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Metric trees in the Gromov–Hausdorff space

YOSHITO ISHIKI

Abstract. Using the wedge sum of metric spaces, for all compact metrizable spaces, we construct a topological embedding of the compact metrizable space into the set of all metric trees in the Gromov–Hausdorff space with finite prescribed values. As its application, we show that the set of all metric trees is path-connected and all its nonempty open subsets have infinite topological dimension.

Keywords: metric tree; Gromov–Hausdorff distance

Classification: 53C23, 51F99

1. Introduction

In [6], by constructing continuum many geodesics in the Gromov–Hausdorff space, parametrized by a Hilbert cube, the author proved that sets of all spaces satisfying some of the doubling property, the uniform disconnectedness, the uniform perfectness, and sets of all infinite-dimensional spaces, and the set of all metric spaces homeomorphic to the Cantor set have infinite topological dimension.

In [8], by constructing topological embeddings of compact metrizable spaces into the Gromov–Hausdorff space, the author proved that the set of all compact metrizable spaces possessing prescribed topological dimension, Hausdorff dimension, packing dimension, upper box dimension, and Assouad dimension, and the set of all compact ultrametric spaces are path-connected and have infinite topological dimension. The proof is based on the direct sum of metric spaces.

In [7], by a similar method to [8] (constructing a topological embedding of compact metrizable spaces), the author proved that each of the sets of all connected, path-connected, geodesic, and CAT(0) compact metric spaces is path-connected and all their nonempty open subsets have infinite topological dimension in the Gromov–Hausdorff space. The proof is based on the l^2 -product metric of the direct product of metric spaces.

As a related work to these author's papers [6], [8] and [7], in the present paper, we prove that the set of all metric trees is path-connected and all its nonempty open subsets have infinite topological dimension in the Gromov–Hausdorff spaces. In contrast to [8] and [7], we use the wedge sum of metric spaces in the proof.

Let (X, d) be a metric space. Let $x, y \in X$. A subset S of a metric space is said to be a *geodesic segment connecting x and y* if there exist a closed interval $[a, b]$ of \mathbb{R} and an isometric embedding $f: [a, b] \rightarrow X$ such that $f(a) = x$, $f(b) = y$, and $S = f([a, b])$. A metric space is said to be a *geodesic space* if for all two points, there exists a geodesic segment connecting them. A metric space (X, d) is said to be a *metric tree* or *\mathbb{R} -tree* if it is a geodesic space and if geodesic segments G_1 and G_2 connecting x, y and y, z with $G_1 \cap G_2 = \{y\}$ satisfies that $G_1 \cup G_2$ is a geodesic segment connecting x and z for all distinct $x, y, z \in X$, see [3]. For a metric space (Z, h) , and for subsets A, B of Z , we denote by $\mathcal{HD}(A, B; Z, h)$ the *Hausdorff distance of A and B in (Z, h)* . For metric spaces (X, d) and (Y, e) , the *Gromov–Hausdorff distance $\mathcal{GH}((X, d), (Y, e))$* between (X, d) and (Y, e) is defined as the infimum of all values $\mathcal{HD}(i(X), j(Y); Z, h)$, where (Z, h) is a metric space, and $i: X \rightarrow Z$ and $j: Y \rightarrow Z$ are isometric embeddings. We denote by \mathcal{M} the set of all isometry classes of nonempty compact metric spaces. The space $(\mathcal{M}, \mathcal{GH})$ is called the *Gromov–Hausdorff space*. By abuse of notation, we represent an element of \mathcal{M} as a pair (X, d) of a set X and a metric d rather than its isometry class. We denote by \mathcal{T} the set of all metric trees in \mathcal{M} . Our main result is the following theorem, which is an analogue of [8, Theorem 1.3] and [7, Theorem 1.1] for metric trees.

Theorem 1.1. *Let $n \in \mathbb{Z}_{\geq 1}$. Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be a sequence in \mathcal{T} such that $\mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0$ for all distinct i, j . Let H be a compact metric space and $\{v_i\}_{i=1}^{n+1}$ be $n + 1$ different points in H . Then, there exists a topological embedding $\Phi: H \rightarrow \mathcal{T}$ such that $\Phi(v_i) = (X_i, d_i)$.*

Applying Theorem 1.1 to $H = [0, 1]^{\aleph_0}$, we obtain:

Corollary 1.2. *The set \mathcal{T} is path-connected and all its nonempty open subsets have infinite topological dimension.*

We can also obtain an analogue of Theorem 1.1 for rooted (pointed) proper metric trees, see Subsection 2.5. Since it can be proven by the same method as Theorem 1.1, we omit the proof.

