

Jindřich Zapletal

Coloring triangles and rectangles

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 64 (2023), No. 1, 83–96

Persistent URL: <http://dml.cz/dmlcz/151803>

## Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 2023

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

# Coloring triangles and rectangles

JINDŘICH ZAPLETAL

*Abstract.* It is consistent that  $ZF + DC$  holds, the hypergraph of rectangles on a given Euclidean space has countable chromatic number, while the hypergraph of equilateral triangles on  $\mathbb{R}^2$  does not.

*Keywords:* real algebraic geometry; algebraic hypergraph; chromatic number; geometric set theory; consistency result

*Classification:* 03E35, 14P99, 05C15

## 1. Introduction

This paper continues the study of algebraic hypergraphs on Euclidean spaces from the point of view of their chromatic numbers in the context of choiceless  $ZF + DC$  (Zermelo–Fraenkel set theory with the axiom of dependence choice) set theory. In the context of  $ZFC$  (Zermelo–Fraenkel set theory with the axiom of choice), such hypergraphs were completely classified by J. H. Schmerl regarding their countable chromatic number in [8]. In the choiceless context, the study becomes much more difficult and informative; in particular, the arity and dimension of the hypergraphs concerned begin to play much larger role. In this paper, we compare chromatic numbers of equilateral triangles with that of rectangles.

**Definition 1.1.** Let  $\Delta$  denote the hypergraph of arity three consisting of equilateral triangles on  $\mathbb{R}^2$ . Let  $n \geq 2$  be a number and  $\Gamma_n$  denote the hypergraph of arity four consisting of rectangles on  $\mathbb{R}^n$ .

In the base theory  $ZFC$ , these hypergraphs are well-understood. By an old result of [1],  $\Delta$  has countable chromatic number. On the other hand, for every number  $n \geq 2$  the chromatic number of  $\Gamma_n$  is countable if and only if the continuum hypothesis holds; this is a conjunction of [2] and [3, Theorem 2]. In the base theory  $ZF + DC$ , we present an independence result:

**Theorem 1.2.** *Let  $n \geq 2$ . It is consistent relative to an inaccessible cardinal that  $ZF + DC$  holds, the chromatic number of  $\Gamma_n$  is countable, yet every non-null subset of  $\mathbb{R}^2$  contains all vertices of an equilateral triangle.*

Note that the conclusion implies that the chromatic number of  $\Delta$  is not countable: in a partition of  $\mathbb{R}^2$  into countably many pieces, not all of them can be Lebesgue null. The consistency result can be achieved simultaneously for all  $n \geq 2$ . The proof seems to use the algebra and geometry of both rectangles and equilateral triangles in a way which does not allow an easy generalization.

The paper follows the set theoretic standard of [5]. The calculus of geometric set theory and balanced pairs in Suslin forcings is developed in [6, Section 5.2]. If  $n > 0$  is a natural number,  $A \subset \mathbb{R}^n$  and  $F \subset \mathbb{R}$  are sets, the set  $A$  is *algebraic over  $F$*  if there is a polynomial  $p(\bar{x})$  with  $n$  many variables and coefficients in  $F$  such that  $A$  is the set of all solutions to the equation  $p(\bar{x}) = 0$ . The set  $A$  is *semialgebraic over  $F$*  if there is a formula  $\phi$  of real closed fields with parameters in  $F$  and  $n$  free variables such that  $A = \{\bar{x} \in \mathbb{R}^n : \mathbb{R} \models \phi(\bar{x})\}$ . If  $X, Y, C$  are sets and  $C \subset X \times Y$  and  $x \in X$  is an element, then  $C_x$  is the vertical section of  $C$  associated with  $x$ ,  $C_x = \{y \in Y : \langle x, y \rangle \in C\}$ . Let  $X$  be a Polish space and  $\mu$  a  $\sigma$ -finite Borel measure on it. If  $M$  is a transitive model of ZFC and  $x \in X$  is a point, then  $x$  is *random over  $M$*  if it belongs to no  $\mu$ -null Borel subset of  $X$  coded in  $M$ . By the Fubini theorem, for points  $x_0, x_1 \in X$  the following are equivalent: (a)  $x_0$  is random over  $M$  and  $x_1$  is random over  $M[x_0]$ , (b)  $x_1$  is random over  $M$  and  $x_0$  is random over  $M[x_1]$ , and (c) the pair  $\langle x_0, x_1 \rangle$  is random over  $M$  for the product measure on  $X \times X$ . In each case, we will say that  $x_0, x_1$  are mutually random over  $M$ . The only measure appearing in this paper is the Lebesgue measure on  $\mathbb{R}^2$  and the word null always pertains to it. DC denotes the axiom of dependent choice.

## 2. A preservation theorem

In accordance with the methodology of geometric set theory [6], the model for Theorem 1.2 is obtained as a generic extension of the Solovay model by Suslin forcing. Here, a forcing  $\langle P, \leq \rangle$  is *Suslin*, see [4], if for some ambient Polish space  $X$ ,  $P \subset X$  is an analytic set, “ $\leq$ ” is a transitive relation on  $P$  which is analytic as a subset of  $X \times X$ , and incompatibility in “ $\leq$ ” is again an analytic relation. It is necessary to isolate a class of Suslin forcings with a suitable preservation property. First, recall the main concepts of [9].

**Definition 2.1** ([9, Section 2]). Let  $X, Y$  be Polish spaces.

- (1) A closed set  $C \subset Y \times X$  is a *Noetherian subbasis* if there is no infinite sequence  $\langle a_n : n \in \omega \rangle$  of finite subsets of  $Y$  such that the sets  $D_n = \bigcap_{y \in a_n} C_y$  are strictly decreasing with respect to inclusion;
- (2) if  $M$  is a transitive model of set theory containing the code for  $C$  and  $A \subset X$  is a set, let  $C(M, A) = \bigcap \{C_y : y \in Y \cap M \text{ and } A \subset C_y\}$ ;

- (3) generic extensions  $V[G_0]$  and  $V[G_1]$  are *mutually Noetherian* if for all Polish spaces  $X, Y$  and Noetherian subbases  $C \subset Y \times X$  coded in the ground model, if  $A_0 \subset X$  is a set in  $V[G_0]$  then  $C(V[G_1], A_0) = C(V, A_0)$ , and if  $A_1 \subset X$  is a set in  $V[G_1]$  then  $C(V[G_0], A_1) = C(V, A_1)$ .

For example, mutually generic extensions are mutually Noetherian [9, Corollary 2.10], and if  $x_0, x_1$  are mutually random reals, then  $V[x_0]$  and  $V[x_1]$  are mutually Noetherian [9, Corollary 3.14]. Another easy and important observation is that the Noetherian assumption on the subbasis  $C$  implies that the intersection defining the set  $C(M, A)$  is always equal to the intersection of a finite subcollection; therefore, the set  $C(M, A)$  is coded in  $M$  no matter whether  $A \in M$  or not.

