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EXISTENCE OF RENORMALIZED SOLUTIONS FOR SOME DEGENERATE AND NON-COERCIVE ELLIPTIC EQUATIONS

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Abstract. This paper is devoted to the study of some nonlinear degenerated elliptic equations, whose prototype is given by

$$\begin{aligned} &-\operatorname{div}(b(|u|)|\nabla u|^{p-2}\nabla u)+d(|u|)|\nabla u|^{p}=f-\operatorname{div}(c(x)|u|^{\alpha}) && \text{in } \Omega,\\ &u=0 && \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N\geqslant 2)$ with $1<p<N$ and $f\in L^1(\Omega)$, under some growth conditions on the function $b(\cdot)$ and $d(\cdot)$, where $c(\cdot)$ is assumed to be in $L^{N/(p-1)}(\Omega)$. We show the existence of renormalized solutions for this non-coercive elliptic equation, also, some regularity results will be concluded.

Keywords: renormalized solution; nonlinear elliptic equation; non-coercive problem

MSC 2020: 35J60, 46E30, 46E35

1. Introduction

In [7], Boccardo et al. have studied the quasilinear elliptic problem with degenerate coercivity

$$(1.1)\qquad \begin{cases} -\operatorname{div}(A(x,u)\nabla u)=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where the data f is assumed to be in $L^m(\Omega)$ with $m\geqslant 1$. They have proved the existence and some regularity results; we refer the reader to [2], [9], and also [16] for the case of measure data.

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Alvino et al. have considered in [1] the nonlinear degenerated elliptic problem of the form

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases} \tag{1.2}$$

they have proved the existence of solutions and some regularity results for f a measurable function in $L^m(\Omega)$ with $m \geqslant 1$.

In [19], Murat has proved the existence of renormalized solutions for the quasilinear elliptic problem

$$\begin{cases} \lambda u - \operatorname{div}(A(x)\nabla u + \phi(u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $f \in L^1(\Omega)$ and $\lambda > 0$. The uniqueness of the solution was concluded under some locally Lipschitz continuous conditions on the vector field $\phi(\cdot)$. We refer also to [10], where Del Vecchio et al. have proved the existence of weak solutions for the non-coercive problem by using the symmetrization method.

In [13], Droniou has studied the nonlinear non-coercive elliptic problems

$$\begin{cases} Au - \operatorname{div}\phi(x,u) = f(x) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

with $Au = -\operatorname{div} a(x,u,\nabla u)$ being a Leray-Lions operator on $W_0^{1,p}(\Omega)$, and $\Phi(x,s)$ being convection term with growth properties, where $f \in W^{-1,p'}(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$. He has proved the existence and some regularity results. Also, the author has proved in [12] the existence and uniqueness of solutions for some elliptic problems.

In [5], Bensoussan, Boccardo and Murat have studied the nonlinear elliptic problem

$$Au + g(x,u,\nabla u) = f \quad \text{in } \Omega,$$

where A is a Leray-Lions operator acted from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$, where g is a Carathéodory function satisfying the sign and growth conditions, the data f belongs to $W^{-1,p'}(\Omega)$. They proved the existence of the solution in the sense of distributions $u \in W_0^{1,p}(\Omega)$ such that $g(x,u,\nabla u) \in L^1(\Omega)$ and $g(x,u,\nabla u)u \in L^1(\Omega)$.

In the case of $f \in L^1(\Omega)$, Boccardo and Gallouet (see [8]) have proved the existence of solutions $u \in W_0^{1,p}(\Omega)$ with $g(x,u,\nabla u) \in L^1(\Omega)$ under the additional assumption:

$$\text{There exist } \sigma > 0,\ \gamma > 0 \text{ such that } |g(x,s,\xi)| \geqslant \gamma|\xi|^p \text{ for } |s| \geqslant \sigma.$$

In [3], Ben Cheikh Ali and Guibé have studied some quasilinear elliptic equations of the type

$$(1.5)\qquad \begin{cases} \lambda(x,u) - \operatorname{div}(a(x,\nabla u) + \Phi(x,u)) = f & \text{in } \Omega, \\ (a(x,\nabla u) + \Phi(x,u)) \cdot n = 0 & \text{on } \Gamma_n, \\ u = 0 & \text{on } \Gamma_d, \end{cases}$$

where $Au = -\operatorname{div} a(x,\nabla u)$ is a Leray-Lions type operator and the Carathéodory functions $\lambda(x,s)\colon\ \Omega\times\mathbb{R} \mapsto \mathbb{R}$ and $\Phi(x,s)\colon\ \Omega\times\mathbb{R} \mapsto \mathbb{R}^N$ satisfy only some growth conditions. They have proved the existence of renormalized solutions for this equation. Moreover, the uniqueness of solution is obtained under some additional conditions on the function $\Phi(x,s)$; we refer the reader to [11], [18], [22], [23].

In [15], Guibé et al. have studied a class of nonlinear elliptic problems whose prototype is

$$(1.6)\qquad \begin{cases} -\Delta_p u - \operatorname{div}(c(x)|u|^{\gamma}) + b(x)|\nabla u|^{\lambda} = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ_p is the p-Laplace operator $1 < p < N$, and μ is a Radon measure with bounded variation on Ω. They have proved the existence of renormalized solutions in the case of $0 \leqslant \gamma \leqslant p-1$ and $0 \leqslant \lambda \leqslant p-1$ (see also [14]).

In this paper, we are interested in proving the existence of renormalized solutions to the following nonlinear elliptic problem having a degenerate coercivity:

$$(1.7)\qquad \begin{cases} Au + g(x,u,\nabla u) = f - \operatorname{div}\phi(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N \geqslant 2)$ with $1 < p < N$ and $Au = -\operatorname{div} a(x,u,\nabla u)$ is a non-coercive Leray-Lions operator acting from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$, the Carathéodory functions $g(x,s,\xi)$ and $\phi(x,s)$ verify only some growth conditions, and the data f is assumed to belong to $L^1(\Omega)$. Under such assumptions, the solution u may not be finite in general. This means that, at least for solutions obtained through approximation, such solutions may reach the values ∞ and $-\infty$. For more details we refer the reader to [6].

The novelty of this work is the fact of overcoming several difficulties at the same time. We prove the existence of renormalized solutions for the strongly nonlinear and non-coercive elliptic problem (1.7), the existence result is obtained by using an approximation procedure and some a priori estimate. The functions test used in this work are essentially inspired from the standard analysis; we refer the reader for example to [1], [3], [13], [20], [21].

This paper is organized as follows: In Section 2, we present some non-standard assumptions on the Carathéodory functions $a(x,s,\xi)$, $g(x,s,\xi)$ and $\phi(x,s)$ for which our nonlinear elliptic problem (1.7) has a renormalized solution. In Section 3, we will state the main results. Section 4 is devoted entirely to prove the existence of renormalized solutions for our nonlinear elliptic equation, also, some regularity results will be proved. Finally in Section 5, we will prove Proposition 4.1.

2. Essential assumptions

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geqslant 2)$ and let $1 < p < N$. We consider a Leray-Lions operator A from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$, defined by the formula

$$Au = -\operatorname{div} a(x,u,\nabla u), \tag{2.1}$$

where $a(x,s,\xi)\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is a Carathéodory function (i.e., measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) and verifies the following conditions:

$$|a(x,s,\xi)| \leqslant \beta(a_0(x) + |s|^{p-1} + |\xi|^{p-1}) \tag{2.2}$$

for a positive function $a_0(x) \in L^{p'}(\Omega)$, and $\beta > 0$;

$$(a(x,s,\xi) - a(x,s,\eta)) \cdot (\xi - \eta) > 0 \quad \text{for any } \xi \neq \eta. \tag{2.3}$$

There exists a positive decreasing function $b\colon [0,\infty[\;\to\;]0,\infty[$, and two constants $b_0, s_0 > 0$ such that

$$a(x,s,\xi) \cdot \xi \geqslant b(|s|)|\xi|^p \quad \text{with } b(|s|) \geqslant \frac{b_0}{(1+|s|)^\lambda} \text{ for all } |s| > s_0, \tag{2.4}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $0 \leqslant \lambda \leqslant p-1$. As a consequence of (2.4) and the continuity of the function $a(x,s,\cdot)$ with respect to ξ, we have

$$a(x,s,0) = 0.$$

The lower order term $g(x,s,\xi)\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfies only the growth condition

$$|g(x,s,\xi)| \leqslant f_0(x) + d(|s|)|\xi|^p, \tag{2.5}$$

where $f_0(x)$ is assumed to be a positive measurable function in $L^1(\Omega)$, and $d(\cdot)\colon \mathbb{R} \mapsto \mathbb{R}^+$ is a continuous decreasing function such that $d(|\cdot|)/b(|\cdot|)$ is decreasing and belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

The Carathéodory function $\phi(\cdot,\cdot)\colon\ \Omega\times\mathbb{R}\mapsto\mathbb{R}^N$ satisfies the growth condition

$$|\phi(x,s)|\leqslant c(x)(1+|s|)^{\alpha}, \tag{2.6}$$

where $0\leqslant\alpha\leqslant p-1-\lambda$ and $c(x)$ is a positive function in $L^{N/(p-1)}(\Omega)$.

We consider the strongly nonlinear and non-coercive elliptic Dirichlet problem

$$\begin{cases} Au+g(x,u,\nabla u)=f-\operatorname{div}\phi(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

where the data f is assumed to be in $L^1(\Omega)$.

Definition 2.1. Let $k>0$, the truncation function $T_k(\cdot)\colon\ \mathbb{R}\mapsto\mathbb{R}$ is given by

$$T_k(s)=\begin{cases} s & \text{if } |s|\leqslant k,\\ k\dfrac{s}{|s|} & \text{if } |s|>k, \end{cases}$$

and we define

$$\mathcal{T}_0^{1,p}(\Omega):=\{u\colon\ \Omega\mapsto\mathbb{R} \text{ measurable, such that } T_k(u)\in W_0^{1,p}(\Omega) \text{ for any } k>0\}.$$

Proposition 2.1 (cf. [4])**.** *Let $u\in\mathcal{T}_0^{1,p}(\Omega)$. There exists a unique measurable function $v\colon\ \Omega\mapsto\mathbb{R}^N$ such that*

$$\nabla T_k(u)=v\chi_{\{|u|<k\}} \quad \text{a.e. in } \Omega \text{ for any } k>0,$$

where χ_A denotes the characteristic function of a measurable set A. The function v is called the weak gradient of u and is still denoted by ∇u. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the gradient of u, that is $v=\nabla u$.