2. Proof of Theorem

2.1 Metric trees. To prove our results, we first discuss the basic properties of metric trees. A metric space (X, d) is said to be *0-hyperbolic* or satisfy the *four*

point condition if

$$d(x, y) + d(z, t) \leq \max\{d(x, z) + d(y, t), d(y, z) + d(x, t)\}$$

for all $x, y, z, t \in X$. The next is proven in [3, Theorem 3.40].

Proposition 2.1. *A metric space is a metric tree if and only if it is connected and 0-hyperbolic.*

All metric trees are uniquely geodesic, i.e., for each pair of points, there exists a unique geodesic segment connecting the two points, see [3, Lemmas 3.5 and 3.20]. Let (X, d) be a metric tree. Based on the fact mentioned above, for $x, y \in X$, we denote by $[x, y]$ the geodesic segment connecting x and y . We also put $[x, y]^\circ = [x, y] \setminus \{x, y\}$.

Since all metric subspaces of a 0-hyperbolic space is 0-hyperbolic, by Proposition 2.1, we obtain:

Lemma 2.2. *A connected subset S of a metric tree is a metric tree itself. In particular, for all $x, y \in S$ we have $[x, y] \subset S$.*

The next is proven in [3, Lemma 3.20].

Lemma 2.3. *Let (X, d) be a metric tree. For all $o, x, y \in X$, there exists a unique $q \in X$ such that $[o, x] \cap [o, y] = [o, q]$.*

Let X be a topological space and $x \in X$. We denote by $\deg(x; X)$ the cardinality of connected components of $X \setminus \{x\}$. We put

$$\mathcal{Y}_3(X) = \{x \in X : \deg(x; X) \geq 3\}, \quad \text{and} \quad \mathcal{I}_2(X) = \{x \in X : \deg(x; X) \leq 2\}.$$

Note that $\mathcal{I}_2(X) = X \setminus \mathcal{Y}_3(X)$, and note that $\mathcal{Y}_3(X)$ and $\mathcal{I}_2(X)$ are invariant under homeomorphisms.

Lemma 2.4. *Let (X, d) be a metric tree. Let C be a connected component of $\mathcal{I}_2(X)$. Let $o, x, y \in C$. Then, we have $[o, x] \cap [o, y] = \{o\}$, or $[o, x] \subset [o, y]$, or $[o, y] \subset [o, x]$.*

PROOF: It suffices to show that the negation of the first conclusion ($[o, x] \cap [o, y] \neq \{o\}$) implies either of the other conclusions. By Lemma 2.3, there exists $q \in X$ such that $[o, x] \cap [o, y] = [o, q]$. By $[o, x] \cap [o, y] \neq \{o\}$, we have $q \neq o$. Suppose that $q \neq x$ and $q \neq y$. Then we obtain $\deg(q; X) \geq 3$. Lemma 2.2 implies that $q \in C$, and hence $q \in \mathcal{I}_2(X)$. This is a contradiction. Thus $q = x$ or $q = y$, which leads to the lemma. \square

Proposition 2.5. *Let (X, d) be a metric tree. If a connected component C of $\mathcal{I}_2(X)$ contains at least two points, then C is isometric to an interval of \mathbb{R} .*

PROOF: Since C is connected, we only need to show the existence of an isometric embedding of C into \mathbb{R} . Take points $o, a, b \in C$ such that $o \in [a, b]^\circ \subset C$. We define a map $f: C \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} d(o, x), & \text{if } b \in [o, x] \text{ or } x \in [o, b]; \\ -d(o, x), & \text{if } a \in [o, x] \text{ or } x \in [o, a]. \end{cases}$$

By Lemma 2.4, and by $o \notin \mathcal{Y}_3(X)$, the map f is well-defined. By the definitions of f and metric trees, the map f is an isometric embedding. This finishes the proof. \square

Corollary 2.6. *Let (X, d) be a metric tree. Then, every connected component C of $\mathcal{I}_2(X)$ is isometric to either a singleton or an (non-degenerate) interval of \mathbb{R} .*

For a metric space (X, d) and a subset A , we denote by $\text{diam}_d(A)$ the diameter of A .