We will need the following strengthening of balance of Suslin forcings.

**Definition 2.2.** Let  $P$  be a Suslin forcing.

- (1) A pair  $\langle Q, \sigma \rangle$  is *(3, 2)-Noetherian balanced over  $P$*  if  $Q \Vdash \sigma \in P$  and for every collection  $\langle V[G_i] : i \in 3 \rangle$  of pairwise mutually Noetherian extensions of  $V$ , every collection  $\langle H_i : i \in 3 \rangle$  of filters on  $Q$ -generic over  $V$  in the respective models  $V[G_i]$ , every tuple  $\langle p_i : i \in 3 \rangle$  of conditions in  $P$  stronger than  $\sigma/H_i$  in the respective models  $V[G_i]$  has a common lower bound;
- (2)  $P$  is *(3, 2)-Noetherian balanced* if for every condition  $p \in P$  there is a *(3, 2)-Noetherian balanced* pair  $\langle Q, \sigma \rangle$  such that  $Q \Vdash \sigma \leq \check{p}$ .

In item (1), the words “over  $P$ ” will be omitted when the poset  $P$  is understood from context. The common lower bound is found in the model  $V[G_0, G_1, G_2]$  or any larger ambient forcing extension; by the Shoenfield absoluteness its existence is absolute among all forcing extensions containing the conditions  $p_i$  for  $i \in 3$ . The following theorem is stated using the parlance of [6, Convention 1.7.18].

**Theorem 2.3.** *In every forcing extension of the choiceless Solovay model by a cofinally (3, 2)-Noetherian balanced Suslin forcing, every non-null subset of  $\mathbb{R}^2$  contains all vertices of an equilateral triangle.*

PROOF: Let  $\kappa$  be an inaccessible cardinal. Let  $P$  be a Suslin forcing which is (3, 2)-Noetherian balanced cofinally in  $\kappa$ . Let  $W$  be a choiceless Solovay model derived from  $\kappa$ . Work in  $W$ . Suppose that  $\tau$  is a  $P$ -name and  $p \in P$  is a condition which forces  $\tau$  to be a non-null subset of  $\mathbb{R}^2$ . We must find an equilateral triangle  $\{x_0, x_1, x_2\} \subset \mathbb{R}^2$  and a condition in  $P$  stronger than  $p$  which forces  $\check{x}_0, \check{x}_1, \check{x}_2 \in \tau$ .

To do that, note that both  $p, \tau$  are definable from a ground model parameter and an additional parameter  $z \in 2^\omega$ . Let  $V[K]$  be an intermediate extension obtained by a poset of cardinality smaller than  $\kappa$  such that  $z \in V[K]$  and  $P$  is  $(3, 2)$ -Noetherian balanced in  $V_\kappa[K]$ . Work in  $V[K]$ . Let  $\langle Q, \sigma \rangle$  be a  $(3, 2)$ -Noetherian balanced pair for the poset  $P$  such that  $|Q| < \kappa$  and  $Q \Vdash \sigma \leq \check{p}$ . Let  $R$  be the usual random poset of non-null closed subsets of  $\mathbb{R}^2$  ordered by inclusion, adding an element  $\dot{x}_{\text{gen}} \in \mathbb{R}^2$ .

**Claim 2.4.** *There is a poset  $S$  of cardinality smaller than  $\kappa$ , a  $Q \times R \times S$ -name  $\eta$  for a condition in  $P$  stronger than  $\sigma$ , and conditions  $q \in Q$ ,  $r \in R$ ,  $s \in S$  such that*

$$\langle q, r, s \rangle \Vdash_{Q \times R \times S} \text{Coll}(\omega, < \kappa) \Vdash \eta \Vdash_P \dot{x}_{\text{gen}} \in \tau.$$

PROOF: Suppose towards a contradiction that this fails. Since any condition in  $P$  in the  $\text{Coll}(\omega, < \kappa)$  extension is found in an extension by a poset of cardinality  $\kappa$ , this means that the following statement holds in  $V[K]$ :

$$Q \times R \times \text{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \dot{x}_{\text{gen}} \notin \tau.$$

In the model  $W$ , let  $B \subset \mathbb{R}^2$  be the set of all points random over the model  $V[K]$ ; thus, the complement of  $B$  is null. Choose a filter  $H_0 \subset Q$  generic over  $V[K]$  and consider the condition  $\sigma/H_0 \leq p$  in the poset  $P$ . We will show that  $\sigma/H_0 \Vdash_P \tau \cap B = 0$ , in contradiction to the initial assumptions of the condition  $p \in P$ .

To show this, let  $x \in B$  be a point and  $p_0 \leq \sigma/H_0$  be a condition; it will be enough to find a condition  $p_1 \in P$  compatible with  $p_0$  which forces  $\dot{x} \notin \tau$ . Let  $H_1 \subset Q$  be a filter generic over the model  $V[K][H_0, x, p_0]$ . The contradictory assumption shows that  $p_1 = \sigma/H_1 \Vdash_P \dot{x} \notin \tau$ . At the same time,  $V[K][H_0, x, p_0]$  and  $V[K][H_1]$  are mutually generic extensions of the model  $V[K]$ . By the balance assumption on the pair  $\langle Q, \sigma \rangle$ , the conditions  $p_0$  and  $p_1$  are compatible in  $P$ . This concludes the proof.  $\square$

Pick  $S, \eta$  and  $q \in Q$ ,  $r \in R$ ,  $s \in S$  as in the claim and move to the model  $W$ . Let  $x_0 \in r$  be a point random over  $V[K]$ . Since  $x_0$  is a point of density of the set  $r$ , there must be a real number  $\varepsilon > 0$  such that, writing  $D \subset \mathbb{R}^2$  for the closed disc centered at  $x_0$  of radius  $\varepsilon$ , the relative mass of  $r$  in  $D$  is greater than  $1/2$ . Consider the measure-preserving self-homeomorphism  $h$  of  $\mathbb{R}^2$  rotating the whole plane around the point  $x_0$  by angle  $\pi/3$  counterclockwise. The disc  $D$  is invariant under  $h$ ; by the choice of  $D$ ,  $r \cap h^{-1}r \cap D$  is a closed set of positive mass. Let  $x_1 \in r \cap h^{-1}r \cap D$  be a point random over  $V[K][x_0]$ , and let  $x_2 = h(x_1)$ . Clearly, the points  $x_0, x_1, x_2 \in r$  form an equilateral triangle.