3. Main result

We begin by introducing the definition of renormalized solutions for the elliptic equation (2.7):

Definition 3.1. A measurable function u is called a renormalized solution of the strongly nonlinear elliptic problem (2.7) if $u\in\mathcal{T}_0^{1,p}(\Omega)$, $g(x,u,\nabla u)\in L^1(\Omega)$, and

$$\lim_{h\to\infty}\frac{1}{h}\int_{\{|u|\leqslant h\}}a(x,u,\nabla u)\nabla u\,\mathrm{d}x=0 \tag{3.1}$$

such that u satisfies the equality

$$\int_\Omega a(x,u,\nabla u)\cdot(S'(u)\varphi\nabla u+S(u)\nabla\varphi)\,\mathrm{d}x+\int_\Omega g(x,u,\nabla u)S(u)\varphi\,\mathrm{d}x \\ =\int_\Omega fS(u)\varphi\,\mathrm{d}x+\int_\Omega \phi(x,u)\cdot(S'(u)\varphi\nabla u+S(u)\nabla\varphi)\,\mathrm{d}x \tag{3.2}$$

for every $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and for any smooth function $S(\cdot)\in W^{1,\infty}(\mathbb{R})$ with a compact support.

The goal of the present paper is to prove the following existence result:

Theorem 3.1. *Let $f\in L^1(\Omega)$, assuming that conditions* (2.2)–(2.6) *hold true, and one of the following additional hypothesis holds true:*

(i) *There exist two positive constants s_0 and d_0 such that $d(|s|)\leqslant d_0/(1+|s|)^p$ for any $s\geqslant s_0$.*

(ii) $0\leqslant\alpha<p-1-\lambda$.

(iii) $\|c(x)\|_{L^{N/(p-1)}(\Omega)}\leqslant c_0$, *where c_0 is small enough.*

Then there exists at least one renormalized solution for the strongly nonlinear and non-coercive elliptic problem (2.7).

R e m a r k 3.1. Note that, under the assumption of Theorem 3.1, the renormalized solution of problem (2.7) belongs to $u\in\mathcal{T}_0^{1,p}(\Omega)$ so that $d(|u|)^{1/p}u$ belongs to $L^p(\Omega)$.

In all remaining parts of this paper, we will denote by C_p the constant of Poincaré's inequality, and by C_s the constant of Sobolev's inequality. The real constants C_i for $i=0,1,\ldots$ are different in each step of the proof of Theorem 3.1.

4. Proof of Theorem 3.1

The proof will be divided into several steps.

Step 1: Approximate problems. We consider a sequence of smooth functions $(f_n)_{n\in\mathbb{N}^*}$ in $W^{-1,p'}(\Omega)\cap L^1(\Omega)$ that converges strongly to f in $L^1(\Omega)$, such that $|f_n|\leqslant|f|$ (for example $f_n=T_n(f)$).

For any $n\in\mathbb{N}^*$ we consider the approximate problem

$$\begin{cases} A_nu_n+g_n(x,u_n,\nabla u_n)=f_n-\operatorname{div}\phi_n(x,u_n) & \text{in } \Omega,\\ u_n=0 & \text{on } \partial\Omega,\end{cases} \tag{4.1}$$

with $\phi_n(x,s)=\phi(x,T_n(s))$ and $g_n(x,s,\xi)=T_n(g(x,s,\xi))$, where the operator A_n is given by

$$A_nv=-\operatorname{div}a(x,T_n(v),\nabla v)\quad\text{for any } v\in W_0^{1,p}(\Omega).$$

Note that the operator A_n is coercive and satisfies the classical Leray-Lions conditions.

Indeed, by using condition (2.4) we have

$$\langle A_n v, v\rangle = \int_\Omega a(x, T_n(v), \nabla v)\cdot \nabla v \,\mathrm{d}x \geqslant b(n)\int_\Omega |\nabla v|^p \,\mathrm{d}x$$

for all $v \in W_0^{1,p}(\Omega)$ with $b(n) > 0$.

In view of the classical results of Leray-Lions (see [17]), there exists at least one weak solution $u_n \in W_0^{1,p}(\Omega)$ for the approximate problem (4.1); we refer the reader also to [15] for more details.

Proposition 4.1. *Assume that conditions* (2.2)–(2.6) *hold true, and let* u_n *be a weak solution of the approximate problem* (4.1). *If one of assumptions* (i)–(iii) *in Theorem* 3.1 *is satisfied, then for any* $n \in \mathbb{N}^*$, *the weak solution of approximate problem* (4.1) *verifies*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \operatorname{meas}\{|u_n| > k\} \to 0. \tag{4.2}$$

The proof of Proposition 4.1 is in Appendix.

Step 2: Weak convergence of truncations. Let $k \geqslant 1$, we set $B(s) = T_k(s)(1+|T_k(s)|)^\lambda$ and $H(s) = 2\int_0^s d(|\tau|)/b(|\tau|)\,\mathrm{d}\tau$.

We have $d(|\cdot|)/b(|\cdot|) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $0 \leqslant H(\infty) := 2\int_0^\infty d(|\tau|)/b(|\tau|)\,\mathrm{d}\tau$ is a finite real number. Then $B(u_n)\mathrm{e}^{H(|u_n|)} \in W_0^{1,p}(\Omega)$.

By taking $B(u_n)\mathrm{e}^{H(|u_n|)}$ as a test function for the approximate problem (4.1), we have

$$\begin{aligned}
&\int_\Omega a(x, T_n(u_n), \nabla u_n)\cdot \nabla T_k(u_n) B'(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\qquad + 2\int_\Omega a(x, T_n(u_n), \nabla u_n)\cdot \nabla u_n \frac{d(|u_n|)}{b(|u_n|)}|B(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\qquad + \int_\Omega g_n(x, u_n, \nabla u_n) B(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad = \int_\Omega f_n B(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x + \int_\Omega \phi(x, T_n(u_n))\cdot \nabla T_k(u_n) B'(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\qquad + 2\int_\Omega \phi(x, T_n(u_n))\cdot \nabla u_n \frac{d(|u_n|)}{b(|u_n|)}|B(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x.
\end{aligned}$$

In view of (2.4), (2.5) and (2.6), and since $(1+|T_k(s)|)^\lambda \leqslant B'(s) \leqslant (\lambda+1)\times(1+|T_k(s)|)^\lambda$ for $|s|<k$, we conclude that

(4.3)
$$\int_{\{|u_n|\leqslant k\}} b(|u_n|)|\nabla T_k(u_n)|^p(1+|u_n|)^\lambda\,\mathrm{d}x + 2\int_\Omega d(|u_n|)|\nabla u_n|^p|B(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x$$
$$\leqslant (1+k)^{\lambda+1}\mathrm{e}^{H(\infty)}\int_\Omega(|f_n|+|f_0|)\,\mathrm{d}x + \int_\Omega d(|u_n|)|\nabla u_n|^p|B(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x$$
$$+(\lambda+1)\mathrm{e}^{H(\infty)}\int_{\{|u_n|\leqslant k\}} c(x)(1+|T_k(u_n)|)^\alpha|\nabla T_k(u_n)|(1+|T_k(u_n)|)^\lambda\,\mathrm{d}x$$
$$+2\int_\Omega c(x)(1+|u_n|)^\alpha|\nabla u_n|\frac{d(|u_n|)}{b(|u_n|)}|B(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x.$$

Thanks to (2.4), we have $b(|u_n|)(1+|u_n|)^\lambda \geqslant b_0$ for $|u_n|\geqslant s_0$. We set

$$b_1 = \min\Big\{b_0, \inf_{|s|\leqslant k} b(|s|)(1+|s|)^\lambda\Big\},$$

then we get

$$b_1\int_{\{|u_n|\leqslant k\}}|\nabla T_k(u_n)|^p\,\mathrm{d}x + \int_\Omega d(|u_n|)|\nabla u_n|^p|B(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x$$
$$\leqslant (1+k)^{\lambda+1}\mathrm{e}^{H(\infty)}(\|f\|_{L^1(\Omega)}+\|f_0\|_{L^1(\Omega)})$$
$$+(\lambda+1)\mathrm{e}^{H(\infty)}\int_{\{|u_n|\leqslant k\}} c(x)|\nabla T_k(u_n)|(1+|T_k(u_n)|)^{\alpha+\lambda}\,\mathrm{d}x$$
$$+2\mathrm{e}^{H(\infty)}\int_\Omega c(x)(1+|u_n|)^\alpha|\nabla u_n|\frac{d(|u_n|)}{b(|u_n|)}|B(u_n)|\,\mathrm{d}x.$$

Using Young's inequality, we obtain

$$\frac{b_1}{2}\int_{\{|u_n|\leqslant k\}}|\nabla T_k(u_n)|^p\,\mathrm{d}x + \frac{1}{2}\int_\Omega d(|u_n|)|\nabla u_n|^p|T_k(u_n)|(1+|T_k(u_n)|)^\lambda\,\mathrm{d}x$$
$$\leqslant C_0(1+k)^{\lambda+1} + C_1\int_{\{|u_n|\leqslant k\}}|c(x)|^{p'}(1+|T_k(u_n)|)^{(\alpha+\lambda)p'}\,\mathrm{d}x$$
$$+C_2\int_\Omega|c(x)|^{p'}(1+|u_n|)^{\alpha p'}\frac{d(|u_n|)}{b(|u_n|)^{p'}}|B(u_n)|\,\mathrm{d}x.$$

Let $R\geqslant 1$, since $0\leqslant \alpha+\lambda\leqslant p-1$, having in mind (2.4), it follows that

(4.4)
$$\frac{b_1}{2}\int_{\{|u_n|\leqslant k\}}|\nabla T_k(u_n)|^p\,\mathrm{d}x + \frac{1}{2}\int_\Omega d(|u_n|)|\nabla u_n|^p|T_k(u_n)|(1+|T_k(u_n)|)^\lambda\,\mathrm{d}x$$
$$\leqslant C_0(1+k)^{\lambda+1} + C_1\int_{\{|u_n|\leqslant k\}}|c(x)|^{p'}(1+|T_k(u_n)|)^{(\alpha+\lambda)p'}\,\mathrm{d}x$$
$$+C_3\int_\Omega|c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'}d(|u_n|)|B(u_n)|\,\mathrm{d}x$$

$$\leqslant C_0(1+k)^{\lambda+1} + C_4 \int_{\{R<|u_n|\leqslant k\}} |c(x)|^{p'} |T_k(u_n)|^p \, dx + C_5 \int_{\{R<|u_n|\}} |c(x)|^{p'} |u_n|^p d(|u_n|)|B(u_n)| \, dx + C_5.$$

Using Hölder's inequality, we obtain

(4.5)

$$\begin{aligned}\frac{b_1}{2}\int_{\{|u_n|\leqslant k\}} |\nabla T_k(u_n)|^p \, dx &+ \frac{1}{2}\int_\Omega d(|u_n|)|\nabla u_n|^p |T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx \\ &\leqslant C_0(1+k)^{\lambda+1} + C_4 \|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})} \|T_k(u_n)\|^p_{L^{p^*}(\Omega)} \\ &\quad + C_5 \|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})} \|u_n d(|u_n|)^{1/p} |B(u_n)|^{1/p}\|^p_{L^{p^*}(\Omega)} + C_6,\end{aligned}$$

where C_6 is a constant depending on R, p and $\|c(\cdot)\|_{L^{p'}(\Omega)}$.