Corollary 2.7. *Let (X, d) be a metric tree. Then, there exist a set I and points $\{a_l\}_{l \in I}$ and $\{b_l\}_{l \in I}$ in X such that $\mathcal{I}_2(X) = \bigcup_{l \in I} [a_l, b_l]$ and the set $[a_l, b_l] \cap [a_{l'}, b_{l'}]$ contains only at most one point for all distinct $l, l' \in I$, and $\text{diam}_d([a_l, b_l]) \leq 1$ for all $l \in I$.*

PROOF: Since every interval of \mathbb{R} can be represented as the union of an at most countable family of closed intervals with diameter less than or equal to 1 such that the intersection of each pair of different members in the family contains only at most one point, we obtain the corollary by Proposition 2.5. \square

Remark 2.1. In Corollary 2.7, a segment $[a_l, b_l]$ can be degenerated, i.e., it can happen that $a_l = b_l$. By recursively applying Proposition 2.13 to the metric tree $[0, 1]$, we obtain a metric tree (X, d) such that the set $\mathcal{Y}_3(X)$ is dense in X . For this metric tree, we observe that $\mathcal{I}_2(X) \neq \emptyset$ and all $[a_l, b_l]$ should be degenerated.

Remark 2.2. In Corollary 2.7, it can happen that I is empty. The metric space R defined in [12, page 120] is a metric tree such that $\mathcal{I}_2(R) = \emptyset$, see also [1].

2.2 Specific metric trees. To show the existence of a topological embedding stated in Theorem 1.1, we construct specific metric trees.

Definition 2.1. We put $\mathbb{I} = [0, 1]$. We construct a family of comb-shaped metric trees parametrized by \mathbb{I} . In what follows, we fix a sequence $\{c_n: \mathbb{R} \rightarrow \mathbb{R}\}_{n \in \mathbb{Z}_{\geq 0}}$ of continuous functions such that for each $n \in \mathbb{Z}_{\geq 0}$, we have $c_n(s) = 0$ for all $s \in [2^{-n}, \infty)$, and $c_n(s) \in (0, 1]$ for all $s \in (-\infty, 2^{-n})$. To simplify our description, we represent an element (x, s) of $\mathbb{I} \times \mathbb{I}$ as x_s . For example, $0_0 = (0, 0)$, and $(1/3)_{1/2} = (1/3, 1/2)$. Let w denote the metric on $\mathbb{I} \times \mathbb{I}$ defined by $w(x_s, y_t) = s + |x - y| + t$ if

$s \neq t$; otherwise $w(x_s, y_t) = |x - y|$. Then, the space $(\mathbb{I} \times \mathbb{I}, w)$ becomes a metric tree. For each $n \in \mathbb{Z}_{\geq 0}$, put $I_n = \{m \cdot 2^{-(n+1)} : m \in \{0, \dots, 2^{n+1}\}\}$. We also put $J_0 = I_0$ and $J_{n+1} = I_{n+1} \setminus I_n$ for $n \in \mathbb{Z}_{\geq 0}$. For each $s \in \mathbb{I}$, we define a subset $B(s)$ of $\mathbb{I} \times \mathbb{I}$ by

$$B(s) = \mathbb{I} \times \{0\} \cup \bigcup_{n \in \mathbb{Z}_{\geq 0}} \bigcup_{a \in J_n} \{a\} \times [0, s \cdot c_n(s)].$$

Let $w[s] = w|_{B(s)^2}$. Then $(B(s), w[s])$ is a compact metric tree for all $s \in \mathbb{I}$. Note that $(B(0), w[0])$ is isometric to \mathbb{I} .

By the definition of $B(s)$, we obtain the next two lemmas.