**Claim 2.5.** *The points  $x_0, x_1, x_2 \in \mathbb{R}^2$  are pairwise random over  $V[K]$ .*

PROOF: The point  $x_1$  is chosen to be random over  $V[K][x_0]$ , therefore the points  $x_0, x_1$  are mutually random over  $V[K]$ . The point  $x_2$  is the image of  $x_1$  under a measure-preserving self-homeomorphism in  $V[K][x_0]$ . Therefore,  $x_2$  is random over  $V[K][x_0]$ , and  $x_0, x_2$  are mutually random over  $V[K]$ . Finally, the point  $x_2$  is the image of  $x_0$  under the measure-preserving rotation around  $x_1$  by angle  $\pi/3$ . Since  $x_0$  is random over  $V[K][x_1]$ , so is  $x_2$ , and the points  $x_1, x_2$  are mutually random over  $V[K]$  as well.  $\square$

Now, for each  $i \in 3$  let  $H_i \subset Q$  and  $G_i \subset S$  for  $i \in 3$  be filters mutually generic over the model  $V[K][x_j: j \in 3]$ , containing the conditions  $q, s$ , respectively. The models  $V[K][x_i]$  for  $i \in 3$  are pairwise mutually Noetherian extensions of  $V[K]$  by [9, Corollary 3.14]. The models  $V[K][x_i][G_i][H_i]$  for  $i \in 3$  are then pairwise mutually Noetherian extensions of  $V[K]$  as well by [9, Proposition 2.9]. For each  $i \in 3$  let  $p_i = \eta/H_i, x_i, G_i \in P$ . By the balance assumption on the pair  $\langle Q, \sigma \rangle$ , the conditions  $p_i$  for  $i \in 3$  have a common lower bound in the poset  $P$ . By the forcing theorem applied in the respective models  $V[K][x_i][G_i][H_i]$ , this common lower bound forces  $\{x_i: i \in 3\} \subset \tau$  as desired.  $\square$

### 3. The coloring poset

Let  $n \geq 2$  be a number, and write  $\Gamma_n$  for the hypergraph of rectangles in  $\mathbb{R}^n$ . To prove Theorem 1.2, we must produce a  $(3, 2)$ -Noetherian balanced Suslin poset adding a total  $\Gamma_n$ -coloring. The definition of the poset uses, as a technical parameter, a Borel ideal  $I$  on  $\omega$  which contains all singletons and which is not generated by countably many sets. Further properties of the ideal  $I$  seem to be irrelevant; the summable ideal will do.

**Definition 3.1.** Let  $n \geq 2$  be a number. The poset  $P_n$  consists of partial functions  $p: \mathbb{R}^n \rightarrow \omega$  such that there is a countable real closed subfield  $\text{supp}(p) \subset \mathbb{R}$  such that  $\text{dom}(p) = \text{supp}(p)^n$ , and  $p$  is a  $\Gamma_n$ -coloring. The ordering is defined by  $p_1 \leq p_0$  if

- (1)  $p_0 \subset p_1$ ;
- (2) for every hypersphere  $S \subset \mathbb{R}^n$  algebraic over  $\text{supp}(p_0)$  and any two points  $x, y \in \text{dom}(p_1 \setminus p_0)$ , if  $x, y$  are opposite points on  $S$  then  $p_1(x) \neq p_1(y)$ ;
- (3) for any two parallel hyperplanes  $P, Q \subset \mathbb{R}^n$  algebraic over  $\text{supp}(p_0)$  and any two points  $x, y \in \text{dom}(p_1 \setminus p_0)$ , if  $x, y$  are opposite points on the respective hyperplanes  $P, Q$  then  $p_1(x) \neq p_1(y)$ ;
- (4) if  $a \subset \text{supp}(p_1)$  is a finite set, then  $p_1''\delta(p_0, p_1, a) \in I$  where  $\delta(p_0, p_1, a) = \{x \in \text{dom}(p_1 \setminus p_0): x \text{ is algebraic over } \text{supp}(p_0) \cup a\}$ .

**Proposition 3.2.** *The ordering “ $\leq$ ” is a  $\sigma$ -closed transitive relation.*

PROOF: For the transitivity, suppose that  $r \leq q \leq p$  are conditions in the poset  $P_n$ ; we must show that  $r \leq p$ . Checking the items of Definition 3.1, (1) is obvious. For (2), suppose that  $S$  is a hypersphere algebraic over  $p$  and  $x, y$  are opposite points on it in  $\text{dom}(r \setminus p)$ . By the closure properties of  $\text{dom}(q)$ , either both  $x, y$  belong to  $\text{dom}(q)$  or both do not. In the former case (2) is confirmed by an application of (2) of  $q \leq p$ , in the latter case (2) is confirmed by an application of (2) of  $r \leq q$ . (3) is verified in a similar way. For (4), suppose that  $a \subset \text{supp}(r)$  is a finite set. Let  $b \subset \text{supp}(q)$  be an inclusion maximal set of points algebraic over  $\text{supp}(p) \cup a$  which is algebraically independent over  $\text{supp}(p)$ . A transcendence dimension argument over the field  $\text{supp}(p)$  shows that  $|b| \leq |a|$  holds. Note that  $\delta(p, r, a) \subseteq \delta(p, q, b) \cup \delta(q, r, a)$  and  $r''\delta(p, r, a) \subseteq q''\delta(p, q, b) \cup r''(q, r, a)$ . Thus, the set  $r''\delta(p, r, a)$  belongs to  $I$ , since it is covered by two sets which are in  $I$  by an application of (4) of  $q \leq p$  and  $r \leq q$ .

For the  $\sigma$ -closure, let  $\langle p_i : i \in \omega \rangle$  be a descending sequence of conditions in  $P_n$ , and let  $q = \bigcup_i p_i$ ; we will show that  $q$  is a common lower bound of the sequence. Let  $i \in \omega$  be arbitrary and work to show  $q \leq p_i$ . For brevity, we deal only with item (4) of Definition 3.1. Let  $a \subset \text{supp}(q)$  be a finite set. There must be  $j \in \omega$  greater than  $i$  such that  $a \subset \text{supp}(p_j)$ . By the closure properties of  $\text{dom}(p_j)$ , it follows that  $\delta(p_i, q, a) = \delta(p_i, p_j, a)$ . Thus,  $q''\delta(p_i, q, a) = p_j''\delta(p_i, p_j, a)$  and the latter set belongs to  $I$  by an application of (4) of  $p_j \leq p_i$ .  $\square$

Further analysis of the poset  $P_n$  depends on a characterization of compatibility of conditions.