Recall that if $g(s)$ is positive and decreasing function on the set $[0,r]$, then $rg(r) \leqslant \int_0^r g(s)\, ds$.

We have that $d(|s|)$ is a decreasing function, and $B(|s|)$ is a constant function on the set $\{|s| > k\}$. Therefore, the function $d(|s|)B(|s|)$ is a decreasing function for $\{|s| > k\}$, and we obtain

$$|u_n| d(|u_n|)^{1/p} |B(u_n)|^{1/p} \leqslant \int_0^{|u_n|} d(|\tau|)^{1/p} |B(\tau)|^{1/p} \, d\tau + C_7 k^{(p+\lambda+1)/p} \quad \text{a.e. in } \Omega.$$

Thanks to Sobolev inequality and (4.2), we conclude that

(4.6)

$$\begin{aligned}\frac{b_1}{2}\int_{\{|u_n|\leqslant k\}} |\nabla T_k(u_n)|^p \, dx &+ \frac{1}{2}\int_\Omega |\nabla u_n|^p d(|u_n|)|T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx \\ &\leqslant C_9(1+k)^{p+\lambda+1} + C_4 \|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})} \|T_k(u_n)\|^p_{L^{p^*}(\Omega)} \\ &\quad + C_8 \|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})} \left\| \int_0^{|u_n|} d(|\tau|)^{1/p} |B(\tau)|^{1/p} \, d\tau \right\|^p_{L^{p^*}(\Omega)} \\ &\leqslant C_9(1+k)^{p+\lambda+1} + C_4 C_S^p \|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})} \|\nabla T_k(u_n)\|^p_{L^p(\Omega)} \\ &\quad + C_8 C_S^p \|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})} \left\| d(|u_n|)^{1/p} |B(u_n)|^{1/p} \nabla u_n \right\|^p_{L^p(\Omega)}.\end{aligned}$$

Since $\operatorname{meas}\{|u_n| > R\} \to 0$ as R tends to ∞, we can choose $R \geqslant 1$ large enough such that

$$C_4 C_S^p \|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})} \leqslant \frac{b_1}{4} \quad \text{and} \quad C_8 C_S^p \|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})} \leqslant \frac{1}{4}.$$

We obtain

$$(4.7)\quad \frac{b_1}{2}\int_{\{|u_n|\leqslant k\}}|\nabla T_k(u_n)|^p\,\mathrm{d}x+\frac{1}{2}\int_\Omega|\nabla u_n|^p d(|u_n|)|T_k(u_n)|(1+|T_k(u_n)|)^\lambda\,\mathrm{d}x \leqslant C_9(1+k)^{p+\lambda+1}+\frac{b_1}{4}\int_{\{|u_n|\leqslant k\}}|\nabla T_k(u_n)|^p\,\mathrm{d}x + \frac{1}{4}\int_\Omega|\nabla u_n|^p d(|u_n|)|T_k(u_n)|(1+|T_k(u_n)|)^\lambda\,\mathrm{d}x.$$

It follows that

$$(4.8)\quad \frac{b_1}{4}\int_{\{|u_n|\leqslant k\}}|\nabla T_k(u_n)|^p\,\mathrm{d}x+\frac{1}{4}\int_\Omega|\nabla u_n|^p d(|u_n|)|T_k(u_n)|(1+|T_k(u_n)|)^\lambda\,\mathrm{d}x \leqslant C_{10}k^{p+\lambda+1}.$$

Then the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,p}(\Omega)$, and there exists a measurable function $\eta_k\in W_0^{1,p}(\Omega)$ such that

$$(4.9)\quad \begin{cases} T_k(u_n)\rightharpoonup\eta_k & \text{weakly in } W_0^{1,p}(\Omega),\\ T_k(u_n)\to\eta_k & \text{strongly in } L^p(\Omega) \text{ and a.e. in } \Omega.\end{cases}$$

In view of (4.2) and (4.9), and following the same approach as in [4], we conclude that the sequence of weak solutions $(u_n)_n$ converges almost everywhere to a measurable function u, and thanks to (4.9), we obtain

$$(4.10)\quad \begin{cases} T_k(u_n)\rightharpoonup T_k(u) & \text{weakly in } W_0^{1,p}(\Omega),\\ T_k(u_n)\to T_k(u) & \text{strongly in } L^p(\Omega) \text{ and a.e. in } \Omega.\end{cases}$$

Moreover, taking $k=1$ in inequality (4.8) and since $d(|\cdot|)$ is a decreasing function,

$$(4.11)\quad \int_\Omega d(|u_n|)|u_n|^p\,\mathrm{d}x\leqslant\int_\Omega\left|\int_0^{|u_n|}d(s)^{1/p}\,\mathrm{d}s\right|^p\mathrm{d}x\leqslant C_p^p\int_\Omega d(|u_n|)|\nabla u_n|^p\,\mathrm{d}x\leqslant C_{11},$$

where C_{11} is a constant that does not depend on n. Then the sequence $(d(|u_n|)^{1/p}\times|u_n|)_n$ is uniformly bounded in $L^p(\Omega)$ and it follows that

$$(4.12)\quad d(|u_n|)^{1/p}|u_n|\rightharpoonup d(|u|)^{1/p}|u|\quad\text{weakly in } L^p(\Omega).$$

Step 3: Some regularity results. In this step, we denote by $\varepsilon_i(n)$, for $i=1,2,\ldots$, some real-valued functions that converge to 0 as n tends to infinity. Similarly, we define $\varepsilon_i(h)$ and $\varepsilon_i(n,h)$.

In this step, we will show the following estimate:

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\leqslant h\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x = 0. \tag{4.13}$$

Indeed, by taking $h^{-1}T_h(u_n)\mathrm{e}^{H(|u_n|)}\in W_0^{1,p}(\Omega)$ as a test function in the approximate problem (4.1), we have

$$\begin{aligned}
&\frac{1}{h}\int_{\{|u_n|\leqslant h\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\
&\qquad+\frac{1}{h}\int_\Omega g_n(x,u_n,\nabla u_n)T_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\
&\qquad+\frac{2}{h}\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n|T_h(u_n)|\frac{d(|u_n|)}{b(|u_n|)}\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\
&\quad=\frac{1}{h}\int_\Omega f_nT_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x+\frac{1}{h}\int_{\{|u_n|\leqslant h\}}\phi(x,T_n(u_n))\cdot\nabla u_n\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\
&\qquad+\frac{2}{h}\int_\Omega\phi(x,T_n(u_n))\cdot\nabla u_n|T_h(u_n)|\frac{d(|u_n|)}{b(|u_n|)}\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x.
\end{aligned}$$

In view of (2.5) and (2.6), we conclude that

$$\begin{aligned}
&\frac{1}{h}\int_{\{|u_n|\leqslant h\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\
&\qquad+\frac{2}{h}\int_\Omega d(|u_n|)|\nabla u_n|^p|T_h(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\
&\quad\leqslant\int_\Omega(|f_n|+|f_0|)\frac{|T_h(u_n)|}{h}\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\
&\qquad+\frac{1}{h}\int_\Omega d(|u_n|)|\nabla u_n|^p|T_h(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\
&\qquad+\frac{1}{h}\int_{\{|u_n|\leqslant h\}} c(x)(1+|T_n(u_n)|)^\alpha|\nabla u_n|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\
&\qquad+\frac{2}{h}\int_\Omega c(x)(1+|T_n(u_n)|)^\alpha|\nabla u_n||T_h(u_n)|\frac{d(|u_n|)}{b(|u_n|)}\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x.
\end{aligned}$$

Using Young's inequality, we deduce that

$$\begin{aligned}
&\frac{1}{h}\int_{\{|u_n|\leqslant h\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x+\frac{1}{h}\int_\Omega d(|u_n|)|\nabla u_n|^p|T_h(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\
&\quad\leqslant\mathrm{e}^{H(\infty)}\int_\Omega(|f_n|+|f_0|)\frac{|T_h(u_n)|}{h}\,\mathrm{d}x+\frac{1}{2h}\int_{\{|u_n|\leqslant h\}} b(|u_n|)|\nabla u_n|^p\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\
&\qquad+\frac{C_0}{h}\int_{\{|u_n|\leqslant h\}}|c(x)|^{p'}\frac{(1+|u_n|)^{\alpha p'}}{b(|u_n|)^{p'/p}}\,\mathrm{d}x+\frac{1}{2h}\int_\Omega d(|u_n|)|\nabla u_n|^p|T_h(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\
&\qquad+\frac{C_0}{h}\int_\Omega|c(x)|^{p'}(1+|u_n|)^{\alpha p'}|T_h(u_n)|\frac{d(|u_n|)}{b(|u_n|)^{p'}}\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x.
\end{aligned}$$