Lemma 2.8. *Let $s \in [0, 1)$. Then the following hold true.*

- (1) *If $s = 0$, for all $t \in \mathbb{I}$ we have $\mathcal{HD}(B(t), B(0); \mathbb{I} \times \mathbb{I}, w) \leq t$.*
- (2) *If $s \neq 0$, taking $n \in \mathbb{Z}_{\geq 0}$ with $2^{-(n+1)} \leq s < 2^{-n}$, for all $t \in \mathbb{I}$ with $|s - t| < 2^{-(n+2)}$, we have*

$$\mathcal{HD}(B(t), B(s); \mathbb{I} \times \mathbb{I}, w) \leq \max_{0 \leq i \leq n+1} |s \cdot c_i(s) - t \cdot c_i(t)|.$$

Lemma 2.9. *Let $n \in \mathbb{Z}_{\geq 0}$ and let $s \in (0, 2^{-n})$. Let C be a connected component of $\mathcal{I}_2(B(s))$. Then we have $\text{diam}_{w[s]}(C) < 2^{-n}$.*

A topological space is said to be a *Hilbert cube* if it is homeomorphic to the countable power of the closed unit interval $[0, 1]$ of \mathbb{R} .

We now introduce a family of star-shaped metric trees parametrized by a Hilbert cube (Definition 2.2), which was first constructed in [7].

We put $\mathbf{C} = \prod_{i=1}^{\infty} [2^{-2i}, 2^{-2i+1}]$. Note that every $\mathbf{a} = \{a_i\}_{i \in \mathbb{Z}_{\geq 1}} \in \mathbf{C}$ satisfies $a_i < 1$ and $a_{i+1} < a_i$ for all $i \in \mathbb{Z}_{\geq 1}$ and $\lim_{i \rightarrow \infty} a_i = 0$. We define a metric τ on \mathbf{C} by $\tau(x, y) = \sup_{i \in \mathbb{Z}_{\geq 1}} |x_i - y_i|$. Then, τ generates the topology which makes \mathbf{C} a Hilbert cube.

Definition 2.2. Let $\mathbf{a} = \{a_i\}_{i \in \mathbb{Z}_{\geq 1}} \in \mathbf{C}$. We supplementally put $a_0 = 1$. Put $\Upsilon = \{(0, 0)\} \cup (0, 1) \times \mathbb{Z}_{\geq 0}$. To simplify our description, we represent an element (s, i) of Υ as s_i . For example, $0_0 = (0, 0)$, $1_n = (1, n)$, and $(1/2)_3 = (1/2, 3)$. We define a metric $R[\mathbf{a}]$ on Υ by

$$R[\mathbf{a}](s_i, t_j) = \begin{cases} a_i |s - t|, & \text{if } i = j; \\ a_i s + a_j t, & \text{otherwise.} \end{cases}$$

Then the space $(\Upsilon, R[\mathbf{a}])$ is a compact metric tree. Note that even if $\mathbf{a} \neq \mathbf{b}$, the metrics $R[\mathbf{a}]$ and $R[\mathbf{b}]$ generate the same topology on Υ .

The following propositions are [7, Propositions 2.2 and 2.3].

Proposition 2.10. *Let $\mathbf{a} = \{a_i\}_{i \in \mathbb{Z}_{\geq 1}}$ and $\mathbf{b} = \{b_i\}_{i \in \mathbb{Z}_{\geq 1}}$ be in \mathbf{C} . Let $K, L \in (0, \infty)$. If $(\Upsilon, K \cdot R[\mathbf{a}])$ and $(\Upsilon, L \cdot R[\mathbf{b}])$ are isometric to each other, then $\mathbf{a} = \mathbf{b}$.*

Proposition 2.11. *For all $\mathbf{a}, \mathbf{b} \in \mathbf{C}$, we obtain*

$$\sup_{x, y \in \Upsilon} |R[\mathbf{a}](x, y) - R[\mathbf{b}](x, y)| \leq 2\tau(\mathbf{a}, \mathbf{b}).$$

2.3 Amalgamation of metrics. The following proposition shows a way of constructing the wedge sum of metric spaces. The statement (2.12) is deduced from [5, Proposition 3.2]. Proposition 2.12 follows from [11, Proposition 2.6] and the definition of metric trees. We refer the readers to [2, Lemma 5.24, page 67] for more general setting.