**Proposition 3.3.** *Let  $p_0, p_1 \in P_n$  be conditions. The following are equivalent:*

- (1)  $p_0, p_1$  are compatible;
- (2) for every point  $x_0 \in \mathbb{R}^n$  there is a common lower bound of  $p_0, p_1$  containing  $x_0$  in its domain;
- (3) the conjunction of the following:
  - (a)  $p_0 \cup p_1$  is a function and a  $\Gamma_n$ -coloring;
  - (b) whenever  $S$  is a hypersphere algebraic over  $\text{supp}(p_0)$  and  $x, y \in \text{dom}(p_1 \setminus p_0)$  are opposite points on  $S$ , then  $p_1(x) \neq p_1(y)$ ;
  - (c) whenever  $P, Q$  are parallel hyperplanes algebraic over  $\text{supp}(p_0)$  and  $x, y \in \text{dom}(p_1 \setminus p_0)$  are opposite points on them, then  $p_1(x) \neq p_1(y)$ ;
  - (d) for every finite set  $a \subset \text{supp}(p_1)$ ,  $p_1''\delta(p_0, p_1, a) \in I$ ;
  - (e) items above with subscripts 0, 1 interchanged.

PROOF: (2) implies (1), which in turn implies (3) by Definition 3.1. The hard implication is the remaining one: (3) implies (2). Suppose that all items in (3) hold and  $x_0 \in \mathbb{R}^n$  is a point. To find a common lower bound of  $p_0, p_1$  which

contains  $x_0$  in its domain, let  $F \subset \mathbb{R}$  be a countable real closed field containing  $x_0$  as an element and  $\text{supp}(p_0), \text{supp}(p_1)$  as subsets. The common lower bound  $q$  will be constructed in such a way that  $\text{dom}(q) = F^n$ . Write  $d = F^n \setminus (\text{dom}(p_0) \cup \text{dom}(p_1))$ . For every point  $x \in d$  and every  $i \in 2$ , let  $\alpha(x, i) = \{y \in \text{dom}(p_i) \setminus \text{dom}(p_{1-i}) : y \text{ and } x \text{ are mutually algebraic over } \text{supp}(p_{1-i})\}$ . Here, points  $x, y$  are mutually algebraic over  $\text{supp}(p_{1-i})$  if each coordinate of  $x$  is algebraic over  $\text{supp}(p_{1-i}) \cup \{y\}$  and vice versa.

**Claim 3.4.** *For each  $x \in d$  and  $i \in 2$ , the set  $p_i''\alpha(x, i)$  belongs to the ideal  $I$ .*

PROOF: For definiteness set  $i = 1$ . The set  $\alpha(x, 1)$  is a subset of  $\delta(p_0, p_1, a)$  where  $a$  is the set of coordinates of any point in  $\alpha(x, 1)$ . The claim then follows from assumption (3) (d).  $\square$

Now, use the claim to find a set  $b \subset \omega$  in the ideal  $I$  which cannot be covered by finitely many elements of the form  $p_i''\alpha(x, i)$  for  $x \in d$  and  $i \in 2$  and finitely many singletons. Let  $q: F^n \rightarrow \omega$  be a function extending  $p_0 \cup p_1$  such that  $q \upharpoonright d$  is an injection and for every  $x \in d$ ,  $q(x) \in b \setminus (p_0''\alpha(x, 0) \cup p_1''\alpha(x, 1))$ . Such a function exists by the choice of the set  $b$ . We will show that  $q \in P_n$  and  $q$  is a lower bound of  $p_0, p_1$ .

To see that  $q \in P_n$ , let  $R \subset \text{dom}(q)$  be a rectangle and work to show that  $R$  is not monochromatic. The treatment splits into cases.

*Case 1.* Let  $R \subset \text{dom}(p_0) \cup \text{dom}(p_1)$ . By the closure properties of the sets  $\text{dom}(p_0)$  and  $\text{dom}(p_1)$ , there are two subcases.

*Case 1.1.* Let  $R$  be entirely contained in one of the two conditions. Then  $R$  is not monochromatic as both  $p_0, p_1$  are  $\Gamma_n$ -colorings.

*Case 1.2.* There are exactly two vertices of  $R$  in  $\text{dom}(p_0 \setminus p_1)$  and exactly two vertices of  $R$  in  $\text{dom}(p_1 \setminus p_0)$ . There are again two subcases.

*Case 1.2.1.* If the two vertices in  $\text{dom}(p_0 \setminus p_1)$  are opposite on the rectangle  $R$ , then they determine a hypersphere visible from  $\text{supp}(p_0)$  on which the other two vertices are opposite as well. Then the other two vertices receive distinct  $p_1$ -colors by assumption (3) (b).

*Case 1.2.2.* If the two vertices in  $\text{dom}(p_0 \setminus p_1)$  are next to each other on the rectangle  $R$ , then they determine parallel hyperplanes visible from  $\text{supp}(p_0)$  on which the other two vertices are opposite as well. Then the other two vertices receive distinct  $p_1$ -colors by assumption (3) (c).

*Case 2.* Let  $R$  contain exactly one vertex in the set  $d$ ; call it  $x$ . By the closure properties of the sets  $\text{dom}(p_0)$  and  $\text{dom}(p_1)$ , the remaining three vertices of  $R$  cannot all belong to the same condition. Thus, there must be two vertices contained in (say)  $\text{dom}(p_0)$  and one vertex, call it  $y$ , in  $\text{dom}(p_1 \setminus p_0)$ . Then  $y, x$  are mutually algebraic over  $\text{dom}(p_0)$ . Thus  $y \in \alpha(x, 1)$  and  $q(x) \neq q(y)$  by



the initial assumptions on the function  $q$ . In conclusion, the rectangle  $R$  is not monochromatic.

*Case 3.* Let  $R$  contain more than one vertex in the set  $d$ . Then  $R$  is not monochromatic as  $q \upharpoonright d$  is an injection.

This shows that  $q \in P_n$  holds. We must show that  $q \leq p_1$ ; the proof of  $q \leq p_0$  is symmetric. To verify Definition 3.1 (2), suppose that  $S$  is a hypersphere algebraic over  $\text{dom}(p_0)$  and  $x, y \in \text{dom}(q \setminus p_0)$  are opposite points on  $S$ . If  $x, y \in \text{dom}(p_1)$  then item (3) (b) shows that  $q(x) \neq q(y)$ . If  $x \in d$  and  $y \in \text{dom}(p_1)$  (or vice versa) then  $y \in \alpha(x, 0)$  and  $q(x) \neq q(y)$  by the choice of the color  $q(x)$ . Finally, if  $x, y \in d$  then  $q(x) \neq q(y)$  as  $q \upharpoonright d$  is an injection.

