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Maximal independent sets, variants of chain/antichain principle and cofinal subsets without AC

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Abstract. In set theory without the axiom of choice (AC), we observe new relations of the following statements with weak choice principles.

- ∘ $\mathcal{P}_{\mathrm{lf,c}}$ (Every locally finite connected graph has a maximal independent set).
- ∘ $\mathcal{P}_{\mathrm{lc,c}}$ (Every locally countable connected graph has a maximal independent set).
- ∘ $\mathrm{CAC}_1^{\aleph_\alpha}$ (If in a partially ordered set all antichains are finite and all chains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$) if $\aleph_\alpha$ is regular.
- ∘ CWF (Every partially ordered set has a cofinal well-founded subset).
- ∘ $\mathcal{P}_{G,H_2}$ (For any infinite graph $G = (V_G, E_G)$ and any finite graph $H = (V_H, E_H)$ on 2 vertices, if every finite subgraph of G has a homomorphism into H, then so has G).
- ∘ If $G = (V_G, E_G)$ is a connected locally finite chordal graph, then there is an ordering "$<$" of V_G such that $\{w < v \colon \{w, v\} \in E_G\}$ is a clique for each $v \in V_G$.

Keywords: variants of chain/antichain principle; graph homomorphism; maximal independent sets; cofinal well-founded subsets of partially ordered sets; axiom of choice; Fraenkel–Mostowski (FM) permutation models of ZFA + ¬ AC

Classification: 03E25, 03E35, 06A07, 05C69

1. Introduction

As usual, ZF denotes the Zermelo–Fraenkel set theory without the axiom of choice (AC), and ZFA is ZF with the axiom of extensionality weakened to allow the existence of atoms. In this note, we observe new relations of some combinatorial statements with certain weak forms of AC. Complete definitions of the choice forms will be given in Definition 2.4.

1.1 Maximal independent sets. H. M. Friedman in [4, Theorem 6.3.2, Theorem 2.4] proved that AC is equivalent to the statement *"Every graph has a maximal independent set."* (abbreviated here as $\mathcal{P}$) in ZF. C. Spanring in [21] gave a different argument to prove the result. Consider the following weaker formulations of $\mathcal{P}$.

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- Fix $n \in \omega\backslash\{0,1\}$. We denote by P_{K_n}, the class of those graphs whose only components are K_n (complete graph on n vertices). We denote by $\mathcal{P}_n$ the statement *"Every graph from the class P_{K_n} has a maximal independent set"*.
- We denote by $\mathcal{P}_{\mathrm{lf,c}}$ the statement *"Every locally finite connected graph has a maximal independent set"*.
- We denote by $\mathcal{P}_{\mathrm{lc,c}}$ the statement *"Every locally countable connected graph has a maximal independent set"*.

In this note, we observe the following.

(1) AC_n is equivalent to $\mathcal{P}_n$ for every $n \in \omega\backslash\{0,1\}$ in ZF (cf. Section 3, Proposition 3.2).
(2) $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ is equivalent to $\mathcal{P}_{\mathrm{lf,c}}$ in ZF (cf. Section 3, Proposition 3.3).
(3) $\mathrm{UT}(\aleph_0,\aleph_0,\aleph_0)$ implies $\mathcal{P}_{\mathrm{lc,c}}$, and $\mathcal{P}_{\mathrm{lc,c}}$ implies $\mathrm{AC}^{\aleph_0}_{\aleph_0}$ in ZF (cf. Section 3, Proposition 3.4).

1.2 A variant of chain/antichain principle. A famous application of the infinite Ramsey's theorem is the *chain/antichain principle* (abbreviated here as CAC), which states that *"Any infinite partially ordered set contains either an infinite chain or an infinite antichain"*. E. Tachtsis in [22] investigated the possible placement of CAC in the hierarchy of weak choice principles. P. Komjáth and V. Totik in [17] proved the following generalized versions of CAC, applying Zorn's lemma.

- *If in a partially ordered set all antichains are finite and all chains are countable, then the set is countable* (cf. [17, Chapter 11, Problem 8]).
- *If in a partially ordered set all chains are finite and all antichains are countable, then the set is countable* (cf. [17, Chapter 11, Problem 7]).

For each regular $\aleph_\alpha$, we denote by $\mathrm{CAC}^{\aleph_\alpha}_1$ the statement *"if in a partially ordered set all antichains are finite and all chains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$"* and we denote by $\mathrm{CAC}^{\aleph_\alpha}$ the statement *"if in a partially ordered set all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$"*. In [1], we observed that for any regular $\aleph_\alpha$ and any $2 \leq n < \omega$, $\mathrm{CAC}^{\aleph_\alpha}$ does not imply AC^-_n in ZFA. In [1], we also observed that $\mathrm{CAC}^{\aleph_\alpha}$ does not imply *"there are no amorphous sets"* in ZFA. In this note, we observe the following.

(1) Let $n \in \omega\backslash\{0,1\}$. The statement "For every regular $\aleph_\alpha$, $\mathrm{CAC}^{\aleph_\alpha}_1$" implies neither AC^-_n nor "there are no amorphous sets" in ZFA (cf. Section 4, Theorem 4.3).
(2) $\mathrm{CAC}^{\aleph_0}_1$ implies $\mathrm{PAC}^{\aleph_1}_{\mathrm{fin}}$ (Every $\aleph_1$-sized family $\mathcal{A}$ of nonempty finite sets has an $\aleph_1$-sized subfamily $\mathcal{B}$ with a choice function.) in ZF (cf. Section 4, Theorem 4.5).
(3) DC does not imply $\mathrm{CAC}^{\aleph_0}_1$ in ZF (cf. Section 4, Corollary 4.6).

1.3 Cofinal well-founded subsets and consistency results. L. Halbeisen and E. Tachtsis in [9, Theorem 10 (ii)], constructed a model of ZFA and proved that LOC_2^- does not imply LOKW_4^- in ZFA. We construct a similar model of ZFA and observe the following.

- (LOC_2^- + MC) does not imply LOC_n^- in ZFA if $n \in \omega$ such that $n = 3$ or $n > 4$ (cf. Section 5, Theorem 5.3).
- (LOC_2^- + MC) does not imply $\mathrm{CAC}_1^{\aleph_0}$ in ZFA (cf. Section 5, Corollary 5.4).

We also observe that under certain hypotheses on the group $\mathcal{G}$ and the normal filter $\mathcal{F}$, CWF and CS are true in the resulting permutation model $\mathcal{N}$ (cf. Lemma 5.1).

1.4 A generalized formulation of the n-coloring theorem. Fix a natural number $n \in \omega\backslash\{0, 1\}$. P. Komjáth sketched a proof of the following generalization of the *n-coloring theorem* applying BPI: *"For any infinite graph $G = (V_G, E_G)$ and any finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G."* abbreviated here as $\mathcal{P}_{G,H}$. We denote by $\mathcal{P}_{G,H_n}$ the above statement if H has n vertices for $n \in \omega\backslash\{0, 1\}$. Clearly, for every $n \in \omega\backslash\{0, 1\}$, $\mathcal{P}_{G,H_n}$ implies the n-coloring theorem in ZF (consider the finite graph H to be K_n), and the n-coloring theorem implies AC_n in ZF, see [20]. We note that for any integer $n \geq 3$, the n-coloring theorem is equivalent to BPI in ZF, as shown by H. Läuchli in [18]. Consequently, $\mathcal{P}_{G,H_n}$ is equivalent to BPI in ZF for every integer $n \geq 3$. In [1], we observed that if $X \in \{\mathrm{AC}_3, \mathrm{AC}_{\mathrm{fin}}^{\omega}\}$, then $\mathcal{P}_{G,H_2}$ does not imply X in ZFA. In this note, we observe that AC_2 is equivalent to $\mathcal{P}_{G,H_2}$ in ZF (cf. [Section 3, Proposition 3.6]).

1.5 Locally finite connected graphs. Locally finite connected graphs are studied extensively in graph theory (cf. [3]). We list some graph-theoretical statements restricted to locally finite connected graphs, which follow from $\mathrm{AC}_{\mathrm{fin}}^{\omega}$ in ZF (cf. [Section 3, Remark 3.9]). Moreover, we prove the following.

(1) $\mathrm{AC}_{\mathrm{fin}}^{\omega}$ implies $\mathcal{P}_{G,H}$ in ZF, if G is a locally finite connected graph (cf. [Section 3, Proposition 3.7]).
(2) $\mathrm{AC}_{\mathrm{fin}}^{\omega}$ implies the statement *"If $G = (V_G, E_G)$ is a connected locally finite chordal graph, then there is an ordering "$<$" of V_G such that $\{w < v \colon \{w, v\} \in E_G\}$ is a clique for each $v \in V_G$"* in ZF (cf. [Section 3, Proposition 3.8]).