Proposition 2.12. *Let $k \in \mathbb{Z}_{\geq 2}$. Let $\{(X_i, d_i)\}_{i=1}^k$ be a sequence of metric spaces. Assume that there exists a point p such that $X_i \cap X_j = \{p\}$ for all distinct $i, j \in \{1, \dots, k\}$. We define a symmetric function $h: (\bigcup_{i=1}^k X_i)^2 \rightarrow [0, \infty)$ by*

$$h(x, y) = \begin{cases} d_i(x, y), & \text{if } x, y \in X_i; \\ d_i(x, p) + d_j(p, y), & \text{if } (x, y) \in X_i \times X_j \text{ and } i \neq j. \end{cases}$$

Then, the following statements hold true.

- (1) *The function h is a metric with $h|_{X_i^2} = d_i$ for all $i \in \{1, \dots, k\}$.*
- (2) *If each (X_i, d_i) is a geodesic metric space (metric tree, respectively), then so is $(\bigcup_{i=1}^k X_i, h)$.*

To prove our theorem, we need an operation of replacing edges of a metric tree by other metric trees.

Definition 2.3. Let (X, d) be a metric tree, and $\{a_l\}_{l \in I}$ and $\{b_l\}_{l \in I}$ be families of points in X satisfying that the segment $[a_l, b_l]$ in X is contained in $\mathcal{I}_2(X)$ for all $l \in I$ and the intersection of the segments $[a_l, b_l]$ and $[a_{l'}, b_{l'}]$ in X is at most one point for all distinct $l, l' \in I$. Let $\{(T_l, e_l, \alpha_l, \beta_l)\}_{l \in I}$ be a family of quadruples of metric trees (T_l, e_l) and two specified points $\alpha_l, \beta_l \in T_l$ such that $e_l(\alpha_l, \beta_l) = d(a_l, b_l)$. Now we remove the sets $[a_l, b_l]^\circ$ from X , and identify a_l, b_l with α_l, β_l , respectively, and consider that $X \cap T_l = \{a_l, b_l\}$ and $T_l \cap T_{l'} = \emptyset$ for all distinct l, l' . Let Y denote the resulting set. For $x \in Y$, let $E(x) = \{a_l, b_l\}$ and $h_x = e_l$ if $x \in [a_l, b_l]$; otherwise, $E(x) = \{x\}$ and $h_x = d$. For each $x, y \in Y$, we define $u_{(x,y)} \in E(x)$ and $v_{(x,y)} \in E(y)$ by the points such that $d(u_{(x,y)}, v_{(x,y)})$ is equal to the distance between the sets $E(x)$ and $E(y)$. Note that the points $u_{(x,y)}$ and $v_{(x,y)}$ uniquely exist and $u_{(x,y)} = v_{(y,x)}$ and $v_{(x,y)} = u_{(y,x)}$. We define a symmetric function D on Y^2 by $D(x, y) = e_l(x, y)$ if $x, y \in [a_l, b_l]$ for some l ; otherwise $D(x, y) = h_x(x, u_{(x,y)}) + d(u_{(x,y)}, v_{(x,y)}) + h_y(v_{(x,y)}, y)$. Then D is

a metric and the space (Y, D) is a metric tree. We call this space the *metric tree induced from (X, d) replaced by $\{(T_l, e_l, \alpha_l, \beta_l)\}_{l \in I}$ with respect to $\{a_l\}_{l \in I}$ and $\{b_l\}_{l \in I}$* . Note that since $[a_l, b_l]$ is isometric to $[\alpha_l, \beta_l]$, the space (Y, D) contains the original metric tree (X, d) as a metric subspace.

Proposition 2.13. *Let (X, d) be a metric tree. Let $\{a_l\}_{l \in I}$ and $\{b_l\}_{l \in I}$ be points stated in Corollary 2.7. Put $M_l = d(a_l, b_l)$. For each $s \in \mathbb{I}$, let $(Y(s), D[s])$ be the metric tree induced from (X, d) replaced by $\{(B(s), M_l \cdot w[s], 0_0, 1_0)\}_{l \in I}$ with respect to $\{a_l\}_{l \in I}$ and $\{b_l\}_{l \in I}$. Then the following statements hold true.*

- (1) *The space $(Y(0), D[0])$ is isometric to (X, d) .*
- (2) *For all $s \in \mathbb{I}$, we have $\lim_{t \rightarrow s} \mathcal{GH}((Y(s), D[s]), (Y(t), D[t])) = 0$.*