In view of (4.2), we have that $\operatorname{meas}\{|u_n| > h\} \to 0$ as h tends to infinity, thus $|T_h(u_n)|/h \rightharpoonup 0$ weak-$*$ in $L^\infty(\Omega)$. Thanks to the Lebesgue dominated convergence theorem, we obtain

$$\varepsilon_1(h) = \lim_{h\to\infty} \mathrm{e}^{H(\infty)} \int_\Omega (|f_n| + |f_0|) \frac{|T_h(u_n)|}{h} \,\mathrm{d}x \to 0 \quad \text{as } h \to \infty. \tag{4.14}$$

Let $R \geqslant 1$ and since $0 \leqslant (\alpha+\lambda)p' \leqslant p$, thanks to (2.4) and (4.2) and Hölder's inequality, we conclude that

$$\begin{aligned}
&\frac{1}{2h}\int_{\{|u_n|\leqslant h\}} a(x, T_n(u_n), \nabla u_n)\cdot\nabla u_n \,\mathrm{d}x + \frac{1}{2h}\int_\Omega d(|u_n|)|\nabla u_n|^p |T_h(u_n)|\,\mathrm{d}x \\
&\leqslant \varepsilon_1(h) + \frac{C_1}{h}\int_{\{|u_n|\leqslant h\}} |c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'} b(|u_n|)\,\mathrm{d}x \\
&\quad + \frac{C_1}{h}\int_\Omega |c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'}|T_h(u_n)| d(|u_n|)\,\mathrm{d}x \\
&\leqslant \varepsilon_2(h) + \frac{C_2}{h}\int_{\{|u_n|\leqslant h\}} |c(x)|^{p'}|u_n|^p b(|u_n|)\,\mathrm{d}x \\
&\quad + \frac{C_2}{h}\int_\Omega |c(x)|^{p'}|u_n|^p|T_h(u_n)| d(|u_n|)\,\mathrm{d}x \\
&\leqslant \varepsilon_2(h) + \frac{C_2}{h}\int_{\{|u_n|\leqslant R\}} |c(x)|^{p'}|u_n|^p b(|u_n|)\,\mathrm{d}x \\
&\quad + \frac{C_2}{h}\int_{\{|u_n|\leqslant R\}} |c(x)|^{p'}|u_n|^p|T_h(u_n)| d(|u_n|)\,\mathrm{d}x \\
&\quad + \frac{C_2}{h}\int_{\{R<|u_n|\leqslant h\}} |c(x)|^{p'}|u_n|^p b(|u_n|)\,\mathrm{d}x \\
&\quad + \frac{C_2}{h}\int_{\{R<|u_n|\}} |c(x)|^{p'}|u_n|^p|T_h(u_n)| d(|u_n|)\,\mathrm{d}x \\
&\leqslant \varepsilon_2(h) + \varepsilon_3(h) + \frac{C_2}{h}\|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})} \\
&\quad \times (\|u_n b(|u_n|)^{1/p}\|^p_{L^{p^*}(\{|u_n|\leqslant h\})} + \|u_n|T_h(u_n)|^{1/p} d(|u_n|)^{1/p}\|^p_{L^{p^*}(\Omega)}),
\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}
\varepsilon_2(h) &= \varepsilon_1(h) + \frac{2^{p-1}C_1}{h}\left(\int_{\{|u_n|\leqslant h\}} |c(x)|^{p'} b(|u_n|)\,\mathrm{d}x + \int_\Omega |c(x)|^{p'}|T_h(u_n)| d(|u_n|)\,\mathrm{d}x\right) \\
&= \varepsilon_1(h) + \frac{2^{p-1}C_1}{h}\Bigg(\|b(\cdot)\|_{L^\infty(\mathbb{R})}\int_{\{|u_n|\leqslant h\}} |c(x)|^{p'}\,\mathrm{d}x \\
&\quad + \|d(\cdot)\|_{L^\infty(\mathbb{R})}\int_\Omega |c(x)|^{p'}|T_h(u_n)|\,\mathrm{d}x\Bigg) \to 0 \quad \text{as } h\to\infty,
\end{aligned}$$

and

$$\begin{aligned}\varepsilon_3(h) &= \frac{C_2}{h}\int_{\{|u_n|\leqslant R\}} |c(x)|^{p'}|u_n|^p b(|u_n|)\,\mathrm{d}x\\ &\quad + \frac{C_2}{h}\int_{\{|u_n|\leqslant R\}} |c(x)|^{p'}|u_n|^p|T_h(u_n)|d(|u_n|)\,\mathrm{d}x\\ &= \frac{C_2}{h}(R^p\|b(\cdot)\|_{L^\infty(\mathbb{R})} + R^{p+1}\|b(\cdot)\|_{L^\infty(\mathbb{R})})\int_\Omega |c(x)|^{p'}\,\mathrm{d}x \to 0 \quad \text{as } h\to\infty.\end{aligned}$$

We have that $b(|s|)$ and $d(|s|)$ are two decreasing functions, and having in mind that $d(|\cdot|)/b(|\cdot|)$ is a decreasing function and belongs to $L^1(\mathbb{R})$, then there exist two positive constants μ and s_0 such that $d(|s|)/b(|s|) \leqslant \mu/|s|$ for any $|s| \geqslant s_0$. Then $|T_h(s)|^{1/p}d(|s|)^{1/p} \leqslant \mu^{1/p}b(|s|)^{1/p} \in L^1(\mathbb{R})$ and we obtain

$$\begin{aligned}&|s||T_h(s)|^{1/p}d(|s|)^{1/p}\cdot\chi_{\{|s|>s_0\}}\\ &\quad\leqslant \mu^{1/p}|s|b(|s|)^{1/p}\cdot\chi_{\{|s|\leqslant h\}} + |s||T_h(s)|^{1/p}d(|s|)^{1/p}\cdot\chi_{\{|s|>h\}}\\ &\quad\leqslant \mu^{1/p}\int_0^{|T_h(s)|} b(\tau)^{1/p}\,\mathrm{d}\tau + \int_0^{|s|}|T_h(s)|^{1/p}d(|s|)^{1/p}\cdot\chi_{\{|s|>h\}}\,\mathrm{d}\tau.\end{aligned}$$

This follows that there exists a positive constant C_3 such that

$$|s||T_h(s)|^{1/p}d(|s|)^{1/p} \leqslant C_3 + \mu^{1/p}\int_0^{|T_h(s)|} b(\tau)^{1/p}\,\mathrm{d}\tau + \int_0^{|s|}|T_h(\tau)|^{1/p}d(\tau)^{1/p}\,\mathrm{d}\tau. \tag{4.16}$$

Thanks to (4.15), we conclude that

$$\begin{aligned}&\frac{1}{2h}\int_{\{|u_n|\leqslant h\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x + \frac{1}{2h}\int_\Omega d(|u_n|)|\nabla u_n|^p|T_h(u_n)|\,\mathrm{d}x\\ &\quad\leqslant \varepsilon_4(h) + \frac{C_4}{h}\|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})}\bigg(\bigg\|\int_0^{|u_n|} b(\tau)^{1/p}\,\mathrm{d}\tau\bigg\|^p_{L^{p^*}(\{|u_n|\leqslant h\})}\\ &\qquad + \bigg\|\int_0^{|u_n|}|T_h(\tau)|^{1/p}d(\tau)^{1/p}\,\mathrm{d}\tau\bigg\|^p_{L^{p^*}(\Omega)} + 1\bigg)\\ &\quad\leqslant \varepsilon_5(h) + \frac{C_s^pC_4}{h}\|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})}\\ &\qquad\times\bigg(\bigg\|\nabla\int_0^{|u_n|} b(\tau)^{1/p}\,\mathrm{d}\tau\bigg\|^p_{L^p(\{|u_n|\leqslant h\})} + \bigg\|\nabla\int_0^{|u_n|}|T_h(\tau)|^{1/p}\,\mathrm{d}(\tau)^{1/p}\,\mathrm{d}\tau\bigg\|^p_{L^p(\Omega)}\bigg)\\ &\quad= \varepsilon_5(h) + \frac{C_s^pC_4}{h}\|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})}\\ &\qquad\times\bigg(\int_\Omega b(|u_n|)|\nabla T_h(u_n)|^p\,\mathrm{d}x + \int_\Omega d(|u_n|)|\nabla u_n|^p|T_h(u_n)|\,\mathrm{d}x\bigg).\end{aligned} \tag{4.17}$$

By taking R large enough such that $C_s^p C_4 \|c(x)\|^{p'}_{L^{N/(p-1)}(\{R<|u_n|\})} \leqslant \frac{1}{4}$, we conclude that

(4.18)
$$\frac{1}{4h}\int_{\{|u_n|\leqslant h\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x + \frac{1}{4h}\int_\Omega d(|u_n|)|\nabla u_n|^p|T_h(u_n)|\,\mathrm{d}x \leqslant \varepsilon_5(h)$$

for any $n\in\mathbb{N}^*$. Thus, by letting h tend to infinity in (4.18), we conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\leqslant h\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x = 0. \tag{4.19}$$

Moreover, we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_\Omega d(|u_n|)|\nabla u_n|^p|T_h(u_n)|\,\mathrm{d}x = 0, \tag{4.20}$$

then

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}} d(|u_n|)|\nabla u_n|^p\,\mathrm{d}x = 0. \tag{4.21}$$

Step 4: Convergence of the gradients. Let $h\geqslant k\geqslant 1$, we set

$$S_h(s) = 1-\frac{|T_{2h}(s)-T_h(s)|}{h} \quad\text{and}\quad H(s) = 2\int_0^s \frac{d(|\tau|)}{b(|\tau|)}\,\mathrm{d}\tau. \tag{4.22}$$

Let $\varphi(s) = s\exp(\frac{1}{2}\gamma^2 s^2)$, where $\gamma = \frac{7}{2}\|d(|\cdot|)/b(|\cdot|)\|_{L^\infty(\mathbb{R})}$. It is obvious that

$$\varphi'(s) - \gamma|\varphi(s)| \geqslant \frac{1}{2} \quad \text{for all } s\in\mathbb{R}.$$