Definition 3.1 (3) is verified in the same way. For item (4) of Definition 3.1, let  $a \subset F$  be a finite set. Let  $a' \subset \text{supp}(p_0)$  be a maximal set of reals in  $\text{supp}(p_0)$  algebraic over  $\text{supp}(p_1) \cup a$  which is algebraically free over  $\text{supp}(p_1)$ . By a transcendence dimension argument over the field  $\text{supp}(p_1)$ ,  $|a'| \leq |a|$  holds, in particular  $a'$  is finite. Now,  $q''\delta(q, p_1, a) \subset q''\delta(p_1, p_0, a') \cup b$ , the first set on the right belongs to  $I$  by assumption (3) (d), so the whole union belongs to  $I$  as required.  $\square$

**Corollary 3.5.**  $P_n$  is a Suslin poset.

PROOF: It is clear from Definition 3.1 that the underlying set and the ordering of the poset  $P_n$  are Borel. Proposition 3.3 shows that the (in)compatibility relation is Borel as well.  $\square$

**Corollary 3.6.**  $P_n$  forces the union of the generic filter to be a total  $\Gamma_n$ -coloring.

PROOF: By a genericity argument, it is enough to show that for every condition  $p \in P_n$  and very point  $x_0 \in \mathbb{R}^n$  there is a stronger condition containing  $x_0$  in its domain. This follows from Proposition 3.3 with  $p = p_0 = p_1$ .  $\square$

It is time for the balance proof. It uses the following general fact.

**Proposition 3.7.** Let  $V[G_0], V[G_1]$  be mutually Noetherian extensions.

- (1) Let  $n_0 \in \omega$  be a number and  $A \subset \mathbb{R}^{n_0}$  be a set algebraic over  $V[G_1]$ . Suppose that  $\bar{x}_0 \in V[G_0] \cap \mathbb{R}^{n_0}$  is a point in  $A$ . Then there is a set  $B \subseteq A$  algebraic over  $V$  such that  $\bar{x}_0 \in B$ ;
- (2) same as (1) except for semialgebraic sets;
- (3) if  $a \subset \mathbb{R} \cap V[G_1]$  is a finite set and  $r \in \mathbb{R} \cap V[G_1]$  is a real algebraic over  $(\mathbb{R} \cap V[G_0]) \cup a$ , then  $r$  is algebraic over  $(\mathbb{R} \cap V) \cup a$ ;
- (4)  $\mathbb{R} \cap V[G_0] \cap V[G_1] = \mathbb{R} \cap V$ .

PROOF: For (1), let  $n_1 \in \omega$  be a number and  $\phi(\bar{v}_0, \bar{v}_1)$  be a polynomial with integer coefficients and  $n_0 + n_1$  many free variables, and let  $\bar{x}_1 \in V[G_1]$  be an  $n_1$ -tuple of real numbers such that  $A = \{\bar{y} \in \mathbb{R}^{n_0} : \phi(\bar{y}, \bar{x}_1) = 0\}$ . Let  $C \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_0}$

be the set of all pairs  $\langle \bar{y}_1, \bar{y}_0 \rangle$  such that  $\phi(\bar{y}_0, \bar{y}_1) = 0$ . This is a Noetherian subbasis by the Hilbert basis theorem. Since  $C(V[G_1], \{\bar{x}_0\}) \subseteq A = C_{\bar{x}_1}$  holds by the definitions and  $C(V[G_1], \{\bar{x}_0\}) = C(V, \{\bar{x}_0\})$  holds by the initial assumption on the generic extensions, (1) is witnessed by  $B = C(V, \{\bar{x}_0\})$ .

For (2), let  $A \subset \mathbb{R}^{n_0}$  be a set semialgebraic over  $V[G_1]$ . By the elimination of quantifiers for real closed fields [7, Section 3.3],  $A$  is defined by a quantifier free formula  $\theta$  with parameters in  $V[G_1]$ . The formula can be rearranged so that its atomic subformulas compare a value of a polynomial with zero. Let  $\{\phi_i : i \in m\}$  be a list of all polynomials mentioned in  $\theta$ . Let  $\bar{x}_0 \in V[G_0] \cap \mathbb{R}^{n_0}$  be a point in  $A$ . Let  $a \subset m$  be the set of all  $i$  such that  $\phi_i(\bar{x}_0) = 0$ . Let  $O \subset \mathbb{R}^{n_0}$  be a rational open box around  $\bar{x}_0$  in which the polynomials  $\phi_i$  for  $i \notin a$  do not change sign. Use (1) to find a set  $C$  algebraic over  $V$  such that  $\bar{x}_0 \in C$  and for all  $\bar{y} \in C$  and all  $i \in a$ ,  $\phi_i(\bar{y}) = 0$ . It is clear that the set  $B = C \cap O \subseteq A$  works as required in (2).

For (3), let  $\phi(\bar{x}_0, \bar{x}_1, v)$  be a nonzero polynomial with parameters  $\bar{x}_0$  in  $V[G_0]$  and  $\bar{x}_1 \in a$  and a free variable  $v$  such that  $\phi(\bar{x}_0, \bar{x}_1, r) = 0$  holds. There is an open neighborhood  $O$  of  $\bar{x}_0$  such that for every  $\bar{x}'_0 \in O$ , the polynomial  $\phi(\bar{x}'_0, \bar{x}_1, v)$  remains nonzero. Let  $A = \{\bar{y}_0 : \phi(\bar{y}_0, \bar{x}_1, r) = 0\}$ , use (1) to find a set  $B \subseteq A$  algebraic over  $V$  such that  $\bar{x}_0 \in B$ , and use a Mostowski absoluteness argument to find a tuple  $\bar{x}'_0 \in O \cap B$  in the ground model. (3) is then witnessed by the tuple  $x'_0$ . Finally, (4) immediately follows from (3).  $\square$

**Theorem 3.8.** *Let  $n \geq 2$  be a number. In the poset  $P_n$ ,*

- (1) *for every total  $\Gamma_n$ -coloring  $c: \mathbb{R}^n \rightarrow \omega$ , the pair  $\langle \text{Coll}(\omega, \mathbb{R}), \check{c} \rangle$  is (3, 2)-Noetherian balanced;*
- (2) *if the continuum hypothesis holds then the poset  $P_n$  is (3, 2)-Noetherian balanced.*

The fine details of this proof are the reason behind the rather mysterious Definition 3.1.

PROOF: For item (1), let  $c: \mathbb{R}^n \rightarrow \omega$  be a total  $\Gamma_n$ -coloring. Let  $V[G_i]$  for  $i \in 3$  be pairwise mutually Noetherian extensions of  $V$ . Suppose that  $p_i \leq c$  is a condition in  $P_n$  in the model  $V[G_i]$  for each  $i \in 3$ ; we must find a common lower bound of all  $p_i$  for  $i \in 3$ .