PROOF: Since $(B(0), w[0])$ is isometric to \mathbb{I} , the statement (1) holds true. The statement (2) follows from Lemma 2.8 and $M_l \leq 1$ and the inequality

$$\mathcal{GH}((Y(s), D[s]), (Y(t), D[t])) \leq \sup_{l \in I} M_l \cdot \mathcal{HD}(B(t), B(s); \mathbb{I} \times \mathbb{I}, w).$$

This finishes the proof. □

2.4 Topological embeddings. For a metric space (X, d) , $o \in X$, and $r \in [0, \infty]$, we denote by $B(o, r)$ the set of all $x \in X$ with $d(o, x) \leq r$. Note that $B(x, 0) = \{x\}$ and $B(x, \infty) = X$. The following lemma is deduced from [9, Lemma 1.16].

Lemma 2.14. *Let (X, d) be a geodesic space. Let $o \in X$. Then, for all $r, r' \in [0, \infty)$ we have $\mathcal{HD}(B(o, r), B(o, r'); X, d) \leq |r - r'|$.*

For every $n \in \mathbb{Z}_{\geq 1}$, we denote by \widehat{n} the set $\{1, \dots, n\}$. In what follows, we consider that the set \widehat{n} is equipped with the discrete topology.

The following proposition has an essential role in the proof of Theorem 1.1. Using this proposition, Theorem 1.1 can be proven by an elementary argument such as the pigeonhole principle. Similar propositions are shown in [8, Proposition 4.4] and [7, Propositions 3.6 and 4.2], proofs of which are based on the direct sum and direct product of metric spaces, respectively. Unlike these propositions, the following is based on the wedge sum of metric spaces discussed in Proposition 2.12.

Proposition 2.15. *Let $n \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}_{\geq 2}$. Let H be a compact metrizable space, and $\{v_i\}_{i=1}^{n+1}$ be $n + 1$ different points in H . Put $H^\times = H \setminus \{v_i : i = 1, \dots, n + 1\}$. Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be a sequence of compact metric spaces in \mathcal{T} satisfying that $\mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0$ for all distinct i, j . Then there exists a continuous map $F: H \times \widehat{m} \rightarrow \mathcal{T}$ such that*

- (1) *for all $i \in \widehat{n + 1}$ and $k \in \widehat{m}$ we have $F(v_i, k) = (X_i, d_i)$;*
- (2) *for all $(u, k), (u', k') \in H^\times \times \widehat{m}$ with $(u, k) \neq (u', k')$, we have $F(u, k) \neq F(u', k')$.*

PROOF: In what follows, we consider that the set $[0, \infty]$ is equipped with the canonical topology homeomorphic to $[0, 1]$. Since every metrizable space is perfectly normal, and since $[0, \infty]$ is homeomorphic to $[0, 1]$, for each $i \in \widehat{n+1}$ we can take a continuous function $\sigma_i: H \rightarrow [0, \infty]$ such that $\sigma_i^{-1}(0) = \{v_j: j \neq i\}$ and $\sigma_i^{-1}(\infty) = \{v_i\}$. We can also take a continuous function $\varphi: H \rightarrow [0, 1/2]$ with $\varphi^{-1}(0) = \{v_i: i = 1, \dots, n+1\}$. We put $\xi(u) = 32 \cdot \varphi(u)$. Since $H \times \widehat{m}$ is compact and metrizable, there exists a topological embedding $\varrho: H \times \widehat{m} \rightarrow \mathbf{C}$ (this is the Urysohn metrization theorem, see [10]).

For each $i \in \widehat{n+1}$, let $\{a_{i,l}\}_{l \in I}$ and $\{b_{i,l}\}_{l \in I}$ be points in (X_i, d_i) stated in Corollary 2.7. Put $M_{i,l} = d(a_{i,l}, b_{i,l})$. Then, we have $M_{i,l} \leq 1$. For each $s \in \mathbb{I}$, we denote by $(Y_i(s), D_i[s])$ the metric tree induced from (X_i, d_i) replaced by $\{(B(s), M_{i,l} \cdot w[s], 0_0, 1_0)\}_{l \in I}$ with respect to $\{a_{i,l}\}_{l \in I}$ and $\{b_{i,l}\}_{l \in I}$.