By taking $\varphi(T_k(u_n)-T_k(u))S_h(u_n)\mathrm{e}^{H(|u_n|)} \in W_0^{1,p}(\Omega)$ as a test function in the approximate problem (4.1), we obtain

$$\begin{aligned}
&\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(\varphi(T_k(u_n)-T_k(u))S_h(u_n)\mathrm{e}^{H(|u_n|)})\,\mathrm{d}x\\
&\qquad + \int_\Omega g_n(x,u_n,\nabla u_n)(\varphi(T_k(u_n)-T_k(u))S_h(u_n)\mathrm{e}^{H(|u_n|)})\,\mathrm{d}x\\
&\quad = \int_\Omega f_n(\varphi(T_k(u_n)-T_k(u))S_h(u_n)\mathrm{e}^{H(|u_n|)})\,\mathrm{d}x\\
&\qquad + \int_\Omega \phi(x,u_n)\cdot\nabla(\varphi(T_k(u_n)-T_k(u))S_h(u_n)\mathrm{e}^{H(|u_n|)})\,\mathrm{d}x.
\end{aligned}$$

We have $S_h(u_n) = 1$ on the set $\{|u_n| \leqslant h\}$, and since $\varphi(T_k(u_n) - T_k(u))$ has the same sign as u_n on the set $\{|u_n| \geqslant k\}$, in view of assumptions (2.4)–(2.6), we conclude that

(4.23)
$$\begin{aligned}
&\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot (\nabla T_k(u_n) - \nabla T_k(u))\varphi'(T_k(u_n) - T_k(u))S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad - 2\int_{\{|u_n|\leqslant k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \frac{d(|u_n|)}{b(|u_n|)} \\
&\quad \times |\varphi(T_k(u_n) - T_k(u))|S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + 2\int_{\{|u_n|>k\}} d(|u_n|)|\nabla u_n|^p|\varphi(T_k(u_n) - T_k(u))|S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\leqslant \mathrm{e}^{H(\infty)}\int_\Omega (|f_0| + |f_n|)|\varphi(T_k(u_n) - T_k(u))|S_h(u_n)\,\mathrm{d}x \\
&\quad + \int_\Omega d(u_n)|\nabla u_n|^p|\varphi(T_k(u_n) - T_k(u))|S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + \frac{1}{h}\int_{\{h<|u_n|\leqslant 2h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n|\varphi(T_k(u_n) - T_k(u))|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad - \frac{1}{h}\int_{\{h<|u_n|\leqslant 2h\}} c(x)(1 + |u_n|)^\alpha|\nabla u_n||\varphi(T_k(u_n) - T_k(u))|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + \int_\Omega c(x)(1 + |u_n|)^\alpha|\nabla T_k(u_n) - \nabla T_k(u)|\varphi'(T_k(u_n) - T_k(u))S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + 2\int_{\{|u_n|\leqslant 2h\}} c(x)(1 + |u_n|)^\alpha|\nabla u_n|\frac{d(|u_n|)}{b(|u_n|)} \\
&\quad \times |\varphi(T_k(u_n) - T_k(u))|S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x.
\end{aligned}$$

For the first term on the right-hand side of (4.23) we have $\varphi(T_k(u_n) - T_k(u)) \rightharpoonup 0$ weak-$*$ in $L^\infty(\Omega)$, and since f_n converges strongly to f in $L^1(\Omega)$ as n goes to infinity, we obtain

$$\begin{aligned}
\text{(4.24)}\qquad \varepsilon_1(n) &= \left|\int_\Omega (|f_0| + |f_n|)|\varphi(T_k(u_n) - T_k(u))|S_h(u_n)\,\mathrm{d}x\right| \\
&\leqslant \int_\Omega (|f_0| + |f_n|)|\varphi(T_k(u_n) - T_k(u))|\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Concerning the third and fourth terms on the right-hand side of (4.23), in view of (4.19), we have

(4.25)
$$\begin{aligned}
\varepsilon_2(n) &= \frac{1}{h}\int_{\{h<|u_n|\leqslant 2h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n|\varphi(T_k(u_n) - T_k(u))|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\leqslant \frac{\varphi(2k)\mathrm{e}^{H(\infty)}}{h}\int_{\{h<|u_n|\leqslant 2h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

and thanks to Young's inequality, since $b(s)$ is a decreasing function, and similarly as in (4.17), we can show that

$$
\begin{aligned}
(4.26)\quad \varepsilon_3(h) &= \left|\frac{1}{h}\int_{\{h<|u_n|\leqslant 2h\}} c(x)(1+|u_n|)^{\alpha}|\nabla u_n||\varphi(T_k(u_n)-T_k(u))|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\right| \\
&\leqslant \frac{\mathrm{e}^{H(\infty)}}{h}\int_{\{h<|u_n|\leqslant 2h\}} |c(x)|^{p'}\frac{(1+|u_n|)^{\alpha p'}}{b(|u_n|)^{p'-1}}|\varphi(T_k(u_n)-T_k(u))|\,\mathrm{d}x \\
&\quad + \frac{\mathrm{e}^{H(\infty)}}{h}\int_{\{h<|u_n|\leqslant 2h\}} b(|u_n|)|\nabla u_n|^p|\varphi(T_k(u_n)-T_k(u))|\,\mathrm{d}x \\
&\leqslant \frac{\varphi(2k)\mathrm{e}^{H(\infty)}}{h}\int_{\{h<|u_n|\leqslant 2h\}} |c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'}b(|u_n|)\,\mathrm{d}x \\
&\quad + \frac{\varphi(2k)\mathrm{e}^{H(\infty)}}{h}\int_{\{h<|u_n|\leqslant 2h\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x \\
&\leqslant \frac{2^{p-1}\varphi(2k)k\mathrm{e}^{H(\infty)}}{h}\int_{\{|u_n|\leqslant 2h\}} |c(x)|^{p'}|u_n|^p b(|u_n|)\,\mathrm{d}x + \varepsilon_4(h) \\
&\leqslant \frac{2^{p-1}\varphi(2k)k\mathrm{e}^{H(\infty)}}{h}\|c(x)\|^{p'}_{L^{N/(p-1)}(\Omega)} \\
&\quad \times \int_{\Omega} b(|u_n|)|\nabla T_{2h}(u_n)|^p\,\mathrm{d}x + \varepsilon_4(h) \to 0 \quad \text{as } h\to\infty.
\end{aligned}
$$

For the fifth term on the right-hand side of (4.23), since $1 \leqslant \varphi'(T_k(u_n)-T_k(u)) \leqslant \varphi'(2k)$, and we have that $(1+|T_{2h}(u_n)|)^{\alpha}$ converges strongly to $(1+|T_{2h}(u)|)^{\alpha}$ in $L^{Np'/(N-p)}(\Omega)$ and $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ in $L^p(\Omega)$, it follows that

$$
\begin{aligned}
(4.27)\qquad \varepsilon_5(n) &= \left|\int_{\Omega} c(x)(1+|u_n|)^{\alpha}|\nabla T_k(u_n)-\nabla T_k(u)| \right.\\
&\qquad \left. \times\,\varphi'(T_k(u_n)-T_k(u))S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\right| \\
&\leqslant \mathrm{e}^{H(\infty)}\varphi'(2k)\int_{\Omega}|c(x)|(1+|T_{2h}(u_n)|)^{\alpha} \\
&\qquad \times|\nabla T_k(u_n)-\nabla T_k(u)|\,\mathrm{d}x \to 0 \quad \text{as } h\to\infty.
\end{aligned}
$$

Concerning the last term on the right-hand side of (4.23), we have

$$
\begin{aligned}
(4.28)\ &\int_{\{|u_n|\leqslant 2h\}} c(x)(1+|u_n|)^{\alpha}|\nabla u_n|\frac{d(|u_n|)}{b(|u_n|)}|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\leqslant \frac{1}{4}\int_{\{|u_n|\leqslant 2h\}} d(|u_n|)|\nabla u_n|^p|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + \int_{\{|u_n|\leqslant 2h\}} |c(x)|^{p'}(1+|u_n|)^{\alpha p'}\frac{d(|u_n|)}{b(|u_n|)^{p'}} \\
&\quad \times|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x
\end{aligned}
$$

$$\leqslant \frac{1}{4}\int_{\{|u_n|\leqslant 2h\}} d(|u_n|)|\nabla u_n|^p|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x$$
$$+\mathrm{e}^{H(\infty)}\int_{\{|u_n|\leqslant 2h\}} |c(x)|^{p'}(1+|T_{2h}(u_n)|)^p d(|u_n|)|\varphi(T_k(u_n)-T_k(u))|\,\mathrm{d}x$$
$$\leqslant \frac{1}{4}\int_{\{|u_n|\leqslant 2h\}} d(|u_n|)|\nabla u_n|^p|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x+\varepsilon_6(n)$$

with

$$\varepsilon_6(n)=\mathrm{e}^{H(\infty)}\int_{\{|u_n|\leqslant 2h\}} |c(x)|^{p'}(1+|T_{2h}(u_n)|)^p d(|u_n|)|\varphi(T_k(u_n)-T_k(u))|\,\mathrm{d}x\to 0$$

as $n\to\infty$. By combining (4.23) and (4.24)–(4.28), we conclude that

(4.29)
$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(T_k(u_n)-T_k(u))S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x$$
$$+\frac{1}{2}\int_{\{|u_n|>k\}} d(|u_n|)|\nabla u_n|^p|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x$$
$$-\frac{7}{2}\int_{\{|u_n|\leqslant k\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\frac{d(|u_n|)}{b(|u_n|)}|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x$$
$$\leqslant \varepsilon_7(n,h).$$

For the first term on the left-hand side of (4.29), we have $a(x,T_k(u_n),\nabla T_k(u_n))=0$ on the set $\{k<|u_n|\}$. Then

(4.30)
$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(T_k(u_n)-T_k(u))S_h(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x$$
$$=\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u))$$
$$\times\varphi'(T_k(u_n)-T_k(u))\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x$$
$$-\int_{\{k<|u_n|\leqslant 2h\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u)\varphi'(T_k(u_n)-T_k(u))\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x$$
$$=\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))$$
$$\times\varphi'(T_k(u_n)-T_k(u))\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x$$
$$+\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))$$
$$\times\varphi'(T_k(u_n)-T_k(u))\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x$$
$$-\int_{\{k<|u_n|\leqslant 2h\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u)\varphi'(T_k(u_n)-T_k(u))\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x.$$