2. Notation, definitions, and known results

Definition 2.1 (Graph-theoretical definitions, and notation)**.** The *degree* of a vertex $v \in V_G$ of a graph $G = (V_G, E_G)$ is the number of edges emerging from v. A graph $G = (V_G, E_G)$ is *locally finite* if every vertex of G has finite degree. We say that a graph $G = (V_G, E_G)$ is *locally countable* if for every $v \in V_G$, the set of neighbors of v is countable. Given a nonnegative integer n, a *path of length n* in the graph $G = (V_G, E_G)$ is a one-to-one finite sequence $\{x_i\}_{0\leq i\leq n}$ of vertices such that for each $i < n$, $\{x_i, x_{i+1}\} \in E_G$; such a path joins x_0 to x_n. The graph G is *connected* if any two vertices are joined by a path of finite length. A homomorphism from a graph $G = (V_G, E_G)$ to a graph $H = \{V_H, E_H\}$ is a map f from V_G to V_H, such that if $\{v_1, v_2\} \in E_G$ then $\{f(v_1), f(v_2)\} \in E_H$. A *good coloring* of a graph $G = (V_G, E_G)$ with a color set C is a mapping $f\colon V_G \to C$ such that for every $\{x, y\} \in E_G$, $f(x) \neq f(y)$. Fix a natural number $n \in \omega$. A graph $G = (V_G, E_G)$ is *n-colorable* if there exists a good coloring of G on n colors. We denote by K_n, the complete graph on n vertices. We denote by C_n the circuit of length n. A graph is *chordal* if it does not contain an induced C_n for $n \geq 4$. An *independent set* is a set of vertices in a graph, no two of which are connected by an edge. A set $W_G \subseteq V_G$ is called a *maximal independent set* in $G = (V_G, E_G)$ if and only if it is independent and there is no independent set W'_G such that $W_G \subsetneq W'_G$ (cf. [21]). A *clique* is a set of vertices in a graph, such that any two of them are joined by an edge.

Definition 2.2 (Chain, antichain, cofinal well-founded subsets)**.** Let $(P, \leq)$ be a *partially ordered set* or a *poset*. A subset $D \subseteq P$ is called a *chain* if $(D, \leq\restriction D)$ is linearly ordered. A subset $A \subseteq P$ is called an *antichain* if no two elements of A are comparable under "$\leq$". A subset $C \subseteq P$ is called *cofinal* in P if for every $x \in P$ there is an element $c \in C$ such that $x \leq c$. An element $p \in P$ is *minimal* if for all $q \in P$, $(q \leq p)$ implies $(q = p)$. A subset $W \subseteq P$ is *well-founded* if every nonempty subset V of W has a $\leq$-minimal element.

Definition 2.3 (Amorphous sets)**.** An infinite set X is called *amorphous* if X cannot be written as a disjoint union of two infinite subsets.

Definition 2.4 (A list of choice forms)**.**

(1) The *axiom of choice*, AC (Form 1 in [12]): Every family of nonempty sets has a choice function.
(2) The *axiom of choice for finite sets*, $\mathrm{AC_{fin}}$ (Form 62 in [12]): Every family of nonempty finite sets has a choice function.
(3) $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ (Form 10 in [12]): Every denumerable, i.e. countably infinite, family of nonempty finite sets has a choice function. We recall two equivalent formulations of $\mathrm{AC}^{\omega}_{\mathrm{fin}}$.

- $\mathrm{UT}(\aleph_0, \mathrm{fin}, \aleph_0)$ (Form 10 A in [12]): The union of denumerably many pairwise disjoint finite sets is denumerable.
- $\mathrm{PAC}^{\omega}_{\mathrm{fin}}$ (Form 10 E in [12]): Every denumerable family of finite sets has an infinite subfamily with a choice function.

(4) $\mathrm{AC}^{\aleph_0}_{\aleph_0}$ (Form 32 A in [12]): Every denumerable family of denumerable sets has a choice function. We recall the following equivalent formulation of $\mathrm{AC}^{\aleph_0}_{\aleph_0}$.
- $\mathrm{PAC}^{\aleph_0}_{\aleph_0}$ (Form 32 B in [12]): Every denumerable set of denumerable sets has an infinite subset with a choice function.

(5) AC_2 (Form 88 in [12]): Every family of pairs has a choice function.

(6) AC_n for each $n \in \omega$, $n \geq 2$ (Form 61 in [12]): Every family of n-element sets has a choice function. We denote by AC^-_n the statement *"Every infinite family $\mathcal{A}$ of n-element sets has a partial choice function, i.e., $\mathcal{A}$ has an infinite subfamily $\mathcal{B}$ with a choice function."* (cf. Form 342 (n) in [12]).

(7) LOC^-_n for each $n \in \omega$, $n \geq 2$, see [9]: Every infinite linearly orderable family of n-element sets has a partial choice function. We denote by LOKW^-_n the statement *"Every infinite linearly orderable family $\mathcal{A}$ of n-element sets has a partial Kinna–Wagner selection function, i.e., there exists an infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ and a function f such that* $\mathrm{dom}(f) = \mathcal{B}$ *and for all $B \in \mathcal{B}$, $\emptyset \neq f(B) \subsetneq B$ (f is called a Kinna–Wagner selection function for $\mathcal{B}$)."* (cf. Definition 1 (2) of [9]).

(8) *Van Douwen's choice principle*, vDCP, see [14]: Every family $X = \{(X_i, \leq_i) \colon i \in I\}$ of linearly ordered sets isomorphic with $(\mathbb{Z}, \leq)$ ("$\leq$" is the usual ordering on $\mathbb{Z}$) has a choice function.

(9) The *axiom of multiple choice*, MC (Form 67 in [12]): Every family $\mathcal{A}$ of nonempty sets has a multiple choice function, i.e., there is a function f with domain $\mathcal{A}$ such that for every $A \in \mathcal{A}$, $f(A)$ is a nonempty finite subset of A.

(10) $\mathrm{MC}(n)$ where $n \geq 2$ is an integer, see [14]: For every family $\{X_i \colon i \in I\}$ of nonempty sets, there is a function F with domain I such that for all $i \in I$, we have that $F(i)$ is a finite subset of X_i and $\gcd(n, |F(i)|) = 1$.

(11) LW (Form 90 in [12]): Every linearly-ordered set can be well-ordered.

(12) $\mathrm{AC}^{\mathrm{LO}}$ (Form 202 in [12]): Every linearly ordered family of nonempty sets has a choice function.

(13) $\mathrm{AC}^{\mathrm{WO}}$ (Form 40 in [12]): Every well-ordered family of nonempty sets has a choice function.

(14) DC_κ for an infinite well-ordered cardinal κ (Form 87(κ) in [12]): Let κ be an infinite well-ordered cardinal (i.e., κ is an $\aleph$). Let S be a nonempty

set and let R be a binary relation such that for every $\alpha < \kappa$ and every α-sequence $s = (s_\varepsilon)_{\varepsilon<\alpha}$ of elements of S there exists $y \in S$ such that sRy. Then there is a function $f\colon \kappa \to S$ such that for every $\alpha < \kappa$, $(f \restriction \alpha)Rf(\alpha)$. We note that $\mathrm{DC}_{\aleph_0}$ is a reformulation of DC (the principle of dependent choices (Form 43 in [12])). We denote by $\mathrm{DC}_{<\lambda}$ the assertion DC_η for all $\eta < \lambda$.

(15) UT(WO,WO,WO) (Form 231 in [12]): The union of a well-ordered collection of well-orderable sets is well-orderable.

(16) For all α $\mathrm{UT}(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ (Form 23 in [12]): For every ordinal α, if A and every member of A has cardinality $\aleph_\alpha$, then $\left|\bigcup A\right| = \aleph_\alpha$.

(17) $\aleph_1$ *is regular* (Form 34 in [12]).

(18) The *Boolean prime ideal theorem*, BPI (Form 14 in [12]): Every Boolean algebra has a prime ideal. We recall the following equivalent formulation of BPI.
   - The *n-coloring theorem for* $n \geq 3$, (Form 14 G (n) ($n \in \omega$, $n \geq 3$) in [12]): For every graph $G = (V_G, E_G)$ if every finite subgraph of G is n-colorable then G is n-colorable. This is De Bruijn–Erdős theorem for $n \geq 3$ colorings.

(19) *Marshall Hall's theorem*, MHT (Form 107 in [12]): If S is a set and $\{S_i\}_{i\in I}$ is an indexed family of *finite* subsets of S, then if the following property (P) holds,

(P) For every finite $F \subseteq I$, there is an injective choice function for $\{S_i\}_{i\in F}$.

then there is an injective choice function for $\{S_i\}_{i\in I}$.

(20) *Dilworth's decomposition theorem* for infinite posets of finite width, DT (cf. [24]): If $\mathbb{P}$ is an arbitrary poset, and k is a natural number such that $\mathbb{P}$ has no antichains of size $k+1$ while at least one k-element subset of $\mathbb{P}$ is an antichain, then $\mathbb{P}$ can be partitioned into k chains.

(21) *Rado's selection lemma*, RSL (Form 99 in [12]): Let $\mathcal{F}$ be a family of finite sets and suppose that to every finite subset F of $\mathcal{F}$ there corresponds a choice function φ_F whose domain is F such that $\varphi_F(T) \in T$ for each $T \in F$. Then there is a choice function f whose domain is $\mathcal{F}$ with the property that for every finite subset F of $\mathcal{F}$, there is a finite subset F' of $\mathcal{F}$ such that $F \subseteq F'$ and $f(T) = \varphi_{F'}(T)$ for all $T \in F$.

(22) The *antichain principle* (Form 89 in [12]): Every partially ordered set has a maximal antichain.