For each $i \in \widehat{n+1}$, we take $p_i \in X_i$. For each $(u, k) \in H \times \widehat{m}$, we denote by $Z_i(u, k)$ the set of all $x \in Y_i(\varphi(u))$ with $D_i[\varphi(u)](x, p_i) \leq \sigma_i(u)$. Let $E_i[u, k]$ denote the restricted metric of $D_i[\varphi(u)]$ on $Z_i(u, k)$. Put $(Z_{n+2}(u, k), E_{n+2}[u, k]) = (\Upsilon, \xi(u) \cdot R[\varrho(u, k)])$ and $p_{n+2} = 1_0 \in \Upsilon$.

We identify the $n+2$ many points $\{p_i: i = 1, \dots, n+2\}$ as a single point, say p , and we consider that $Z_i(u, k) \cap Z_{i'}(u, k) = \{p\}$ for all distinct $i, i' \in \widehat{n+1}$. We put $W(u, k) = \bigcup_{i=1}^{n+2} Z_i(u, k)$. Applying Proposition 2.12, we obtain a metric $g[u, k]$ on $W(u, k)$ such that $g[u, k]|_{Z_i(u, k)^2} = E_i[u, k]$. Namely, the space $(W(u, k), g[u, k])$ is the wedge sum of the spaces $\{(Z_i(u, k), E_i[u, k])\}_{i=1}^{n+2}$ with respect to the points $\{p_i: i = 1, \dots, n+2\}$.

By (2) in Proposition 2.12, we see that $(W(u, k), g[u, k])$ is a metric tree for all $(u, k) \in H \times \widehat{m}$. By (1) in Proposition 2.13, note that $(W(v_i, k), g[v_i, k])$ is isometric to (X_i, d_i) for all $i \in \widehat{n+1}$ and $k \in \widehat{m}$. We define $F: H \times \widehat{m} \rightarrow \mathcal{T}$ by

$$F(u, k) = \begin{cases} (X_i, d_i), & \text{if } u = v_i \text{ for some } i \in \widehat{n+1}; \\ (W(u, k), g[u, k]), & \text{otherwise.} \end{cases}$$

By (2) in Proposition 2.13, and Proposition 2.11 and Lemma 2.14, and the continuity of each σ_i , the map F is continuous. By the definition, the condition (1) is satisfied.

To prove the condition (2) in the proposition, we assume that there exists an isometry $f: (W(u, k), g[u, k]) \rightarrow (W(u', k'), g[u', k'])$.

We first show that $f(\Upsilon) = \Upsilon$. Fix arbitrary $(v, l) \in H \times \widehat{m}$. Let $\mathcal{P}(v, l)$ be the set of all connected components of $\mathcal{I}_2(W(v, l))$. Take $a \in \mathbb{Z}_{\geq 0}$ with $2^{-(a+1)} \leq \varphi(v) < 2^{-a}$. Let C be a connected component of $\mathcal{I}_2(\bigcup_{i=1}^{n+1} Z_i(v, l))$. Then by Lemma 2.9, and by $M_{i,l} \leq 1$, we have $\text{diam}_{g[v, l]}(C) < 2^{-a}$. Since $2^{-a} \leq 2\varphi(v) = 2^{-4}\xi(v)$, we obtain $\text{diam}_{g[v, l]}(C) < 2^{-4}\xi(v)$. Since $2^{-4} \leq R[\mathbf{a}](0_0, 1_1)$ for all $\mathbf{a} \in \mathbf{C}$, we have $2^{-4}\xi(v) \leq g[v, l](0_0, 1_1)$. By the definitions of Υ and g , we have

$g[v, l](0_0, 1_{i+1}) < g[v, l](0_0, 1_i)$ for all $i \in \mathbb{Z}_{\geq 0}$. Thus, we conclude that the subset $[0_0, 1_0]^\circ \cup \{1_0\}$ of Υ is the unique set possessing the maximal diameter of elements in $\mathcal{P}(v, l)$, and the subset $[0_0, 1_1]^\circ$ of Υ is the unique set possessing the second maximal diameter of elements in $\mathcal{P}(v, l)$. Putting $(v, l) = (u, k), (u', k')$, since f is an isometry, by the argument discussed above, we obtain $f([0_0, 1_0]) = [0_0, 1_0]$ and $f([0_0, 1_1]) = [0_0, 1_1]$. This implies that $f(0_0) \in \{0_0, 1_0\}$ and $f(0_0) \in \{0_0, 1_1\}$. Thus $f(0_0) = 0_0$, and $f(1_i) = 1_i$ for all $i \in \{0, 1\}$.