For the second term on the right-hand side of (4.30), we have $a(x, T_k(u_n), \nabla T_k(u)) \to a(x, T_k(u), \nabla T_k(u))$ strongly in $(L^{p'}(\Omega))^N$, and since $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L^p(\Omega))^N$,

$$
\begin{aligned}
\varepsilon_8(n) &= \bigg| \int_\Omega a(x, T_k(u_n), \nabla T_k(u)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \\
&\qquad \times \varphi'(T_k(u_n) - T_k(u)) \mathrm{e}^{H(|u_n|)} \, \mathrm{d}x \bigg| \\
&\leqslant \mathrm{e}^{H(\infty)} \varphi'(2k) \int_\Omega |a(x, T_k(u_n), \nabla T_k(u))| \\
&\qquad \times |\nabla T_k(u_n) - \nabla T_k(u)| \, \mathrm{d}x \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{4.31}
$$

Concerning the last term on the right-hand side of (4.30), we have that $(|a(x, T_{2h}(u_n), \nabla T_{2h}(u_n))|)_n$ is bounded in $L^{p'}(\Omega)$. Then there exists a function $\psi_{2h} \in L^{p'}(\Omega)$ such that $|a(x, T_{2h}(u_n), \nabla T_{2h}(u_n))| \rightharpoonup \psi_{2h}$ weakly in $L^{p'}(\Omega)$, which yields that

$$
\begin{aligned}
\varepsilon_8(n) &= \bigg| \int_{\{k<|u_n|\leqslant 2h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u) \varphi'(T_k(u_n) - T_k(u)) \mathrm{e}^{H(|u_n|)} \, \mathrm{d}x \bigg| \\
&\leqslant \mathrm{e}^{H(\infty)} \varphi'(2k) \int_{\{k<|u_n|\leqslant 2h\}} |a(x, T_{2h}(u_n), \nabla T_{2h}(u_n))| |\nabla T_k(u)| \, \mathrm{d}x \\
&\to \mathrm{e}^{H(\infty)} \varphi'(2k) \int_{\{k<|u|\leqslant 2h\}} \psi_{2h} |\nabla T_k(u)| \, \mathrm{d}x = 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{4.32}
$$

By combining (4.30)–(4.32), we conclude that

$$
\begin{aligned}
&\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) S_h(u_n) \varphi'(T_k(u_n) - T_k(u)) \mathrm{e}^{H(|u_n|)} \, \mathrm{d}x \\
&\quad = \int_\Omega (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \\
&\qquad \times (\nabla T_k(u_n) - \nabla T_k(u)) \varphi'(T_k(u_n) - T_k(u)) \mathrm{e}^{H(|u_n|)} \, \mathrm{d}x + \varepsilon_9(n).
\end{aligned} \tag{4.33}
$$

Similarly, we can show that

$$
\begin{aligned}
&\int_{\{|u_n|\leqslant k\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n \frac{d(|u_n|)}{b(|u_n|)} |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) \mathrm{e}^{H(|u_n|)} \, \mathrm{d}x \\
&\qquad \leqslant \int_\Omega (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \\
&\qquad\quad \times \bigg\| \frac{d(|\cdot|)}{b(|\cdot|)} \bigg\|_{L^\infty(\mathbb{R})} |\varphi(T_k(u_n) - T_k(u))| \mathrm{e}^{H(|u_n|)} \, \mathrm{d}x + \varepsilon_{10}(n).
\end{aligned} \tag{4.34}
$$

We have $\varphi'(s)-\gamma|\varphi(s)|\geqslant\frac{1}{2}$ for any $s\in\mathbb{R}$, thus, by combining (4.29), (4.33) and (4.34) we conclude that

(4.35)
$$\begin{aligned}0&\leqslant\frac{1}{2}\int_\Omega(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,\mathrm{d}x\\&\leqslant\int_\Omega(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\\&\quad\times\Big(\varphi'(T_k(u_n)-T_k(u))-\frac{7}{2}\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(\mathbb{R})}|\varphi(T_k(u_n)-T_k(u))|\Big)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x\\&\quad+\varepsilon_{11}(n)\\&\leqslant\varepsilon_7(n,h).\end{aligned}$$

By letting n and h tend to infinity, we obtain

(4.36)
$$\lim_{n\to\infty}\int_\Omega(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,\mathrm{d}x=0.$$

Under assumptions (2.2)–(2.4), it is well known that this implies

$$T_k(u_n)\to T_k(u)\text{ in }W_0^{1,p}(\Omega)\quad\text{and}\quad\nabla u_n\to\nabla u\text{ a.e. in }\Omega.\tag{4.37}$$

Moreover, since $a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n$ tends to $a(x,u,\nabla u)\cdot\nabla u$ almost everywhere in Ω, and in view of Fatou's lemma and (4.19), we conclude that

$$\begin{aligned}\lim_{h\to\infty}\frac{1}{h}\int_{\{|u|\leqslant h\}}&a(x,u,\nabla u)\cdot\nabla u\,\mathrm{d}x\\&\leqslant\lim_{h\to\infty}\liminf_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\leqslant h\}}a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x\\&\leqslant\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\leqslant h\}}a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x=0,\end{aligned}\tag{4.38}$$

which proves (3.1).

Step 5: The equi-integrability of $g_n(x,u_n,\nabla u_n)$. To prove that

$$g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u)\quad\text{strongly in }L^1(\Omega),$$

using Vitali's theorem, it is sufficient to show that the sequence $(g_n(x,u_n,\nabla u_n))_n$ is uniformly equi-integrable. Indeed, thanks to (4.21), we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}}d(|u_n|)|\nabla u_n|^p\,\mathrm{d}x=0.\tag{4.39}$$

Having in mind that $f_0 \in L^1(\Omega)$, we conclude that

$$\int_{\{|u_n|>h\}} |g(x,u_n,\nabla u_n)|\,\mathrm{d}x \leqslant \int_{\{|u_n|>h\}} |f_0|\,\mathrm{d}x + \int_{\{|u_n|>h\}} d(|u_n|)|\nabla u_n|^p\,\mathrm{d}x \to 0$$

as $h \to \infty$, thus, for all $\varepsilon > 0$, there exists $h_0(\varepsilon) > 0$ such that

$$\int_{\{|u_n|>h\}} |g(x,u_n,\nabla u_n)|\,\mathrm{d}x \leqslant \frac{\varepsilon}{2} \quad \text{for any } h \geqslant h_0(\varepsilon). \tag{4.40}$$

On the other hand, for any measurable subset $E \subset \Omega$ we have

$$\begin{aligned} \int_E |g_n(x,u_n,\nabla u_n)|\,\mathrm{d}x \leqslant \int_E &|g_n(x,T_h(u_n),\nabla T_h(u_n))|\,\mathrm{d}x \\ &+ \int_{\{|u_n|>h\}} |g(x,u_n,\nabla u_n)|\,\mathrm{d}x. \end{aligned} \tag{4.41}$$

Thanks to (4.37), there exists $\beta(\varepsilon) > 0$ small enough such that

$$\int_E |g_n(x,T_h(u_n),\nabla T_h(u_n))|\,\mathrm{d}x \leqslant \int_E |f_0(x)|\,\mathrm{d}x + \int_E d(|T_h(u_n)|)|\nabla T_h(u_n)|^p\,\mathrm{d}x \leqslant \frac{\varepsilon}{2}. \tag{4.42}$$

By combining (4.40), (4.41) and (4.42), we deduce that for any $\varepsilon > 0$ there exists $\beta(\varepsilon) > 0$ such that

$$\int_E |g_n(x,u_n,\nabla u_n)|\,\mathrm{d}x \leqslant \varepsilon \quad \text{with } E \subseteq \Omega \text{ such that } \operatorname{meas}(E) \leqslant \beta(\varepsilon). \tag{4.43}$$

We conclude that the sequence $(g_n(x,u_n,\nabla u_n))_n$ is uniformly equi-integrable, and thanks to (4.37), we have

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{a.e. in } \Omega. \tag{4.44}$$

Thus, in view of Vitali's theorem, we obtain

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega). \tag{4.45}$$

Step 6: Passage to the limit. Let $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and let $S(\cdot)$ be a smooth function in $W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp}(S(\cdot)) \subseteq [-M, M]$ for some $M \geqslant 0$.

By choosing $S(u_n)\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as a test function in the approximate problem (4.1), we obtain

$$\begin{aligned} &\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot(\nabla u_n S'(u_n)\varphi + S(u_n)\nabla\varphi)\,\mathrm{d}x + \int_\Omega g_n(x,u_n,\nabla u_n)S(u_n)\varphi\,\mathrm{d}x \\ &\quad = \int_\Omega f_n S(u_n)\varphi\,\mathrm{d}x + \int_\Omega \phi(x,T_n(u_n))\cdot(\nabla u_n S'(u_n)\varphi + S(u_n)\nabla\varphi)\,\mathrm{d}x. \end{aligned} \tag{4.46}$$

We begin by the first term on the left-hand side of (4.46). We have

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot (\nabla u_n S'(u_n)\varphi + S(u_n)\nabla\varphi)\,\mathrm{d}x$$
$$= \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n)) \cdot (S'(u_n)\varphi\nabla T_M(u_n) + S(T_M(u_n))\nabla\varphi)\,\mathrm{d}x.$$

In view of (2.3), we have that $(a(x, T_M(u_n), \nabla T_M(u_n)))_n$ is bounded in $(L^{p'}(\Omega))^N$, and since $a(x, T_M(u_n), \nabla T_M(u_n))$ tends to $a(x, T_M(u), \nabla T_M(u))$ almost everywhere in Ω, it follows that

$$a(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup a(x, T_M(u), \nabla T_M(u)) \quad \text{in } (L^{p'}(\Omega))^N,$$

and since $S'(u_n)\varphi\nabla T_M(u_n) + S(T_M(u_n))\nabla\varphi$ tends strongly to $S'(u)\varphi\nabla T_M(u) + S(T_M(u))\nabla\varphi$ in $(L^p(\Omega))^N$, we deduce that

(4.47)
$$\lim_{n\to\infty} \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot (\nabla u_n S'(u_n)\varphi + S(u_n)\nabla\varphi)\,\mathrm{d}x$$
$$= \lim_{n\to\infty} \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n)) \cdot (\nabla T_M(u_n) S'(T_M(u_n))\varphi + S(T_M(u_n))\nabla\varphi)\,\mathrm{d}x$$
$$= \int_\Omega a(x, T_M(u), \nabla T_M(u)) \cdot (\nabla T_M(u) S'(T_M(u))\varphi + S(T_M(u))\nabla\varphi)\,\mathrm{d}x$$
$$= \int_\Omega a(x, u, \nabla u) \cdot (\nabla u S'(u)\varphi + S(u)\nabla\varphi)\,\mathrm{d}x.$$