(23) The *chain/antichain principle*, CAC (Form 217 in [12]): Every infinite poset has an infinite chain or an infinite antichain.

(24) *There are no amorphous sets* (Form 64 in [12]).

(25) CS, see [13]: Every poset without a maximal element has two disjoint cofinal subsets.
(26) CWF, see [23]: Every poset has a cofinal well-founded subset.
(27) *A weaker form of Łoś's lemma*, LT (Form 253 in [12]): If $\mathcal{A} = \langle A, \mathcal{R}^{\mathcal{A}} \rangle$ is a nontrivial relational $\mathcal{L}$-structure over some language $\mathcal{L}$, and $\mathcal{U}$ be an ultrafilter on a nonempty set I, then the ultrapower $\mathcal{A}^I/\mathcal{U}$ and $\mathcal{A}$ are elementarily equivalent.

2.1 Group-theoretical facts. A group $\mathcal{G}$ *acts on a set* X if for each $g \in \mathcal{G}$ there is a mapping $x \to gx$ of X into itself, such that $1x = x$ for every $x \in X$ and $h(gx) = (hg)x$ for every $g, h \in \mathcal{G}$. Alternatively, actions of a group $\mathcal{G}$ on a set X are the same as group homomorphisms from $\mathcal{G}$ to $\mathrm{Sym}(X)$. Suppose that a group $\mathcal{G}$ acts on a set X. Let $\mathrm{Orb}_{\mathcal{G}}(x) = \{gx\colon g \in \mathcal{G}\}$ be the orbit of $x \in X$ under the action of $\mathcal{G}$, and $\mathrm{Stab}_{\mathcal{G}}(x) = \{g \in \mathcal{G}\colon gx = x\}$ be the stabilizer of x under the action of $\mathcal{G}$. The *Orbit-Stabilizer theorem* states that the size of the orbit is the index of the stabilizer, that is $|\mathrm{Orb}_{\mathcal{G}}(x)| = [\mathcal{G}\colon \mathrm{Stab}_{\mathcal{G}}(x)]$. We also recall that different orbits of the action are disjoint and form a partition of X, i.e., $X = \bigcup\{\mathrm{Orb}_{\mathcal{G}}(x)\colon x \in X\}$. An *alternating group* is the group of even permutations of a finite set. Let $\{G_i\colon i \in I\}$ be an indexed collection of groups. Define the following set:

$$\prod_{i\in I}^{\text{weak}} G_i = \Big\{ f\colon I \to \bigcup_{i\in I} G_i\colon\ (\forall i \in I)\ f(i) \in G_i,\ f(i) = 1_{G_i} \text{ for all but finitely many } i \Big\}. \tag{1}$$

The *weak direct product* of the groups $\{G_i\colon i \in I\}$ is the set $\prod_{i\in I}^{\text{weak}} G_i$ with the operation of component-wise multiplicative defined for all $f, g \in \prod_{i\in I}^{\text{weak}} G_i$ by $(fg)(i) = f(i)g(i)$ for all $i \in I$.

2.2 Permutation models. We start with a ground model M of ZFA + AC where A is a set of atoms. Each permutation of A extends uniquely to a permutation of M by ε-induction. A permutation model $\mathcal{N}$ of ZFA is determined by a group $\mathcal{G}$ of permutations of A and a normal filter $\mathcal{F}$ of subgroups of $\mathcal{G}$. Let $\mathcal{G}$ be a group of permutations of A and $\mathcal{F}$ be a normal filter of subgroups of $\mathcal{G}$. For $x \in M$, we denote the symmetric group with respect to $\mathcal{G}$ by $\mathrm{sym}_{\mathcal{G}}(x) = \{g \in \mathcal{G}\colon g(x) = x\}$. We say x is *$\mathcal{F}$-symmetric* if $\mathrm{sym}_{\mathcal{G}}(x) \in \mathcal{F}$ and x is *hereditarily $\mathcal{F}$-symmetric* if x and all elements of its transitive closure are $\mathcal{F}$-symmetric. We define the permutation model $\mathcal{N}$ with respect to $\mathcal{G}$ and $\mathcal{F}$, to be the class of all hereditarily $\mathcal{F}$-symmetric sets. We recall that $\mathcal{N}$ is a model of ZFA (cf. [15, Theorem 4.1]). If $\mathcal{I} \subseteq \mathcal{P}(A)$ is a normal ideal, then the filter base $\{\mathrm{fix}_{\mathcal{G}} E\colon E \in \mathcal{I}\}$

generates a normal filter over $\mathcal{G}$, where $\text{fix}_{\mathcal{G}}E$ denotes the subgroup $\{\varphi \in \mathcal{G} \colon \forall\, y \in E(\varphi(y) = y)\}$ of $\mathcal{G}$. Let $\mathcal{I}$ be a normal ideal generating a normal filter $\mathcal{F}_{\mathcal{I}}$ over $\mathcal{G}$. Let $\mathcal{N}$ be the permutation model determined by $M, \mathcal{G}$, and $\mathcal{F}_{\mathcal{I}}$. We say $E \in \mathcal{I}$ supports a set $\sigma \in \mathcal{N}$ if $\text{fix}_{\mathcal{G}}E \subseteq \text{sym}_{\mathcal{G}}(\sigma)$.

Lemma 2.5. *The following hold:*

(1) *In every Fraenkel–Mostowski permutation model, CS implies $vDCP$ (cf.* [13, Theorem 3.15 (3)]*).*
(2) *In ZFA, CWF implies LW (cf.* [23, Lemma 5]*).*
(3) *In ZFA, MC implies CS (cf.* [13, Theorem 3.12]*).*

In this paper,

- Fix a natural number $n \geq 2$. We denote by $\mathcal{N}^1_{HT}(n)$ the permutation model constructed in [9, Theorem 8].
- We denote by $\mathcal{N}_1$ the basic Fraenkel model.
- We denote by $\mathcal{N}^2_{HT}$ the permutation model constructed in [9, Theorem 10 (ii)].
- We denote by $\mathcal{N}_2$ the Second Fraenkel model.
- Fix a prime $p \in \omega$. We denote by $\mathcal{N}_{22}(p)$ the permutation model constructed in [14, Section 4.4].
- Fix a natural number n such that $n = 3$ or $n > 4$ and an infinite well-ordered cardinal number κ. We denote by $\mathcal{M}_{\kappa,n}$ the permutation model constructed in Theorem 5.3.

2.3 Loeb's theorem. A topological space (X, τ) is called *compact* if for every $U \subseteq \tau$ such that $\bigcup U = X$ there is a finite subset $V \subseteq U$ such that $\bigcup V = X$.

Lemma 2.6 ([19, Theorem 1])**.** *Let $\{X_i\}_{i \in I}$ be a family of compact spaces which is indexed by a set I on which there is a well-ordering "$\leq$". If I is an infinite set and there is a choice function F on the collection $\{C \colon C$ is closed, $C \neq \emptyset$, $C \subset X_i$ for some $i \in I\}$, then the product space $\prod_{i \in I} X_i$ is compact in the product topology.*

2.4 A theorem of Fulkerson and Gross. D. R. Fulkerson and O. A. Gross in [5] proved the following lemma.

Lemma 2.7 (cf. [16, Lemma 1], [5])**.** *A finite graph (V, X) is chordal if and only if there is an ordering "$<$" of V such that $\{w < v \colon \{w, v\} \in X\}$ is a clique for each $v \in V$.*

3. Graph theoretical observations

3.1 Maximal independent set.

Proposition 3.1 (ZF). *Every graph based on a well-ordered set of vertices has a maximal independent set.*

PROOF: Let $G = (V_G, E_G)$ be a graph on a well-ordered set of vertices $V_G = \{v_\alpha : \alpha < \lambda\}$. Thus we can use transfinite recursion, without using any form of choice, to construct a maximal independent set. Let $M_0 = \emptyset$. Clearly, M_0 is an independent set. For any ordinal α, if M_α is a maximal independent set, then we are done. Otherwise, there is some $v \in V_G \backslash M_\alpha$, where $M_\alpha \cup \{v\}$ is an independent set of vertices. In that case, let $M_{\alpha+1} = M_\alpha \cup \{v\}$. For limit ordinals α, we use $M_\alpha = \bigcup_{i\in\alpha} M_i$. Clearly, $M = \bigcup_{i\in\lambda} M_i$ is a maximal independent set. □

Proposition 3.2 (ZF). *For every $n \in \omega\backslash\{0,1\}$, $\mathcal{P}_n$ is equivalent to AC_n.*

PROOF: ($\Leftarrow$) Fix $n \in \omega\backslash\{0,1\}$, and let us assume AC_n. Let $G = (V_G, E_G)$ be a graph from the class P_{K_n} (cf. Section 1.1 for definition of P_{K_n}). Let $\{G_i\}_{i\in I} = \{(V_{G_i}, E_{G_i})\}_{i\in I}$ be the components of G. By AC_n select $g_i \in V_{G_i}$ for each $i \in I$. We can see that $J = \{g_i : i \in I\}$ is a maximal independent set of G. For any $g_i, g_j \in J$ such that $g_i \neq g_j$, we have $\{g_i, g_j\} \notin E_G$. Consequently, J is an independent set. For the sake of contradiction, suppose J is not a maximal independent set. Then there is an independent set L which must contain two vertices x and y from V_{G_i} for some $i \in I$. Since $\{x, y\} \in E_G$, we obtain a contradiction.