Work in the model  $V[G_i : i \in 3]$ . Let  $F \subset \mathbb{R}$  be a countable real closed field containing  $\text{supp}(p_i)$  for  $i \in 3$ . We will construct a lower bound  $q$  such that  $F = \text{supp}(q)$ . Write  $d = F^n \setminus \bigcup_i \text{dom}(p_i)$ . For each point  $x \in d$  and for each pair  $i, j \in 3$  of distinct indices, define sets  $\alpha(x, i, j), \beta(x, i, j)$  and  $\gamma(x, i, j) \subset \text{dom}(p_i)$  as follows:

- $\alpha(x, i, j) = \{y \in \text{dom}(p_i \setminus c) : \text{there is a hypersphere } S \subset \mathbb{R}^n \text{ algebraic over } \text{supp}(p_j) \text{ such that } x, y \text{ are opposite points on } S\}$ ;
- $\beta(x, i, j) = \{y \in \text{dom}(p_i \setminus c) : \text{there are parallel hyperplanes } P, Q \subset \mathbb{R}^n \text{ algebraic over } \text{supp}(p_j) \text{ such that } x, y \text{ are opposite points on } P, Q, \text{ respectively}\}$ ;
- $\gamma(x, i, j) = \{y \in \text{dom}(p_i \setminus c) : \text{there are points } x_j \in \text{dom}(p_j \setminus c) \text{ and } x_k \in \text{dom}(p_k \setminus c) \text{ such that } x, y, x_j, x_k \text{ are four vertices of a rectangle listed in a clockwise or counterclockwise order}\}$ . Here  $k \in 3$  is the index distinct from  $i$  and  $j$ .

**Claim 3.9.** *There is a finite set  $a \subset \text{supp}(p_i)$  such that  $\alpha(x, i, j)$  consists of points algebraic over  $(\mathbb{R} \cap V) \cup a$ .*

PROOF: This is clear if  $\alpha(x, i, j) = 0$ . Otherwise, let  $y \in \alpha(x, i, j)$  be any point and argue that all other points in  $\alpha(x, i, j)$  are algebraic over  $(\mathbb{R} \cap V) \cup \{y\}$ . To see this, suppose that  $z \in \alpha(x, i, j)$  is any other point. Let  $S_y, S_z$  be hyperspheres algebraic in  $\text{supp}(p_j)$  such that  $x$  is opposite of  $y$  on  $S_y$  and opposite of  $z$  on  $S_z$ . It follows that  $z$  is algebraic over  $\text{supp}(p_j) \cup \{y\}$ : one first derives  $x$  from  $y$  and then  $z$  from  $x$ . By Fact 3.7  $z$  is algebraic over  $(\mathbb{R} \cap V) \cup \{y\}$  as desired.  $\square$

**Claim 3.10.** *There is a finite set  $a \subset \text{supp}(p_i)$  such that  $\beta(x, i, j)$  consists of points algebraic over  $(\mathbb{R} \cap V) \cup a$ .*

PROOF: This is parallel to the previous argument.  $\square$

**Claim 3.11.** *There is a finite set  $a \subset \text{supp}(p_i)$  such that  $\gamma(x, i, j)$  consists of points algebraic over  $(\mathbb{R} \cap V) \cup a$ .*

PROOF: This is the heart of the whole construction and the reason why item (4) appears in Definition 3.1. For each point  $y \in \gamma(x, i, j)$  choose points  $x_j(y) \in \text{dom}(p_j \setminus c)$  and  $x_k(y) \in \text{dom}(p_k \setminus c)$  witnessing the membership relation. Let  $H(y) \subset \mathbb{R}^n$  be the hyperplane passing through  $y$  and perpendicular to the vector  $y - x_j(y)$ ; thus,  $x \in H(y)$ . Write  $H = \bigcap_{y \in \gamma(x, i, j)} H(y)$ . Let  $a \subset \gamma(x, i, j)$  be a set of minimum cardinality such that  $H = \bigcap_{y \in a} H(y)$ ; the set  $a$  is finite. Identify  $a$  with one of its enumerations  $\langle a(l) : l \in m \rangle$ . We will show that every point  $y \in \gamma(x, i, j)$  is algebraic over  $(\mathbb{R} \cap V) \cup a$ . This will prove the claim.

Let  $y \in \gamma(x, i, j)$  be an arbitrary point. Consider the set  $A = \{u \in (\mathbb{R}^n)^{m+1} : \forall z \in \mathbb{R}^n (\forall l \in m (x_j(a(l)) - u(l)) \cdot (z - u(l)) = 0) \rightarrow (x_j(y) - u(m)) \times (z - u(m)) = 0\}$ . The set  $A$  is semialgebraic in parameters from  $\text{supp}(p_j)$  and contains the tuple  $a \hat{\ } y$ . In addition, by Thales's theorem the section  $A_a$  is a subset of the hypersphere of which the segment between  $x_j(y)$  and  $x$ , and also the segment between  $x_k(y)$  and  $y$ , is a diameter. By Fact 3.7 (2) and the Noetherian assumption between  $V[G_i]$  and  $V[G_j]$ , there is a set  $B \subset A$  semialgebraic

over  $\mathbb{R} \cap V$  such that  $a \frown y \in B$ . Let  $C = \{u \in B : u(m) \text{ is the farthest point of } B_{u \upharpoonright m} \text{ from } x_k(y)\}$ . This is a semialgebraic set in parameters from  $\text{supp}(p_k)$ . By Fact 3.7 (2) and the Noetherian assumption on  $V[G_i]$  and  $V[G_k]$ , there is a set  $D \subseteq C$  semialgebraic over  $\mathbb{R} \cap V$  such that  $a \frown y \in D$ . Clearly,  $D_a = \{y\}$ . It follows that  $y$  is algebraic over  $(\mathbb{R} \cap V) \cup a$  as desired.  $\square$

Now, define the set  $f(x) \subset \omega$  of forbidden colors by setting it to the union of  $p_i''(\alpha(x, i, j) \cup \beta(x, i, j) \cup \gamma(x, i, j))$  for all choices of distinct indices  $i, j \in 3$ . By the claims and Definition 3.1 (4) applied to  $p_i \leq c$ ,  $f(x) \in I$ . Let  $b \subset \omega$  be a set in the ideal  $I$  which cannot be covered by finitely many sets of the form  $f(x)$  for  $x \in d$ , and finitely many singletons. Let  $q : F^n \rightarrow \omega$  be any map extending  $\bigcup_i p_i$  and such that  $q \upharpoonright d$  is an injection such that  $q(x) \in b \setminus f(x)$  holds for every  $x \in d$ . We claim that  $q$  is the requested common lower bound of the conditions  $p_i$  for  $i \in 3$ .

**Claim 3.12.** *We have that  $q$  is a  $\Gamma_n$ -coloring.*

PROOF: Let  $R \subset F^n$  be a rectangle; we must show that  $q$  is not constant on it. The proof breaks into numerous cases and subcases.

*Case 1.* Let  $R$  contain no elements of the set  $d$ . Let  $a \subset 3$  be an inclusion minimal set such that  $R \subset \bigcup_{i \in a} \text{dom}(p_i)$ .

*Case 1.1.* Let  $|a| = 1$ . Here,  $R$  is not monochromatic because  $p_i$  is a  $\Gamma_n$ -coloring where  $i$  is the unique element of  $a$ .

