write the isomorphic copy of A simply as A.

Regarding binary coproducts (which is what we are going to be concerned with below), suppose that A is a sublocale of $L_1 \oplus L_2$. Then A is a sublocale of $\mathfrak{c}_L(a_1) \oplus L_2$, in light of the composite

$$\mathfrak{c}_{L_1}(a_1) \oplus L_2 \longrightarrow \mathfrak{c}_{L_1}(a_1) \oplus \mathfrak{c}_{L_2}(a_2) \longrightarrow A,$$

where the first arrow is the onto homomorphism $\mathrm{id}_{\mathfrak{c}_{L_1}(a_1)} \oplus \kappa_{a_2}$ and the second is the left adjoint of the localic embedding $A \rightarrowtail \mathfrak{c}_{L_1}(a_1) \oplus \mathfrak{c}_{L_2}(a_2)$.

Taking a cue from [7], we formulate the following definition.

Definition 6.5. A localic map $f\colon L \to M$ is *cc-closed* if $f[A]$ is closed in M for every closed countably compact sublocale A of L.

Closed localic maps are cc-closed. As in spaces, a localic map with an isocompact domain is cc-closed. Let us digress a little and present characterizations of cc-closed maps even though we will not use them. In [8], Gutiérrez García and Picado define several properties of localic maps by requiring that the map sends certain types of complemented sublocales to certain types of complemented sublocales. Exactly their method of proof (such as that of the propositions on page 289) leads to the following characterizations of cc-closed maps. Recall from Section 4 that $\mathrm{C}(L)$ denotes the set of elements a of L such that $\mathfrak{c}_L(a)$ is countably compact, together with the bottom of L.

Proposition 6.6. *The following statements are equivalent for a localic map* $f\colon L \to M$.

(1) f *is cc-closed.*

(2) *For every nonzero* $a \in \mathrm{C}(L)$, $f[\mathfrak{c}_L(a)] = \mathfrak{c}_M(f(a))$.

(3) *For every nonzero* $a \in \mathrm{C}(L)$ *and* $b \in M$, $f(a \vee h(b)) = f(a) \vee b$, *where* h *denotes the left adjoint of* f.

The following proposition is a localic version of a result of Hasegawa, see [9]. In its proof we are going to apply the Kuratowski-Mrówka theorem for locales ([13], Proposition VII.3.5), which states that a frame L is compact if and only if for every frame M, the projection map $p_2\colon L \oplus M \to M$ is a closed map.

Proposition 6.7. *The following statements are equivalent for isocompact frames* L *and* M.

(1) $L \oplus M$ *is isocompact.*

(2) *The projection map* $p_1\colon L \oplus M \to L$ *is cc-closed.*

(3) *The projection map* $p_2\colon L \oplus M \to M$ *is cc-closed.*

Proof. We prove only the equivalence of statements (1) and (2), and remark that the equivalence of (1) and (3) can be established similarly. That (1) implies (2) follows from the fact that any localic map with an isocompact domain is cc-closed.

Conversely, suppose (2) holds. Let A be a closed countably compact sublocale of $L \oplus M$. The sublocale $p_1[A]$ of L is countably compact, so, by the hypothesis on p_1, $p_1[A]$ is a closed sublocale of L. Since L is isocompact, $p_1[A]$ is compact. Let $\pi\colon p_1[A] \oplus M \to M$ be the projection map to M. Since $p_1[A]$ is compact, π is a closed map. If we set $a = p_1(0_A)$, then, in light of $p_1[A]$ being closed in L, $p_1[A] = \overline{p_1[A]} = \mathfrak{c}_L(a)$. Therefore, as we noted above, A is a closed sublocale of $p_1[A] \oplus M$, and so $\pi[A]$ is a closed sublocale of M. Since $\pi[A]$ is also countably compact and M is isocompact, $\pi[A]$ is compact. Arguments as in the discussion preceding Definition 6.5 show that A is a closed sublocale of $p_1[A] \oplus \pi[A]$, and since the latter frame is compact, A is also compact. Therefore $L \oplus M$ is isocompact. □

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