($\Rightarrow$) Fix $n \in \omega\backslash\{0,1\}$, and let us assume $\mathcal{P}_n$. Consider a system of n-element sets $\mathcal{A} = \{A_i\}_{i\in I}$. We construct a graph $G = (V_G, E_G)$.

Constructing G: Let V_G consist of all the pairs (Y, y) such that $Y \in \mathcal{A}$ and $y \in Y$, and the edge set is defined as follows $\{(Y_1, y_1), (Y_2, y_2)\} \in E_G$ if and only if $Y_1 = Y_2$ and $y_1 \neq y_2$.

Clearly, the components of G are K_n. By $\mathcal{P}_n$, G has a maximal independent set M. Since M is an independent set, for each $Y \in \mathcal{A}$ there is at most one $y \in Y$ such that $(Y, y) \in M$. Since M is a maximal independent set, there is at least one $y \in Y$ such that $(Y, y) \in M$. Consequently, M determines a choice function for $\mathcal{A}$. □

Proposition 3.3 (ZF). *AC^ω_{fin} is equivalent to $\mathcal{P}_{\mathrm{lf,c}}$.*

PROOF: ($\Rightarrow$) We assume $\mathrm{AC}^\omega_{\mathrm{fin}}$. Let $G = (V_G, E_G)$ be some nonempty locally finite, connected graph. Consider some $r \in V_G$. Let $V_0 = \{r\}$. For each integer $n \geq 1$, define $V_n = \{v \in V_G : d_G(r, v) = n\}$ where "$d_G(r, v) = n$" means there are n edges in the shortest path joining r and v. Each V_n is finite by locally finiteness of G, and $V_G = \bigcup_{n\in\omega} V_n$ by connectedness of G. By $\mathrm{UT}(\aleph_0, \mathrm{fin}, \aleph_0)$

(which is equivalent to $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ (cf. Definition 2.4)), V_G is countable. Consequently, V_G is well-ordered. The rest follows from Proposition 3.1.

($\Leftarrow$) We assume $\mathcal{P}_{\mathrm{lf,c}}$. Since $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ is equivalent to its partial version $\mathrm{PAC}^{\omega}_{\mathrm{fin}}$ (cf. Definition 2.4 or [12]), it suffices to show $\mathrm{PAC}^{\omega}_{\mathrm{fin}}$. Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a denumerable set of nonempty finite sets. Without loss of generality, we assume that $\mathcal{A}$ is disjoint. Consider a denumerable sequence $T = \{t_n : n \in \omega\}$ disjoint from $\mathcal{A}$. We construct a graph $G = (V_G, E_G)$.

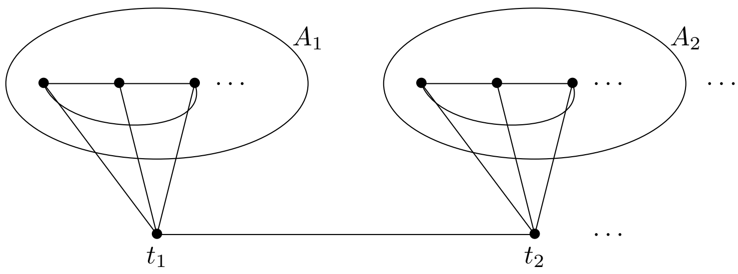

FIGURE 1. The graph G.

Constructing G: Let $V_G = \big(\bigcup_{n\in\omega} A_n\big) \cup T$. For each $n \in \omega$, let $\{t_n, t_{n+1}\} \in E_G$ and $\{t_n, x\} \in E_G$ for every element $x \in A_n$. Also for each $n \in \omega$, and any two $x, y \in A_n$ such that $x \neq y$, let $\{x, y\} \in E_G$, see Figure 1.

Clearly, the graph G is connected and locally finite. By assumption, G has a maximal independent set of vertices, say M. Since M is maximal, M has to be infinite. Moreover, for each $i \in \omega$, either $t_i \in M$ or some $v \in A_i$ is in M. Since M is an independent set, for each $i \in \omega$ there is at most one $v \in A_i$ such that $v \in M$. Define $M' = \{v \in M : v \in A_i \text{ for some } i \in \omega\}$. If M' is finite, then since $\{t_n, t_{n+1}\} \in E_G$ for all $n \in \omega$, it follows that for some $n \in \omega$, $M \cap (A_n \cup \{t_n\}) = \emptyset$. Then for any $u \in A_n$, $M \cup \{u\}$ is an independent set which properly contains M, contradicting M's being a maximal independent set. Thus M' is infinite, which clearly yields a partial choice function for $\mathcal{A}$. $\square$

Proposition 3.4 (ZF). *$UT(\aleph_0, \aleph_0, \aleph_0)$ implies $\mathcal{P}_{\mathrm{lc,c}}$, and $\mathcal{P}_{\mathrm{lc,c}}$ implies $AC^{\aleph_0}_{\aleph_0}$.*

PROOF: In order to prove the first implication, let $G = (V_G, E_G)$ be some non-empty locally countable connected graph. Consider some $r \in V_G$. Let $V_0 = \{r\}$. For each integer $n \geq 1$, define $V_n = \{v \in V_G : d_G(r, v) = n\}$. Since G is locally countable, each V_n is countable by $\mathrm{UT}(\aleph_0, \aleph_0, \aleph_0)$. Also $V_G = \bigcup_{n\in\omega} V_n$ since G is connected. By $\mathrm{UT}(\aleph_0, \aleph_0, \aleph_0)$, V_G is countable. The rest follows from Proposition 3.1. The second assertion follows from the arguments of Proposition 3.3, since $\mathrm{AC}^{\aleph_0}_{\aleph_0}$ is equivalent to $\mathrm{PAC}^{\aleph_0}_{\aleph_0}$ in ZF (cf. Definition 2.4 or [12]). $\square$

Remark 3.5. Fix $n \in \omega\backslash\{0,1\}$. We denote by P_{C_n}, the class of those graphs whose only components are C_n. We denote by $\mathcal{P}'_n$ the statement *"Every graph from the class P_{C_n} has a maximal independent set"*. We remark that AC_{P_n} implies $\mathcal{P}'_n$ in ZF where P_n is the *Perrin number* of n. Perrin numbers are defined by the recurrence relation $P(n) = P(n-2) + P(n-3)$ for $n > 2$, where the initial values are $P(0) = 3$, $P(1) = 0$, and $P(2) = 2$. Let $G = (V_G, E_G)$ be a graph from the class P_{C_n}. Let $\{G_i\}_{i\in I} = \{(V_{G_i}, E_{G_i})\}_{i\in I}$ be the components of P_{C_n}. Let M_i be the collection of different maximal independent sets of G_i for each $i \in I$. Since the number of different maximal independent sets in each component is P_n[1], by AC_{P_n} we can choose a $m_i \in M_i$ for each $i \in I$. Clearly, $\bigcup_{i\in I} m_i$ is a maximal independent set of G.

3.2 The graph homomorphism problem.

Proposition 3.6 (ZF). *$\mathcal{P}_{G,H_2}$ is equivalent to AC_2.*

PROOF: As mentioned in Subsection 1.4, for any $n \in \omega\backslash\{0,1\}$, $\mathcal{P}_{G,H_n}$ implies AC_n. We prove that AC_2 implies $\mathcal{P}_{G,H_2}$ in ZF. Let $H = (V_H, E_H)$ be a graph such that $V_H = \{v_1, v_2\}$, and $G = (V_G, E_G)$ be an infinite graph. We assume that every finite subgraph of G has a homomorphism into H. Let I be the set of components of G.

Case 1. Let $\{v_1, v_1\} \in E_H$ or $\{v_2, v_2\} \in E_H$. If $\{v_1, v_1\} \in E_H$, then for any $G' = (V_{G'}, E_{G'}) \in I$, $f_{G'}\colon G' \to H$ defined by $f_{G'}(x) = v_1$ for every $x \in V_{G'}$, is a homomorphism from G' to H. The function $f\colon G \to H$, defined by $f(x) = f_{G'}(x)$ for $G' = (V_{G'}, E_{G'}) \in I$ and $x \in V_{G'}$, is a homomorphism from G to H. The case $\{v_2, v_2\} \in E_H$ is similar.

Case 2. Let $E_H = \{\{v_1, v_2\}\}$. We follow the proof of the fact that AC_2 implies the 2-coloring problem in ZF (cf. [20]). Fix an arbitrary $G' = (V_{G'}, E_{G'}) \in I$ and select an arbitrary element $a \in V_{G'}$. The function $f_{G'}\colon G' \to H$ defined by $f_{G'}(v) = v_1$ if there is an odd number of vertices between a and v in the shortest path from a to v and $f_{G'}(v) = v_2$ otherwise, is a homomorphism from G' to H. Clearly, the set of homomorphisms $\varphi\colon G' \to H$ contains precisely two elements. By AC_2, there exists a family $\{f_{G'}\}_{G'\in I}$ of homomorphisms $f_{G'}\colon G' \to H$. The function $f\colon G \to H$, defined by $f(x) = f_{G'}(x)$ for $G' = (V_{G'}, E_{G'}) \in I$ and $x \in V_{G'}$ is a homomorphism from G to H.