*Case 1.2.* Let  $|a| = 2$ , containing indices  $i, j \in 3$ . The closure properties of the domains of  $p_i$  and  $p_j$  imply that each set  $\text{dom}(p_i \setminus c)$  and  $\text{dom}(p_j \setminus c)$  contains exactly two points of  $R$ .

*Case 1.2.1.* The two points in  $\text{dom}(p_i \setminus c) \cap R$  are adjacent in  $R$ . Then the hyperplanes containing the two respective points and perpendicular to their connector are algebraic over both  $V[G_i]$  and  $V[G_j]$ , so in  $V$  by Proposition 3.7 (4). The two points are opposite on these planes and therefore they receive distinct  $p_i$  colors by Definition 3.1 (3). Therefore,  $R$  is not monochromatic.

*Case 1.2.2.* The two points in  $\text{dom}(p_i \setminus c) \cap R$  are opposite in  $R$ . Then both the center of the rectangle  $R$  and the real number which is half of the length of the rectangle diagonal belong to both  $V[G_i]$  and  $V[G_j]$ , so to  $V$  by Proposition 3.7 (4). The hypersphere  $S$  they determine is visible from  $V$ , and the two points of  $\text{dom}(p_i \setminus c) \cap R$  are opposite on  $S$ . Applying Definition 3.1 (2) to  $p_i \leq c$ , it is clear that the two points receive distinct  $p_i$  colors and  $R$  is not monochromatic.

*Case 1.3.* Let  $|a| = 3$ . Then there must be index  $i \in 3$  such that  $\text{dom}(p_i)$  contains exactly two points of  $R$  and  $\text{dom}(p_j \setminus c)$  contains exactly one point of  $R$  for each index  $j \neq i$ . We will show that this case cannot occur regardless of the

colors on the rectangle  $R$ . For an index  $j \neq i$ , write  $x_j$  for the unique point in  $R \cap \text{dom}(p_j \setminus c)$ .

*Case 1.3.1.* The two points in  $\text{dom}(p_i) \cap R$  are adjacent in  $R$ . Consider the two hyperplanes  $Q_j, Q_k$  containing these two points, respectively, and perpendicular to their connecting segment, indexed by  $j, k \neq i$ . Reindexing if necessary,  $x_j \in Q_j$  and  $x_k \in Q_k$  holds. By Proposition 3.7 (1), there must be algebraic sets  $Q'_j \subseteq Q_j$  and  $Q'_k \subseteq Q_k$  visible from the ground model and still containing  $x_j$  and  $x_k$ . This means that  $x_k$  can be recovered in  $V[G_j]$  as the closest point to  $x_j$  in  $Q'_k$ . This is impossible as  $\mathbb{R} \cap V[G_j] \cap V[G_k] = \mathbb{R} \cap V$ .

*Case 1.3.2.* The two points in  $\text{dom}(p_i) \cap R$  are opposite in  $R$ . Consider the hypersphere  $S$  in which these two points are opposite.  $S$  then contains  $x_j$  and  $x_k$  and these two points are opposite in  $S$ . By Fact 3.7, there must be algebraic sets  $S_j \subseteq S$  and  $S_k \subseteq S$  visible from the ground model and still containing  $x_j$  and  $x_k$ . This means that  $x_k$  can be recovered in  $V[G_j]$  as the farthest point to  $x_j$  in  $S_k$ . This is impossible as  $\mathbb{R} \cap V[G_j] \cap V[G_k] = \mathbb{R} \cap V$ .

*Case 2.* Let  $R$  contain exactly one point in the set  $d$ ; call this unique point  $x$ . Let  $a \subset 3$  be an inclusion minimal set such that  $R \setminus \{x\} \subset \bigcup_{i \in a} \text{dom}(p_i)$ .

*Case 2.1.* Let  $|a| = 1$ . This cannot occur since  $\text{dom}(p_i)$  would contain  $x$  with the other three vertices of  $R$ , where  $i \in 3$  is the only element of  $a$ .

*Case 2.2.* Let  $|a| = 2$ , containing indices  $i, j \in 3$ . Here, for one of the indices (say  $j$ )  $\text{dom}(p_j)$  has to contain two elements of  $R$  while  $\text{dom}(p_i \setminus c)$  contains just one; denote the latter point by  $x_i$ .

*Case 2.2.1.* The points  $x_i$  and  $x$  are opposite on the rectangle  $R$ . Then  $x_i \in \alpha(x, i, j)$  as the hypersphere on which  $x_i, x$  are opposite points is the same as the one on which the other two points are opposite, and therefore is algebraic over  $\text{supp}(p_j)$ . The choice of the map  $q$  shows that  $q(x) \neq p_i(x_i)$ , so  $R$  is not monochromatic.

*Case 2.2.2.* The points  $x_i$  and  $x$  are adjacent on the rectangle  $R$ . Then  $x_i \in \beta(x, i, j)$  as  $x_i, x$  are opposite points on the hyperplanes passing through the other two points and perpendicular to their connecting segment, and these are algebraic over  $\text{supp}(p_j)$ . The choice of the map  $q$  shows that  $q(x) \neq p_i(x_i)$ , so  $R$  is not monochromatic.

*Case 2.3.* Let  $|a| = 3$ . For each index  $i \in 3$  let  $x_i \in R$  be the unique point in  $\text{dom}(p_i \setminus c)$ . Let  $i, j, k \in 3$  be indices such that the sequence  $x, x_i, x_j, x_k$  goes around the rectangle  $R$ . Then  $x_i \in \gamma(x, i, j)$  holds. The choice of the map  $q$  shows that  $q(x) \neq p_i(x_i)$ , so  $R$  is not monochromatic.

*Case 3.* Let  $R$  contain more than one point in the set  $d$ . Then  $R$  is not monochromatic as  $q \upharpoonright d$  is an injection.  $\square$

Finally, let  $i \in 3$  be an index; we must prove that  $q \leq p_i$  holds. It is clear that  $p_i \leq q$  holds. The following claims verify the other items of Definition 3.1.

**Claim 3.13.** *If  $S \subset \mathbb{R}^n$  is a hypersphere algebraic over  $\text{supp}(p_i)$  and  $x, y \in \text{dom}(q \setminus p_i)$  are opposite points on it, then  $q(x) \neq q(y)$ .*

PROOF: The argument splits into cases.

*Case 1.* If  $x, y$  both belong to the set  $d$ , then  $q(x) \neq q(y)$  as  $q \upharpoonright d$  is an injection.

*Case 2.* If  $x \in d$  and  $y \notin d$ , let  $j \in 3$  be an index distinct from  $i$  such that  $y \in \text{dom}(p_j \setminus c)$ . Then,  $y \in \alpha(x, j, i)$  holds and therefore  $q(x) \neq p_j(y)$  as  $q(x) \notin p_j''\alpha(x, j, i)$ .