Concerning the second term on the right-hand side of (4.46), we have $S(T_M(u_n))\varphi \rightharpoonup S(T_M(u))\varphi$ weak-$*$ in $L^\infty(\Omega)$, and thanks to (4.45), we have that $g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u)$ strongly in $L^1(\Omega)$, which yields that

$$\lim_{n\to\infty} \int_\Omega g_n(x, u_n, \nabla u_n) S(T_M(u_n))\varphi\,\mathrm{d}x = \int_\Omega g(x, u, \nabla u) S(T_M(u))\varphi\,\mathrm{d}x = \int_\Omega g(x, u, \nabla u) S(u)\varphi\,\mathrm{d}x. \tag{4.48}$$

Similarly, we have $f_n \to f$ strongly in $L^1(\Omega)$, then

$$\lim_{n\to\infty} \int_\Omega f_n S(T_M(u_n))\varphi\,\mathrm{d}x = \int_\Omega f S(T_M(u))\varphi\,\mathrm{d}x = \int_\Omega f S(u)\varphi\,\mathrm{d}x. \tag{4.49}$$

For the last term on the right-hand side of (4.46), we have $\phi_n(x, u_n)S(u_n) = \phi(x, T_M(u_n))S(T_M(u_n))$ for n large enough (for example $n \geqslant M$), and since

$\phi(x, T_M(u_n)) \to \phi(x, T_M(u))$ strongly in $(L^{p'}(\Omega))^N$, we obtain

(4.50)

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \phi_n(x,u_n)\cdot(\nabla u_n S'(u_n)\varphi + S(u_n)\nabla\varphi)\,\mathrm{d}x \\ = \lim_{n\to\infty}\int_\Omega \phi(x,T_M(u_n))\cdot(\nabla T_M(u_n)S'(T_M(u_n))\varphi + S(T_M(u_n))\nabla\varphi)\,\mathrm{d}x \\ = \int_\Omega \phi(x,T_M(u))\cdot(\nabla T_M(u)S'(T_M(u))\varphi + S(T_M(u))\nabla\varphi)\,\mathrm{d}x \\ = \int_\Omega \phi(x,u)\cdot(\nabla u S'(u)\varphi + S(u)\nabla\varphi)\,\mathrm{d}x.\end{aligned}$$

By combining (4.46)–(4.50), we conclude that

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\cdot(S'(u)\varphi\nabla u + S(u)\nabla\varphi)\,\mathrm{d}x + \int_\Omega g(x,u,\nabla u)S(u)\varphi\,\mathrm{d}x \\ = \int_\Omega fS(u)\varphi\,\mathrm{d}x + \int_\Omega \phi(x,u)\cdot(S'(u)\varphi\cdot\nabla u + S(u)\nabla\varphi)\,\mathrm{d}x,\end{aligned} \tag{4.51}$$

which completes the proof of Theorem 3.1.

5. Appendix: Proof of Proposition 4.1

Case 1. Assuming that there exist two positive constants d_0 and s_0 such that

$$d(|s|) \leqslant \frac{d_0}{(1+|s|)^p} \quad \text{for any } s \geqslant s_0, \tag{5.1}$$

we have

$$\lim_{k\to\infty}\limsup_{n\to\infty} \operatorname{meas}(\{|u_n| > k\}) = 0.$$

Proof of Case 1. Let $k \geqslant 1$ and n large enough. We consider the function $\psi(\cdot)$ defined by

$$\psi(s) = \frac{1}{p-\lambda-1}\Big(1 - \frac{1}{(1+|T_k(s)|)^{p-\lambda-1}}\Big)\operatorname{sign}(s), \tag{5.2}$$

where $\lambda < p-1$ and $H(s) = 2\int_0^s d(|\tau|)/b(|\tau|)\,\mathrm{d}\tau$. By taking $\psi(u_n)\mathrm{e}^{H(|u_n|)} \in W_0^{1,p}(\Omega)$ as a test function in the approximate problem (4.1), we have

(5.3)

$$\begin{aligned}\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(\psi(u_n)\mathrm{e}^{H(|u_n|)})\,\mathrm{d}x + \int_\Omega g_n(x,u_n,\nabla u_n)\psi(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\ = \int_\Omega f_n\psi(u_n)\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x + \int_\Omega \phi(x,T_n(u_n))\cdot\nabla(\psi(u_n)\mathrm{e}^{H(|u_n|)})\,\mathrm{d}x.\end{aligned}$$

In view of (2.4), (2.5) and (2.6), we conclude that

$$
\begin{aligned}
(5.4)\quad & \int_{\{|u_n|\leqslant k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1+|T_k(u_n)|)^{p-\lambda}} \mathrm{e}^{H(|u_n|)}\,\mathrm{d}x + 2\int_\Omega d(|u_n|)|\nabla u_n|^p|\psi(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\leqslant \int_{\{|u_n|\leqslant k\}} \frac{a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n}{(1+|T_k(u_n)|)^{p-\lambda}} \mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + 2\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n|\psi(u_n)|\frac{d(|u_n|)}{b(|u_n|)}|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\leqslant \int_\Omega (|f_n(x)|+|f_0(x)|)|\psi(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + \int_\Omega d(|u_n|)|\nabla u_n|^p|\psi(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + \int_{\{|u_n|\leqslant k\}} \frac{c(x)(1+|u_n|)^\alpha|\nabla T_k(u_n)|}{(1+|T_k(u_n)|)^{p-\lambda}} \mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + 2\int_\Omega c(x)(1+|u_n|)^\alpha|\nabla u_n|\frac{d(|u_n|)}{b(|u_n|)}|\psi(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x.
\end{aligned}
$$

Thanks to Young's inequality, we obtain

$$
\begin{aligned}
(5.5)\quad & \int_{\{|u_n|\leqslant k\}} \frac{c(x)(1+|u_n|)^\alpha|\nabla T_k(u_n)|}{(1+|T_k(u_n)|)^{p-\lambda}} \mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\leqslant \frac{1}{2}\int_{\{|u_n|\leqslant k\}} \frac{b(|u_n|)|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^{p-\lambda}} \mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + C_0\int_{\{|u_n|\leqslant k\}} \frac{|c(x)|^{p'}(1+|u_n|)^{\alpha p'}}{(1+|T_k(u_n)|)^{p-\lambda}b(|u_n|)^{p'-1}} \mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\leqslant \frac{1}{2}\int_{\{|u_n|\leqslant k\}} \frac{b(|u_n|)|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^{p-\lambda}} \mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + C_1\int_{\{|u_n|\leqslant k\}} \frac{|c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'-\lambda}}{(1+|u_n|)^{p-\lambda}} \mathrm{e}^{H(|u_n|)}\,\mathrm{d}x,
\end{aligned}
$$

and thanks to (2.4), we have

$$
\begin{aligned}
(5.6)\quad & \int_\Omega c(x)(1+|u_n|)^\alpha|\nabla u_n|\frac{d(|u_n|)}{b(|u_n|)}|\psi(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\leqslant C_2\int_\Omega |c(x)|^{p'}(1+|u_n|)^{\alpha p'}\frac{d(|u_n|)}{b(|u_n|)^{p'}}|\psi(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega d(|u_n|)|\nabla u_n|^p|\psi(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\leqslant C_3\int_\Omega d(|u_n|)|c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'}|\psi(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega d(|u_n|)|\nabla u_n|^p|\psi(u_n)|\mathrm{e}^{H(|u_n|)}\,\mathrm{d}x.
\end{aligned}
$$

By combining (5.4), (5.5) and (5.6), we conclude that

$$
\begin{aligned}
(5.7)\qquad & \frac{1}{2}\int_{\{|u_n|\leqslant k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1+|T_k(u_n)|)^{p-\lambda}}\,\mathrm{d}x + \frac{1}{2}\int_\Omega d(|u_n|)|\nabla u_n|^p|\psi(u_n)|\,\mathrm{d}x \\
&\leqslant C_4(\|f\|_{L^1(\Omega)} + \|f_0\|_{L^1(\Omega)}) \\
&\quad + C_1\mathrm{e}^{H(\infty)}\int_{\{|u_n|\leqslant k\}} \frac{|c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'-\lambda}}{(1+|u_n|)^{p-\lambda}}\,\mathrm{d}x \\
&\quad + C_3\mathrm{e}^{H(\infty)}\int_\Omega d(|u_n|)|c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'}|\psi(u_n)|\,\mathrm{d}x \\
&\leqslant C_4(\|f\|_{L^1(\Omega)} + \|f_0\|_{L^1(\Omega)}) + C_1\mathrm{e}^{H(\infty)}\int_{\{|u_n|\leqslant k\}} |c(x)|^{p'}\,\mathrm{d}x \\
&\quad + C_3\mathrm{e}^{H(\infty)}\int_\Omega d(|u_n|)|c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'}|\psi(u_n)|\,\mathrm{d}x \\
&\leqslant C_5 + C_3\mathrm{e}^{H(\infty)}\int_\Omega d(|u_n|)|c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'}|\psi(u_n)|\,\mathrm{d}x.
\end{aligned}
$$

In view of (5.1) we have $d(|s|)(1+|s|)^p \leqslant d_0$ for any $s \geqslant s_0$, which yields that

$$
\begin{aligned}
(5.8)\qquad & \frac{1}{2}\int_{\{|u_n|\leqslant k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1+|T_k(u_n)|)^{p-\lambda}}\,\mathrm{d}x + \frac{1}{2}\int_\Omega d(|u_n|)|\nabla u_n|^p|\psi(u_n)|\,\mathrm{d}x \\
&\leqslant C_5 + C_3 d_0\mathrm{e}^{H(\infty)}\int_\Omega |c(x)|^{p'}\,\mathrm{d}x \leqslant C_6.
\end{aligned}
$$

In view of (2.4), we conclude that

$$
(5.9)\qquad b_0\int_{\{|u_n|\leqslant k\}} \frac{|\nabla u_n|^p}{(1+|T_k(u_n)|)^p}\,\mathrm{d}x \leqslant \int_{\{|u_n|\leqslant k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1+|T_k(u_n)|)^{p-\lambda}}\,\mathrm{d}x \leqslant C_7.
$$