Case 3. Let $E_H = \emptyset$. Then G must be a discrete graph with no edges (by the assumption that every finite subgraph of G has a homomorphism into H) and any possible mapping of vertices from V_G to either v_1 or v_2 gives a homomorphism.

[1]We use the fact that the number of different maximal independent sets in an n-vertex cycle graph is the nth Perrin number for $1 < n < \omega$ (cf. [6]).

Define a function $f\colon G \to H$ such that $x \mapsto v_1$ for each $x \in V_G$. Clearly, f is a homomorphism from G into H without using any form of choice. $\square$

3.3 Locally finite connected graphs.

Proposition 3.7 (ZF). *AC^{ω}_{fin} implies $\mathcal{P}_{G,H}$, if G is locally finite and connected.*

Proof: Let $G = (V_G, E_G)$ be some nonempty locally finite, connected graph. Consider some $r \in V_G$. Let $V_0 = \{r\}$. For each integer $n \geq 1$, define $V_n = \{v \in V_G\colon d_G(r, v) = n\}$. Each V_n is finite by locally finiteness of G, and $V_G = \bigcup_{n\in\omega} V_n$ by connectedness of G. By $\mathrm{AC}^{\omega}_{\text{fin}}$, V_G is countable. We know that $\mathcal{P}_{G,H}$ holds in ZF, if G is based on a well-ordered set of vertices (cf. [1]). $\square$

Proposition 3.8 (ZF). *AC^{ω}_{fin} implies the statement "If (V, X) is a connected locally finite chordal graph, then there is an ordering "$<$" of V such that $\{w < v\colon \{w, v\} \in X\}$ is a clique for each $v \in V$.".*

Proof: We note that by arguments in the proof of Proposition 3.7, it is enough to see that the statement *"If (V, X) is a chordal graph based on a well-orderable set of vertices, then there is an ordering "$<$" of V such that $\{w < v\colon \{w, v\} \in X\}$ is a clique for each $v \in V$."* is provable in ZF. By Lemma 2.7, each finite subgraph $(W, X|W)$ has an ordering such that $\{w < v\colon \{w, v\} \in X \restriction W\}$ is a clique for every $v \in W$. We can encode every total ordering of a set W by a choice of one of "$<, =, >$" for each pair $(x, y) \in W \times W$. Endow $\{<, =, >\}$ with the discrete topology and $T = \{<, =, >\}^{V\times V}$ with the product topology. Since V is well-ordered, $V \times V$ is well-ordered in ZF. Consequently, $\{<, =, >\} \times \{V \times V\}$ is well-ordered in ZF. By Lemma 2.6, T is compact. We use the compactness of T to prove the existence of the desired ordering. $\square$

Remark 3.9. We list some other graph-theoretical statements from different papers, restricted to locally finite connected graphs, which are related to $\mathrm{AC}^{\omega}_{\text{fin}}$.

(1) P. Komjáth and F. Galvin in [7] proved that any graph based on a well-ordered set of vertices has a chromatic number and an irreducible good coloring in ZF. Consequently, the statements *"any locally finite connected graph has a chromatic number"* and *"any locally finite connected graph has an irreducible good coloring"* are provable under $\mathrm{AC}^{\omega}_{\text{fin}}$ in ZF.

(2) A. Hajnal in [8, Theorem 2] proved that if the chromatic number of a graph G_1 is finite (say $k < \omega$), and the chromatic number of another graph G_2 is infinite, then the chromatic number of $G_1 \times G_2$ is k. In [1] we observed that if G_1 is based on a well-ordered set of vertices, then the following statement holds in ZF.

$$\text{“}\chi(E_{G_1}) = k < \omega \text{ and } \chi(E_{G_2}) \geq \omega \text{ implies } \chi(E_{G_1\times G_2}) = k.\text{”}$$

Consequently, under $AC^{\omega}_{\mathrm{fin}}$ the above statement holds in ZF if G_1 is a locally finite connected graph.

(3) C. Delhommé and M. Morillon in [2] proved that $AC^{\omega}_{\mathrm{fin}}$ is equivalent to the statement *"Every locally finite connected graph has a spanning tree."* in ZF.

(4) The n-coloring theorem restricted to locally finite connected graphs is provable under $AC^{\omega}_{\mathrm{fin}}$ in ZF by Proposition 3.7.

4. A variant of CAC

E. Tachtsis communicated to us the following lemma.

Lemma 4.1. *The following holds.*

(1) *UT*$(\aleph_0, \aleph_0, \aleph_0)$ *implies the statement "If* $(P, \leq)$ *is a poset such that* P *is well-ordered, and if all antichains in* P *are finite and all chains in* P *are countable, then* P *is countable".*

(2) "$\aleph_1$ *is regular" implies the statement "If* $(P, \leq)$ *is a poset such that* P *is well-ordered, and if all antichains in* P *are finite and all chains in* P *are countable, then* P *is countable".*

Proof: We prove (1). Let $(P, \leq)$ be a poset such that P is well-ordered, all antichains in P are finite, and all chains are countable. Fix a well-ordering "$\preceq$" of P. By way of contradiction, assume that P is uncountable.[2] We construct an infinite antichain to obtain a contradiction. Since P is well-ordered by "$\preceq$", we may construct (via transfinite induction) a maximal $\leq$-chain, V_0 say, without invoking any form of choice. Since V_0 is countable, it follows that $P - V_0$ is uncountable and every element of $P - V_0$ is incomparable to some element of V_0. Thus $P - V_0 = \bigcup\{W_p : p \in V_0\}$, where W_p is the set of all elements of $P - V_0$ which are incomparable to p. Since $P - V_0$ is uncountable and V_0 is countable, it follows by $\mathrm{UT}(\aleph_0, \aleph_0, \aleph_0)$ that W_p is uncountable for some p in V_0. Let p_0 be the least (with respect to "$\preceq$") such element of V_0. Now, construct a maximal $\leq$-chain in (the uncountable set) W_{p_0}, V_1 say, and let (similarly to the above argument) p_1 be the least (with respect to "$\preceq$") element of V_1 such that the set W_{p_1} of all elements of W_{p_0} which are incomparable to p_1 is uncountable. Continuing in this fashion by induction (and noting that the process cannot stop at a finite

[2] Since we consider set theory without choice, we note that a set X is uncountable if $|X| \not\leq \aleph_0$. We also note that without choice, "uncountable" may not generally have a clear meaning; for example, another definition could be that X is uncountable if $\aleph_0 < |X|$ (meaning that there is an injection from ω into X but not vice versa). The above two definitions are clearly equivalent in ZFC, but they are not equivalent in ZF.

stage), we obtain a countably infinite antichain $\{p_n\colon n \in \omega\}$, contradicting the assumption that all antichains are finite. Therefore, P is countable.

Similarly, we can prove (2). $\square$

Modifying Lemma 4.1, we may observe that $\mathrm{UT}(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ implies the statement *"If $(P, \leq)$ is a poset such that P is well-ordered, and if all antichains in P are finite and all chains in P have size $\aleph_\alpha$, then P has size $\aleph_\alpha$."* for any regular $\aleph_\alpha$ in ZF.

Corollary 4.2. *The statement "If $(P, \leq)$ is a poset such that P is well-ordered, and if all antichains in P are finite and all chains in P are countable, then P is countable." holds in any Fraenkel–Mostowski model.*

Proof: It follows from the fact that the statement "$\aleph_1$ is a regular cardinal" holds in every Fraenkel–Mostowski model (cf. [11, Corollary 1]). $\square$

Theorem 4.3 (ZFA)**.** *Let $n \in \omega\backslash\{0, 1\}$. The statement "For every regular $\aleph_\alpha$, $CAC_1^{\aleph_\alpha}$" implies neither AC_n^- nor "there are no amorphous sets".*

Proof: L. Halbeisen and E. Tachtsis in [9, Theorem 8] constructed a permutation model (we denote by $\mathcal{N}^1_{HT}(n)$) where for arbitrary $n \geq 2$, AC_n^- fails but CAC holds. We fix an arbitrary integer $n \geq 2$ and recall the model constructed in the proof of [9, Theorem 8] as follows.