*Case 3.* If neither of the points  $x, y$  belongs to  $d$ , then there are two subcases.

*Case 3.1.* There is  $j \in 3$  such that both  $x, y$  belong to  $\text{dom}(p_j) \setminus c$ . In such a case, the hypersphere  $S$  is also algebraic over  $\text{supp}(p_j)$ . By the Noetherian assumption on the models  $V[G_i]$  and  $V[G_j]$  and Proposition 3.7 (4), the hypersphere  $S$  is algebraic over the ground model. It follows that  $q(x) = p_j(x) \neq p_j(y) = q(y)$  by Definition 3.1 (2) applied to  $p_j \leq c$ .

*Case 3.2.* Let  $x \in \text{dom}(p_j \setminus c)$  and  $y \in \text{dom}(p_k \setminus c)$  for distinct indices  $j, k$ . By the Noetherian assumption on the models  $V[G_i]$  and  $V[G_j]$  and Proposition 3.7 (1), there is a set  $T \subseteq S$  algebraic over the ground model such that  $x \in T$ . Then  $x$  can be recovered in  $V[G_k]$  as the point on  $T$  farthest away from  $y$ , contradicting the fact that  $V[G_j] \cap V[G_k] = 0$ .  $\square$

**Claim 3.14.** *If  $P, Q \subset \mathbb{R}^n$  are parallel hyperplanes algebraic over  $\text{supp}(p_i)$  and  $x, y \in \text{dom}(q \setminus p_i)$  are opposite points on them, then  $q(x) \neq q(y)$ .*

PROOF: The argument is similar to that for Claim 3.13.  $\square$

**Claim 3.15.** *If  $a \subset F^n$  is a finite set, then  $q''\delta(p_i, q, a) \in I$ .*

PROOF: For each index  $j \in 3$  distinct from  $i$ , let  $a_j \subset \delta(p_i, q, a) \cap \text{dom}(p_j)$  be an inclusion-maximal set which is algebraically free over  $\text{supp}(p_i)$ . Since sets algebraically free over  $\text{supp}(p_i)$  form a matroid,  $|a_j| \leq |a|$ . By Proposition 3.7 (3),  $\delta(p_i, q, a) \cap \text{dom}(p_j) = \delta(c, p_j, a_j)$  holds. This means that  $\delta(p_i, q, a) = \delta(c, p_j, a_j) \cup \delta(c, p_k, a_k) \cup d$  where  $j, k \in 3$  are the two indices distinct from  $i$ . Now,  $p_j''(c, p_j, a_j) \in I$  by Definition 3.1 (4) applied to  $p_j \leq c$ ,  $p_k''(c, p_k, a_k) \in I$  by Definition 3.1 (4) applied to  $p_k \leq c$ , and  $q''d \in I$  as this set is a subset of  $b$ . As the ideal  $I$  is closed under unions and subsets,  $q''\delta(p_i, q, a) \in I$  as desired.  $\square$

This concludes the proof of item (1) of the theorem. For item (2), if CH holds and  $p \in P_n$  is a condition, by (1) it is enough to produce a total  $\Gamma_n$ -coloring  $c$  such that  $\text{Coll}(\omega, \mathbb{R}) \Vdash \check{c} \leq \check{p}$ . To this end, choose an enumeration  $\langle x_\alpha : \alpha \in \omega_1 \rangle$  of  $\mathbb{R}^n$  and by recursion on  $\alpha \in \omega_1$  build conditions  $p_\alpha \in P_n$  so that

- $p = p_0 \geq p_1 \geq \dots$ ;
- $x_\alpha \in \text{dom}(p_{\alpha+1})$ ;
- $p_\alpha = \bigcup_{\beta \in \alpha} p_\beta$  for limit ordinals  $\alpha$ .

The successor step is possible by Corollary 3.6 and the limit step by Proposition 3.2. In the end, let  $c = \bigcup_\alpha p_\alpha$  and observe that  $c$  is a total  $\Gamma_n$ -coloring and  $\text{Coll}(\omega, \mathbb{R}) \Vdash \check{c} \leq \check{p}$ .  $\square$

Finally, we can complete the proof of Theorem 1.2. Let  $n \geq 2$  be a number. Let  $\kappa$  be an inaccessible cardinal. Let  $W$  be the choiceless Solovay model derived from  $\kappa$ . Let  $P_n$  be the Suslin poset of Definition 3.1, and let  $G \subset P_n$  be a filter generic over  $W$ . Then  $W[G]$  is a model of  $\text{ZF} + \text{DC}$  since it is a  $\sigma$ -closed extension of a model of  $\text{ZF} + \text{DC}$ . In  $W[G]$ , the chromatic number of  $\Gamma_n$  is countable by Corollary 3.6. In  $W[G]$ , every non-null subset of the plane contains an equilateral triangle by the conjunction of Theorem 2.3 and Theorem 3.8. The proof is complete.

#### REFERENCES

- [1] Ceder J., *Finite subsets and countable decompositions of Euclidean spaces*, Rev. Roumaine Math. Pures Appl. **14** (1969), 1247–1251.
- [2] Erdős P., Kakutani S., *On non-denumerable graphs*, Bull. Amer. Math. Soc. **49** (1943), 457–461.
- [3] Erdős P., Komjáth P., *Countable decompositions of  $\mathbb{R}^2$  and  $\mathbb{R}^3$* , Discrete Comput. Geom. **5** (1990), no. 4, 325–331.
- [4] Ihoda J. I., Shelah S., *Souslin forcing*, J. Symbolic Logic **53** (1998), no. 4, 1188–1207.
- [5] Jech T., *Set Theory*, Springer Monographs in Mathematics, Springer, Berlin, 2003.
- [6] Larson P., Zapletal J., *Geometric Set Theory*, Mathematical Surveys and Monographs, 248, American Mathematical Society, Providence, 2020.
- [7] Marker D., *Model Theory: An Introduction*, Graduate Texts in Mathematics, 217, Springer, New York, 2002.
- [8] Schmerl J. H., *Avoidable algebraic subsets of Euclidean space*, Trans. Amer. Math. Soc. **352** (2000), no. 6, 2479–2489.
- [9] Zapletal J., *Noetherian spaces in choiceless set theory*, available at arXiv:2101.03434v3 [math.LO] (2022), 23 pages.

J. Zapletal:

DEPARTMENT OF MATHEMATICS, P. O. BOX 118105, UNIVERSITY OF FLORIDA,  
GAINESVILLE, FLORIDA, FL 32611-8105, USA

and

INSTITUTE OF MATHEMATICS, CZECH ACADEMY OF SCIENCES, ŽITNÁ 25, PRAHA,  
CZECH REPUBLIC

*E-mail:* zapletal@ufl.edu

(Received April 12, 2022, revised September 23, 2022)