To prove $f(\Upsilon) = \Upsilon$, for the sake of contradiction, we suppose that there exists $x \in \Upsilon$ with $f(x) \notin \Upsilon$. Take $q \in \mathbb{Z}_{\geq 0}$ such that $x \in [0_0, 1_q]$. Then, by the construction of $W(u, k)$, the segment $[0_0, f(x)]$ must contain 1_0 . Thus, $g[u, k](0_0, 1_0) \leq g[u, k](0_0, x) \leq g[u, k](0_0, 1_q)$. Since $g[u, k](0_0, 1_i) < g[u, k](0_0, 1_0)$ for all $i \neq 0$, we obtain $1_q = 1_0 = x$. This contradicts $f(1_0) = 1_0$. Therefore $f(\Upsilon) \subset \Upsilon$. By replacing the role of f with f^{-1} , we conclude that $f(\Upsilon) = \Upsilon$.

We now prove the condition (2). By the definition of g , and $f(\Upsilon) = \Upsilon$, the spaces $(\Upsilon, \xi(u) \cdot R[\varrho(u, k)])$ and $(\Upsilon, \xi(u') \cdot R[\varrho(u', k')])$ are isometric to each other. Then, by Proposition 2.10, we have $\varrho(u, k) = \varrho(u', k')$, and hence $u = u'$ and $k = k'$. Therefore we obtain the condition (2). This finishes the proof of Proposition 2.15. \square

PROOF OF THEOREM 1.1: The proof of Theorem 1.1 is essentially the same as [7, Theorem 1.1] and [8, Theorem 1.3]. Put $m = n + 2$. Let $F: H \times \widehat{m} \rightarrow \mathcal{T}$ be a map stated in Proposition 2.15. For the sake of contradiction, we suppose that for all $k \in \widehat{m}$ we have $\{(X_i, d_i): i = 1, \dots, n + 1\} \cap F(H^\times \times \{k\}) \neq \emptyset$. Then, by $m = n + 2$, and by the pigeonhole principle, there exists two distinct $j, j' \in \widehat{m}$ such that $(X_i, d_i) \in F(H^\times \times \{j\}) \cap F(H^\times \times \{j'\})$ for some $i \in \widehat{n + 1}$. This contradicts the condition (2) in Proposition 2.15. Thus, there exists $k \in \widehat{m}$ such that $\{(X_i, d_i): i = 1, \dots, n + 1\} \cap F(H^\times \times \{k\}) = \emptyset$. Therefore, the function $\Phi: H \rightarrow \mathcal{T}$ defined by $\Phi(u) = F(u, k)$ is injective, and hence Φ is a topological embedding since H is compact. This completes the proof of Theorem 1.1. \square

2.5 Additional remark. We denote by \mathcal{PM} the set of all proper metric spaces equipped with the pointed Gromov–Hausdorff distance \mathcal{GH}_* , for the definition see [4] or [7]. Let \mathcal{PT} denote the set of all metric trees in \mathcal{PM} . By the same method as the proof of Theorem 1.1, using [4, Lemma 3.4], we obtain an analogue of Theorem 1.1 for proper metric trees. We omit the proof of the following. A similar theorem is proven in [7, Theorem 1.3], and we refer the readers to the proofs of [7, Theorem 1.3] and Theorem 1.1 in the present paper.

Theorem 2.16. *Let $n \in \mathbb{Z}_{\geq 1}$. Let H be a compact metrizable space, and $\{v_i\}_{i=1}^{n+1}$ be $n + 1$ different points in H . Let $\{(X_i, d_i, a_i)\}_{i=1}^{n+1}$ be a sequence in \mathcal{PT}*

such that $\mathcal{GH}*((X_i, d_i, a_i), (X_j, d_j, a_j)) > 0$ for all distinct i, j . Then, there exists a topological embedding $\Phi: H \rightarrow \mathcal{PT}$ such that $\Phi(v_i) = (X_i, d_i, a_i)$.

Corollary 2.17. *The set \mathcal{PT} is path-connected and its all nonempty open subsets have infinite topological dimension.*

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