- Defining the ground model M: We start with a ground model M of ZFA + AC where A is a countably infinite set of atoms written as a disjoint union $\bigcup\{A_i\colon i \in \omega\}$ where for each $i \in \omega$, $A_i = \{a_{i_1}, a_{i_2}, \ldots, a_{i_n}\}$ and $|A_i| = n$.
- Defining the group $\mathcal{G}$ and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$:
  - Defining $\mathcal{G}$: $\mathcal{G}$ is defined in [9] in a way so that *if $\eta \in \mathcal{G}$, then η only moves finitely many atoms* and for all $i \in \omega$, $\eta(A_i) = A_k$ for some $k \in \omega$. We recall the details from [9] as follows. For all $i \in \omega$, let τ_i be the n-cycle $a_{i_1} \mapsto a_{i_2} \mapsto \cdots \mapsto a_{i_n} \mapsto a_{i_1}$. For every permutation ψ of ω, which moves only finitely many natural numbers, let φ_ψ be the permutation of A defined by $\varphi_\psi(a_{i_j}) = a_{\psi(i)_j}$ for all $i \in \omega$ and $j = 1, 2, \ldots, n$. Let $\eta \in \mathcal{G}$ if and only if $\eta = \varrho\varphi_\psi$ where ψ is a permutation of ω which moves only finitely many natural numbers and ϱ is a permutation of A for which there is a finite $F \subseteq \omega$ such that for every $k \in F$, $\varrho \restriction A_k = \tau_k^j$ for some $j < n$, and ϱ fixes A_m pointwise for every $m \in \omega\backslash F$.
  - Defining $\mathcal{F}$: Let $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ generated by $\{\mathrm{fix}_{\mathcal{G}}(E)\colon E \in [A]^{<\omega}\}$.
- Defining the permutation model: Consider the FM-model $\mathcal{N}^1_{HT}(n)$ determined by M, $\mathcal{G}$ and $\mathcal{F}$.

Following point 1 in the proof of [9, Theorem 8], both A and $\mathcal{A} = \{A_i\}_{i\in\omega}$ are amorphous in $\mathcal{N}^1_{HT}(n)$ and no infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ has a Kinna–Wagner selection function. Consequently, AC^-_n fails. We prove that for any regular $\aleph_\alpha$, $\mathrm{CAC}^{\aleph_\alpha}_1$ holds in $\mathcal{N}^1_{HT}(n)$. Let $(P,\leq)$ be a poset in $\mathcal{N}^1_{HT}(n)$ such that all antichains in P are finite and all chains in P have size $\aleph_\alpha$. Let $E\in[A]^{<\omega}$ be a support of $(P,\leq)$. Following the arguments of [22, Claim 3] we can see that for each $p\in P$, the set $\mathrm{Orb}_E(p)=\{\varphi(p)\colon \varphi\in \mathrm{fix}_{\mathcal{G}}(E)\}$ is an anti-chain in P. Following the arguments of [22, Claim 4] we can see that P can be expressed as a well-orderable union of antichains. In fact, $\mathcal{O}=\{\mathrm{Orb}_E(p)\colon p\in P\}$ is a well-ordered partition of P. We note that all antichains in P are finite, and hence well-orderable. Consequently, P is well-orderable in $\mathcal{N}^1_{HT}(n)$ since UT(WO,WO,WO) holds in $\mathcal{N}^1_{HT}(n)$. We also note that UT(WO,WO,WO) implies $\mathrm{UT}(\aleph_\alpha,\aleph_\alpha,\aleph_\alpha)$ in any FM-model (cf. page 176 of [12]). So, we are done by Lemma 4.1 and the point noted in the paragraph after Lemma 4.1 (cf. the arguments of [22, Claim 5] as well). □

Remark 4.4. We can see that in the basic Fraenkel model (labeled as Model $\mathcal{N}_1$ in [12]) the statement "For every regular $\aleph_\alpha$, $\mathrm{CAC}^{\aleph_\alpha}_1$." holds. We recall that UT(WO,WO,WO) holds in $\mathcal{N}_1$ (cf. [12]). Fix a regular $\aleph_\alpha$. Let $(P,\leq)$ be a poset in $\mathcal{N}_1$, and E be a finite support of $(P,\leq)$. By the arguments of the proof of Theorem 4.3, $\mathcal{O}=\{\mathrm{Orb}_E(p)\colon p\in P\}$ is a well-ordered partition of P. Now for each $p\in P$, $\mathrm{Orb}_E(p)$ is an antichain (cf. the proof of [15, claim 9.3]). Thus, by methods from the proof of Theorem 4.3, $\mathrm{CAC}^{\aleph_\alpha}_1$ holds in $\mathcal{N}_1$. In $\mathcal{N}_1$, the statement "there are no amorphous sets" is false. For reader's information we note that AC^-_n and "there are no amorphous sets" are independent of each other.

Theorem 4.5 (ZF)**.** *$CAC^{\aleph_0}_1$ implies $PAC^{\aleph_1}_{\mathrm{fin}}$.*

Proof: Let $\mathcal{A}=\{A_n\colon n\in\aleph_1\}$ be a family of nonempty finite sets. Without loss of generality, we assume that $\mathcal{A}$ is disjoint. Define a binary relation "$\leq$" on $A=\bigcup\mathcal{A}$ as follows: for all $a,b\in A$, let $a\leq b$ if and only if $a=b$ or $a\in A_n$ and $b\in A_m$ and $n<m$. Clearly, "$\leq$" is a partial order on A. Also, A is uncountable. The only antichains of $(A,\leq)$ are the finite sets A_n and subsets of A_n where $n\in\aleph_1$. By $\mathrm{CAC}^{\aleph_0}_1$, A has an uncountable chain, say C. Let $M=\{m\in\aleph_1\colon C\cap A_m\neq\emptyset\}$. Since C is a chain and $\mathcal{A}$ is the family of all antichains of $(A,\leq)$, we have $M=\{m\in\aleph_1\colon |C\cap A_m|=1\}$. Clearly, $f=\{(m,c_m)\colon m\in M\}$, where for $m\in M$, c_m is the unique element of $C\cap A_m$, is a choice function of the uncountable subset $\mathcal{B}=\{A_m\colon m\in M\}$ of $\mathcal{A}$. Thus $\mathcal{B}$ is an $\aleph_1$-sized subfamily of $\mathcal{A}$ with a choice function. □

Corollary 4.6. *There exists a model of ZF in which DC holds and $PAC^{\aleph_1}_{\mathrm{fin}}$ fails, and thus $CAC^{\aleph_0}_1$ also fails.*

Proof: We refer the reader to T. J. Jech [15, Theorem 8.3] by noting that $\aleph_\alpha$ therein can be replaced by $\aleph_1$. We also note that the fact that $\mathrm{PAC}^{\aleph_1}_{\mathrm{fin}}$ is false in the model follows immediately from [15, Theorem 8.3 (iii)]. The rest follows from Theorem 4.5. $\square$

5. Cofinal well-founded subsets in ZFA

E. Tachtsis in [23, Theorem 10 (ii)] proved that CWF holds in the basic Fraenkel model. P. Howard, D. I. Saveliev and E. Tachtsis in [13, Theorem 3.26], proved that CS holds in the basic Fraenkel model. We modify the arguments from [13, Theorem 3.26] and [23, Theorem 10 (ii)] to observe the following.

Lemma 5.1. *Let A be a set of atoms. Let $\mathcal{G}$ be the group of permutations of A such that either each $\eta \in \mathcal{G}$ moves only finitely many atoms or there is an $n \in \omega \backslash \{0, 1\}$, such that for all $\eta \in \mathcal{G}$, $\eta^n = 1_A$. Let $\mathcal{F}$ be the normal filter of subgroups of $\mathcal{G}$ generated by $\{\mathrm{fix}_{\mathcal{G}}(E) \colon E \in [A]^{<\omega}\}$. Then in the Fraenkel–Mostowski model $\mathcal{N}$ determined by $A, \mathcal{G}$, and $\mathcal{F}$, CS and CWF hold. Consequently, vDCP and LW hold.*

Proof: We follow the steps below.

(1) Let $(P, \leq)$ be a poset in $\mathcal{N}$ and $E \in [A]^{<\omega}$ be a support of $(P, \leq)$. We can write P as a disjoint union of $\mathrm{fix}_{\mathcal{G}}(E)$-orbits, i.e., $P = \bigcup\{\mathrm{Orb}_E(p) \colon p \in P\}$, where $\mathrm{Orb}_E(p) = \{\varphi(p) \colon \varphi \in \mathrm{fix}_{\mathcal{G}}(E)\}$ for all $p \in P$. The family $\{\mathrm{Orb}_E(p) \colon p \in P\}$ is well-orderable in $\mathcal{N}$ since $\mathrm{fix}_{\mathcal{G}}(E) \subseteq \mathrm{Sym}_{\mathcal{G}}(\mathrm{Orb}_E(p))$ for all $p \in P$ (cf. the arguments of [22, Claim 4]).

(2) We prove that $\mathrm{Orb}_E(p)$ is an antichain in P for each $p \in P$. Otherwise there is a $p \in P$, such that $\mathrm{Orb}_E(p)$ is not an antichain in $(P, \leq)$. Thus, for some $\varphi, \psi \in \mathrm{fix}_{\mathcal{G}}(E)$, $\varphi(p)$ and $\psi(p)$ are comparable. Without loss of generality, we may assume $\varphi(p) < \psi(p)$. Let $\pi = \psi^{-1}\varphi$. Consequently, $\pi(p) < p$.

Case 1: Suppose there is an $n \in \omega \backslash \{0, 1\}$, such that for every $\eta \in \mathcal{G}$, $\eta^n = 1_A$. So $\pi^n = 1_A$. Thus, $p = \pi^n(p) < \pi^{n-1}(p) < \cdots < \pi(p) < p$. By transitivity of "$<$", $p < p$, which is a contradiction.

Case 2: Suppose each $\eta \in \mathcal{G}$ moves only finitely many atoms. Then for some $k < \omega$, $\pi^k = 1_A$. Rest follows from the arguments in Case 1.

(3) We can follow [13, Theorem 3.26] to see that CS holds in $\mathcal{N}$.