Using the Poincaré inequality, we get

$$
\begin{aligned}
|\log(1+k)|^p \mathrm{meas}(\{|u_n|>k\}) &= \int_{\{|u_n|>k\}} |\log(1+|T_k(u_n)|)|^p\,\mathrm{d}x \\
&\leqslant \int_\Omega |\log(1+|T_k(u_n)|)|^p\,\mathrm{d}x \\
&\leqslant C_p^p\int_\Omega |\nabla\log(1+|T_k(u_n)|)|^p\,\mathrm{d}x \\
&= C_p^p\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^p}\,\mathrm{d}x \\
&\leqslant C_8
\end{aligned}
$$

with C_8 being a positive constant that does not depend on n and k. Thus,

$$
(5.10)\qquad \limsup_{n\to\infty} \mathrm{meas}(\{|u_n|>k\}) \leqslant \frac{C_8}{|\log(1+k)|^p} \to 0 \quad \text{as } k\to\infty,
$$

which concludes the proof of Case 1. □

Case 2. We assume that $0 \leqslant \alpha < p-1-\lambda$. Then

$$\lim_{k\to\infty} \limsup_{n\to\infty} \operatorname{meas}(\{|u_n| > k\}) = 0.$$

Proof of Case 2. By taking $\psi(u_n)\mathrm{e}^{H(|u_n|)} \in W_0^{1,p}(\Omega)$ as a test function in the approximate problem (4.1) and similarly as in (5.7) we can prove that

$$\tag{5.11} \begin{aligned} \frac{1}{2}\int_{\{|u_n|\leqslant k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1+|T_k(u_n)|)^{p-\lambda}}\,\mathrm{d}x + \frac{1}{2}\int_\Omega d(|u_n|)|\nabla u_n|^p|\psi(u_n)|\,\mathrm{d}x \\ \leqslant C_5 + C_3\mathrm{e}^{H(\infty)}\int_\Omega d(|u_n|)|c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'}|\psi(u_n)|\,\mathrm{d}x. \end{aligned}$$

We have that $d(|\cdot|)$ is a decreasing function, then there exists a positive constant $r_0 > 0$ such that $1/r_0 \leqslant |\psi(u_n)| \leqslant 1/(p-\lambda-1)$ on the set $\{|u_n| \geqslant 1\}$, which yields that

$$\begin{aligned} |u_n| d(|u_n|)^{1/p}|\psi(u_n)|^{1/p} &\leqslant \frac{|u_n| d(|u_n|)^{1/p}}{(p-\lambda-1)^{1/p}} \leqslant \frac{1}{(p-\lambda-1)^{1/p}}\int_0^{|u_n|} d(s)^{1/p}\,\mathrm{d}s \\ &\leqslant \frac{1}{(p-\lambda-1)^{1/p}}\int_0^1 d(s)^{1/p}\,\mathrm{d}s \\ &\quad + \frac{r_0^{1/p}}{(p-\lambda-1)^{1/p}}\int_1^{|u_n|} d(s)^{1/p}|\psi(u_n)|^{1/p}\,\mathrm{d}s. \\ &= C_9 + C_{10}\int_0^{|u_n|} d(s)^{1/p}|\psi(u_n)|^{1/p}\,\mathrm{d}s. \end{aligned}$$

Having in mind that $0 \leqslant \alpha < p-1-\lambda$ let $\varepsilon > 0$ and by using Young's and Sobolev inequalities, we conclude that

$$\tag{5.12} \begin{aligned} &\frac{1}{2}\int_{\{|u_n|\leqslant k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1+|T_k(u_n)|)^{p-\lambda}}\,\mathrm{d}x + \frac{1}{2}\int_\Omega d(|u_n|)|\nabla u_n|^p|\psi(u_n)|\,\mathrm{d}x \\ &\quad\leqslant C_5 + C_{11}\int_\Omega d(|u_n|)|c(x)|^{p'}\,\mathrm{d}x + \varepsilon\int_\Omega d(|u_n|)|c(x)|^{p'}|u_n|^p|\psi(u_n)|\,\mathrm{d}x \\ &\quad\leqslant C_{12} + 2^{p-1}C_{10}^p\varepsilon\|c(x)\|^{p'}_{L^{N/(p-1)}(\Omega)}\bigg\|\int_0^{|u_n|} d(s)^{1/p}|\psi(s)|^{1/p}\,\mathrm{d}s\bigg\|^p_{L^{p^*}(\Omega)} \\ &\quad\leqslant C_{12} + 2^{p-1}C_{10}^p\varepsilon C_s^p\|c(x)\|^{p'}_{L^{N/(p-1)}(\Omega)}\bigg\|\nabla\int_0^{|u_n|} d(s)^{1/p}|\psi(s)|^{1/p}\,\mathrm{d}s\bigg\|^p_{L^p(\Omega)} \\ &\quad= C_{12} + 2^{p-1}C_{10}^p\varepsilon C_s^p\|c(x)\|^{p'}_{L^{N/(p-1)}(\Omega)}\int_\Omega d(|u_n|)|\nabla u_n|^p|\psi(u_n)|\,\mathrm{d}x. \end{aligned}$$

By taking $\varepsilon > 0$ small enough such that $2^{p-1}C_{10}^p\varepsilon C_s^p\|c(x)\|^{p'}_{L^{N/(p-1)}(\Omega)} \leqslant \frac{1}{4}$, we obtain

$$\tag{5.13} \frac{1}{2}\int_{\{|u_n|\leqslant k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1+|T_k(u_n)|)^{p-\lambda}}\,\mathrm{d}x + \frac{1}{4}\int_\Omega d(|u_n|)|\nabla u_n|^p|\psi(u_n)|\,\mathrm{d}x \leqslant C_{12}.$$

In view of (2.4), we conclude that

$$b_0 \int_{\{|u_n|\leqslant k\}} \frac{|\nabla u_n|^p}{(1+|T_k(u_n)|)^p}\,\mathrm{d}x \leqslant \int_{\{|u_n|\leqslant k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1+|T_k(u_n)|)^{p-\lambda}}\,\mathrm{d}x \leqslant C_{13}. \tag{5.14}$$

Following the same steps as in the previous case, we conclude that

$$\limsup_n \operatorname{meas}(\{|u_n|>k\}) \leqslant \frac{C_{14}}{|\log(1+k)|^p} \to 0 \quad \text{as } k\to\infty, \tag{5.15}$$

which conclude the proof of Case 2. □

Case 3. We assume that $\|c(x)\|_{L^{N/(p-1)}(\Omega)}$ is small enough. Then

$$\lim_{k\to\infty}\limsup_{n\to\infty} \operatorname{meas}(\{|u_n|>k\}) = 0.$$

Proof of Case 3. By taking $\psi(u_n)\mathrm{e}^{H(|u_n|)} \in W_0^{1,p}(\Omega)$ as a test function in the approximate problem (4.1) and similarly as in (5.7) we can prove that

$$\begin{aligned}\frac{1}{2}\int_{\{|u_n|\leqslant k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1+|T_k(u_n)|)^{p-\lambda}}\,\mathrm{d}x + \frac{1}{2}\int_\Omega d(|u_n|)|\nabla u_n|^p|\psi(u_n)|\,\mathrm{d}x \\ \leqslant C_5 + C_3\mathrm{e}^{H(\infty)}\int_\Omega d(|u_n|)|c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'}|\psi(u_n)|\,\mathrm{d}x\end{aligned} \tag{5.16}$$

with $C_3 = 2^{p'-1}/(p'p^{p'-1})$. For the last term on the right-hand side of (5.16), we have

$$\begin{aligned}&\int_\Omega d(|u_n|)|c(x)|^{p'}(1+|u_n|)^{(\alpha+\lambda)p'}|\psi(u_n)|\,\mathrm{d}x \\ &\quad\leqslant 2^{p-1}\int_\Omega d(|u_n|)|c(x)|^{p'}|\psi(u_n)|\,\mathrm{d}x + 2^{p-1}\int_\Omega d(|u_n|)|c(x)|^{p'}|u_n|^p|\psi(u_n)|\,\mathrm{d}x \\ &\quad\leqslant C_{15} + 2^{p-1}C_{10}^p\|c(x)\|^{p'}_{L^{N/(p-1)}(\Omega)}\bigg\|\int_0^{|u_n|} d(s)^{1/p}|\psi(s)|^{1/p}\,\mathrm{d}s\bigg\|^p_{L^{p^*}(\Omega)} \\ &\quad\leqslant C_{15} + 2^{p-1}C_{10}^pC_s^p\|c(x)\|^{p'}_{L^{N/(p-1)}(\Omega)}\bigg\|\nabla\int_0^{|u_n|} d(s)^{1/p}|\psi(s)|^{1/p}\,\mathrm{d}s\bigg\|^p_{L^{p}(\Omega)} \\ &\quad= C_{15} + 2^{p-1}C_{10}^pC_s^p\|c(x)\|^{p'}_{L^{N/(p-1)}(\Omega)}\int_\Omega d(|u_n|)|\nabla u_n|^p|\psi(u_n)|\,\mathrm{d}x\end{aligned} \tag{5.17}$$

with $C_{10}^p = r_0/(p-\lambda-1)$ and C_s being the constant of the Sobolev inequality. Choosing the measurable function $c(x)$ such that $\|c(x)\|_{L^{N/(p-1)}(\Omega)}$ is small enough, for example

$$\|c(x)\|_{L^{N/(p-1)}(\Omega)}^{p'} \leqslant \frac{1}{2^{p+1}C_{10}^p C_s^p C_3 \mathrm{e}^{H(\infty)}},$$

it follows that

$$(5.18) \qquad b_0 \int_{\{|u_n|\leqslant k\}} \frac{|\nabla u_n|^p}{(1+|T_k(u_n)|)^p}\,\mathrm{d}x \leqslant \int_{\{|u_n|\leqslant k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1+|T_k(u_n)|)^{p-\lambda}}\,\mathrm{d}x \leqslant C_{16}.$$

Using similar process as in the first case, we deduce that

$$(5.19) \qquad \limsup_{n\to\infty} \operatorname{meas}(\{|u_n|>k\}) \leqslant \frac{C_{17}}{|\log(1+k)|^p} \to 0 \quad \text{as } k\to\infty,$$

which proves Case 3. Thus, the proof of Proposition 4.1 is complete. □

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