(4) Although in every Fraenkel–Mostowski model, CS implies vDCP in ZFA (cf. Lemma 2.5), we can recall the arguments from the 1st-paragraph of [13, page 175] to give a direct proof of vDCP in $\mathcal{N}$ without invoking CS.

(5) We can follow [23, Theorem 10 (ii)] to see that CWF holds in $\mathcal{N}$.

(6) Although CWF implies LW in ZFA (cf. Lemma 2.5), we can recall the arguments from the proof of [9, Theorem 10 (ii)] to give a direct proof of LW in $\mathcal{N}$ without invoking CWF. In particular, using a given linear order in $\mathcal{N}$, the fact that an element x of $\mathcal{N}$ is well-orderable in $\mathcal{N}$ if $\mathrm{fix}_{\mathcal{G}}(x) \in \mathcal{F}$ and a similar argument as in Case 1 of step (2), one can verify that LW is true in $\mathcal{N}$ without invoking CWF. $\square$

Remark 5.2. The authors of [13] communicated to the referee that in an unpublished manuscript of theirs, they have shown that MC implies CWF in ZFA. The referee communicated to us the argument with their kind permission. We quote their argument for reader's convenience: "Assume that MC is true. Let $(P, \leq)$ be a (nonempty) poset and also let F be a multiple choice function for $\mathcal{P}(P)\backslash\{\emptyset\}$. Using F, a cofinal well-founded subset of P can be recursively defined as follows: Let $A_0 = P$ and $B_0 = F(A_0)$. Assume that for some ordinal $\alpha > 0$, sets A_β and B_β are defined for all $\beta < \alpha$. Define $A_\alpha = \{p \in P\colon\ \forall\, \beta < \alpha\, \forall\, q \in B_\beta\ (p \not\leq q)\}$ and $B_\alpha = F(A_\alpha)$, if A_α is nonempty. Since On (the class of all ordinal numbers) is a proper class, there is $\gamma \in$ On such that $A_\gamma = \emptyset$. Clearly, $B = \bigcup\{B_\alpha\colon\ \alpha < \gamma\}$ is a cofinal well-founded subset of P."

5.1 A model of ZFA. H. Herrlich, P. Howard, and E. Tachtsis in [10, Theorem 11, Case 1, Case 2] constructed two different classes of permutation models. L. Halbeisen and E. Tachtsis in [9, Theorem 10 (ii)] proved that LOC_2^- does not imply LOKW_4^- in ZFA. For the sake of convenience, we denote by $\mathcal{N}_{HT}^2$, the permutation model of [9, Theorem 10 (ii)]. The model $\mathcal{N}_{HT}^2$ is very similar to the model from [10, Theorem 11, Case 2] except for the fact that in $\mathcal{N}_{HT}^2$ each permutation φ in the group $\mathcal{G}$ of permutations of the sets of atoms can move only finitely many atoms. Fix a natural number n such that $n = 3$ or $n > 4$ and an infinite well-ordered cardinal number κ. We construct a model $\mathcal{M}_{\kappa,n}$ of ZFA similar to the model constructed in [10, Theorem 11, Case 1], where each permutation φ in the group $\mathcal{G}$ of permutations of the sets of atoms can move only finitely many atoms.

Theorem 5.3. *Let n be a natural number such that $n = 3$ or $n > 4$ and κ be an infinite well-ordered cardinal number. Then there is a model $\mathcal{M}_{\kappa,n}$ of ZFA where the following hold.*

(1) *If $X \in \{LOC_2^-, MC\}$, then X holds.*
(2) *LOC_n^- fails.*
(3) *If $X \in \{\mathcal{P}_n, \mathcal{P}_{G,H_n}, DT, LT\}$, then X fails.*

Proof: Fix a natural number n such that $n = 3$ or $n > 4$ and an infinite well-ordered cardinal number κ.

- Defining the ground model M: We start with a ground model M of ZFA + AC where A is a κ-sized set of atoms written as a disjoint union $\bigcup\{A_\alpha\colon \alpha<\kappa\}$, where $A_\alpha=\{a_{\alpha,1},a_{\alpha,2},\dots,a_{\alpha,n}\}$ such that $|A_\alpha|=n$ for all $\alpha<\kappa$.
- Defining the group $\mathcal{G}$ and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$:
  - Defining $\mathcal{G}$: Let $\mathcal{G}$ be the *weak direct product* of $\mathcal{G}_\alpha$'s where $\mathcal{G}_\alpha$ is the alternating group on A_α for each $\alpha<\kappa$. Hence, a permutation η of A is an element of $\mathcal{G}$ if and only if for every $\alpha<\kappa$, $\eta\restriction A_\alpha\in\mathcal{G}_\alpha$, and $\eta\restriction A_\alpha=1_{A_\alpha}$ for all but finitely many ordinals $\alpha<\kappa$. Consequently, *every element $\eta\in\mathcal{G}$ moves only finitely many atoms.*
  - Defining $\mathcal{F}$: Let $\mathcal{F}$ be the normal filter of subgroups of $\mathcal{G}$ generated by $\{\mathrm{fix}_{\mathcal{G}}(E)\colon E\in[A]^{<\omega}\}$.
- Defining the permutation model: Consider the permutation model $\mathcal{M}_{\kappa,n}$ determined by $M,\mathcal{G}$ and $\mathcal{F}$.

(1) If $X\in\{\mathrm{LOC}_2^-,\mathrm{MC}\}$, then X holds in $\mathcal{M}_{\kappa,n}$: We note that MC is true in the model $\mathcal{M}_{\kappa,n}$. The proof is fairly similar to the one that MC is true in the Second Fraenkel model, see [15]. Applying the group-theoretic facts from [10, Theorem 11, Case 1] and following the arguments of the proof of [9, Theorem 10 (ii)] we may observe that LOC_2^- holds in $\mathcal{M}_{\kappa,n}$.

(2) LOC_n^- fails in $\mathcal{M}_{\kappa,n}$: We prove that in $\mathcal{M}_{\kappa,n}$, the well-ordered family $\mathcal{A}=\{A_\alpha\colon\alpha<\kappa\}$ of n-element sets does not have a partial choice function. For the sake of contradiction, let $\mathcal{B}$ be an infinite subfamily of $\mathcal{A}$ with a choice function $f\in\mathcal{M}_n$ and support $E\in[A]^{<\omega}$. Since E is finite, there is an $i<\kappa$ such that $A_i\in\mathcal{B}$ and $A_i\cap E=\emptyset$. Without loss of generality, let $f(A_i)=a_{i_1}$. Consider the permutation π which is the identity on A_j for all $j\in\kappa-i$, and let $(\pi\restriction A_i)(a_{i_1})=a_{i_2}\neq a_{i_1}$. Then π fixes E pointwise, hence $\pi(f)=f$. So, $f(A_i)=a_{i_2}$ which contradicts the fact that f is a function. Thus LOC_n^- fails in $\mathcal{M}_{\kappa,n}$.

(3) If $X\in\{\mathcal{P}_n,\mathcal{P}_{G,H_n},\mathrm{DT},\mathrm{LT}\}$, then X fails in $\mathcal{M}_{\kappa,n}$: Since AC_n fails in the model from the arguments of the previous paragraph, $\mathcal{P}_n$ fails in the model by Proposition 3.2. Since AC_n fails, $\mathcal{P}_{G,H_n}$ fails as well (cf. Section 1.4). Since in $\mathcal{M}_{\kappa,n}$, the linearly-ordered family $\mathcal{A}=\{A_\alpha\colon\alpha<\kappa\}$ of n-element sets does not have a choice function, DT fails in $\mathcal{M}_{\kappa,n}$ by [24, Theorem 3.1 (ii)]. Since in every Fraenkel–Mostowski model of ZFA, LT implies $\mathrm{AC}^{\mathrm{WO}}$ (cf. [25, Theorem 4.6 (i)]), LT fails in $\mathcal{M}_{\kappa,n}$ since the well-ordered family $\mathcal{A}=\{A_\alpha\colon\alpha<\kappa\}$ does not have a choice function. □

Corollary 5.4 (ZFA)**.** *(LOC_2^- + MC) does not imply $CAC_1^{\aleph_0}$.*

PROOF: Consider the permutation model $\mathcal{M}_{\kappa,n}$ constructed in Theorem 5.3 by letting the infinite well-ordered cardinal number κ to be $\aleph_1$. Rest follows from Theorem 4.5 and the arguments of Theorem 5.3 (2). □

We note that $\mathcal{M}_{\aleph_0,n}$ is actually equal to the model of [10, proof of Theorem 11, Case 1]; for an argument, one follows in much the same way the ideas of [9, Remark 2, page 589]. Following the arguments in the proof of Theorem 5.3 (3), we can also observe that DT and LT fail in the model from [9, Theorem 10 (ii)].

Remark 5.5. We recall two more permutation models where MC holds.

- We recall that MC holds in the Second Fraenkel model (labeled as Model $\mathcal{N}_2$ in [12]) (cf. [12]).
- Fix a prime $p \in \omega$. We recall the model $\mathcal{N}_{22}(p)$ from [14, Section 4.4]. Let A be the disjoint union of countably many sets of cardinality p, i.e., $A = \bigcup_{i\in\omega} A_i$ where for each $i \in \omega$, $A_i = \{a_{i,1}, a_{i,2}, \dots, a_{i,p}\}$. Let $\mathcal{G}$ be the group generated by $\{\varphi_i : i \in \omega\}$ where for each $i \in \omega$, φ_i is the cycle $(a_{i,1}, a_{i,2}, \dots, a_{i,p})$. Let $\mathcal{N}_{22}(p)$ be the model determined by $\mathcal{G}$ and the finite support filter $\mathcal{F}$. P. Howard and E. Tachtsis in [14, Theorem 4.7] proved that MC(q) holds in $\mathcal{N}_{22}(p)$ for every prime $q \neq p$.

Fix a prime $p \in \omega$. Let $X \in \{\mathrm{CS}, \mathrm{CWF}, \mathrm{vDCP}, \mathrm{LW}\}$. Since MC is true in the model of the proof of Theorem 5.3, $\mathcal{N}_2$ and $\mathcal{N}_{22}(p)$, X holds in all the mentioned models by Remark 5.2 and Lemma 2.5. However, we note that Lemma 5.1 can play under certain premises, *independently from* MC. In particular, the referee communicated to us that MC may fail in the following permutation model $\mathcal{N}$—for example, take A countably infinite, $\mathcal{G}$ the group of all finitary permutations of A and $\mathcal{F}$ the finite support filter; then the resulting permutation model is equal to the basic Fraenkel model, in which MC is false. On the other hand, X is true in the basic Fraenkel model (cf. [23], [13]). We can use Lemma 5.1 to see that X holds in the model of the proof of Theorem 5.3, $\mathcal{N}_2$ and $\mathcal{N}_{22}(p)$, without invoking MC, because all the models are determined by a group $\mathcal{G}$ with the following properties, and the finite support filter $\mathcal{F}$.

(1) Every permutation $\varphi \in \mathcal{G}$ moves only finitely many atoms in $\mathcal{M}_{\kappa,n}$.
(2) We note that $\mathcal{N}_2$ was constructed via a group $\mathcal{G}$ such that for all $\varphi \in \mathcal{G}$, $\varphi^2 = 1_A$.
(3) We note that $\mathcal{N}_{22}(p)$ was constructed via a group $\mathcal{G}$ such that for all $\varphi \in \mathcal{G}$, $\varphi^p = 1_A$.

6. Conclusion

6.1 Synopsis of theorems, propositions, and corollaries.

- (ZF) $(\forall n \in \omega\backslash\{0,1\})$ $\mathrm{AC}_n \leftrightarrow \mathcal{P}_n$ (cf. Section 3, Proposition 3.2).
- (ZF) $\mathrm{UT}(\aleph_0,\aleph_0,\aleph_0) \to \mathcal{P}_{\mathrm{lc,c}} \to \mathrm{AC}_{\aleph_0}^{\aleph_0} \to \mathrm{AC}_{\mathrm{fin}}^{\omega} \longleftrightarrow \mathcal{P}_{\mathrm{lf,c}}$ (cf. Section 3, Proposition 3.3, Proposition 3.4).
- (ZF) $\mathrm{AC}_2 \longleftrightarrow \mathcal{P}_{G,H_2}$ (cf. Section 3, Proposition 3.6).
- (ZF) $\mathrm{BPI} \longleftrightarrow \mathcal{P}_{G,H_n}$ if $n \geq 3$ (cf. Section 1.4).
- (ZF) Let G be a locally finite and connected graph.
  - $\mathrm{AC}_{\mathrm{fin}}^{\omega} \to \mathcal{P}_{G,H}$ (cf. Section 3, Proposition 3.7).
  - $\mathrm{AC}_{\mathrm{fin}}^{\omega} \to$ *"If $G = (V_G, E_G)$ is a chordal graph, then there is an ordering "$<$" of V_G such that $\{w < v\colon \{w,v\} \in E_G\}$ is a clique for each $v \in V_G$."* (cf. Section 3, Proposition 3.8).
- (ZFA) Let $n \in \omega\backslash\{0,1\}$. The statement "For every regular $\aleph_\alpha$, $\mathrm{CAC}_1^{\aleph_\alpha}$" implies neither AC_n^- nor "there are no amorphous sets." (cf. Section 4, Theorem 4.3).
- (ZF) $\mathrm{CAC}_1^{\aleph_0} \to \mathrm{PAC}_{\mathrm{fin}}^{\aleph_1}$ (cf. Section 4, Theorem 4.5).
- (ZF) $\mathrm{DC} \not\to \mathrm{CAC}_1^{\aleph_0}$ (cf. Section 4, Corollary 4.6).
- (ZFA) Let $n \in \omega$ such that $n = 3$ or $n > 4$. Then $(\mathrm{LOC}_2^- + \mathrm{MC}) \not\to X$, if $X \in \{\mathrm{LOC}_n^-, \mathrm{DT}, \mathrm{LT}\}$ (cf. Section 5, Theorem 5.3).
- (ZFA) $(\mathrm{LOC}_2^- + \mathrm{MC}) \not\to \mathrm{CAC}_1^{\aleph_0}$ (cf. Section 5, Corollary 5.4).

6.2 Questions and further studies. In this paper, we studied the relationship of certain weak choice principles, like AC_n, $\mathrm{AC}_{\mathrm{fin}}^{\omega}$, $\mathrm{AC}_{\aleph_0}^{\aleph_0}$, and $\mathrm{UT}(\aleph_0,\aleph_0,\aleph_0)$, with some weaker formulations of $\mathcal{P}$ (cf. Section 1.1) in ZF. It would be interesting to see if some other weak choice principle, like BPI, is equivalent to some weaker formulation of $\mathcal{P}$ in ZF.

For a natural number $k < \omega$, we denote by $\mathcal{Q}_k$ the following statement.

$$\text{“}\chi(E_{G_1}) = k < \omega \text{ and } \chi(E_{G_2}) \geq \omega \text{ implies } \chi(E_{G_1 \times G_2}) = k.\text{”}$$

We observed that under $\mathrm{AC}_{\mathrm{fin}}^{\omega}$ the above statement holds in ZF if G_1 is a locally finite connected graph (cf. Remark 3.9 (2)). Moreover, we proved that if $X \in \{\mathrm{AC}_3, \mathrm{AC}_{\mathrm{fin}}^{\omega}\}$, then $\mathcal{Q}_k \not\to X$ in ZFA when $k = 3$ (cf. [1, Section 1]). We recall the following problem posted in [1].

Question 6.1 ([1, Question 5.2])**.** If $k > 3$, does BPI follow from $\mathcal{Q}_k$? Otherwise is there any model of ZF or ZFA, where $\mathcal{Q}_k$ holds for $k > 3$, but BPI fails?

In this direction, we ask the following question.

Question 6.2. Is AC_2 equivalent to $\mathcal{Q}_3$? Otherwise is there any model of ZF or ZFA, where $\mathcal{Q}_3$ fails, but AC_2 holds?

Fix $k \in \omega\backslash\{0,1,2\}$ and $n \in \omega\backslash\{0,1\}$. We recall that if $X \in \{\mathcal{P}_{G,H_n}, \mathrm{DT}, \mathrm{MHT}, \mathcal{Q}_k\}$, then BPI implies X in ZF (cf. [8], [24], [12]). It would be interesting to see the interrelationship between the above-mentioned implications of BPI in ZF. For instance, we recall the following open problem posted in [24].

Question 6.3 (cf. [24, Section 4])**.** Does MHT imply DT?

In this direction, we ask the following question.

Question 6.4. Fix $k \in \omega\backslash\{0,1,2\}$. Does MHT imply $\mathcal{Q}_k$ in ZF?

We list three more open problems related to DT from [24].

Question 6.5 (cf. [24, Section 4])**.** Is there a model of ZFA in which $\mathrm{AC}^{\mathrm{LO}}$ is true, but DT is false?

Question 6.6 (cf. [24, Section 4])**.** Does RSL imply DT in ZF?

Question 6.7 (cf. [24, Section 4])**.** Does DT + $\mathrm{AC}_{\mathrm{fin}}$ imply BPI?

Secondly, we studied the relationship of certain weak choice principles, like MC, LOC_2^-, DC, AC_n^-, and "there are no amorphous sets", with $\mathrm{CAC}_1^{\aleph_0}$ in ZFA. It would be interesting to see the relationship of some other weak choice principles with $\mathrm{CAC}_1^{\aleph_0}$ and $\mathrm{CAC}^{\aleph_0}$ in ZFA. We also observed that under certain hypotheses on the group $\mathcal{G}$ and the normal filter $\mathcal{F}$, CWF is true in the resulting permutation model $\mathcal{N}$ (cf. Lemma 5.1). The results that MC implies CWF in ZFA (cf. Remark 5.2) and CWF implies LW in ZFA (cf. Lemma 2.5) are known. It would be interesting to see the relationship of CWF with some other weak choice principles in ZFA. We also list two open problems related to CS from [13].

Question 6.8 ([13, Problem 5.1])**.** Is CS equivalent to AC in ZF? If not, which weak choice principles does CS imply? In particular, does CS imply vDCP?

Question 6.9 ([13, Problem 5.3])**.** Does the antichain principle imply CS?

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