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Abstract. We show that in set theory without the axiom of choice ZF, the statement sH: "Every proper closed subset of a finitary matroid is the intersection of hyperplanes including it" implies $\mathrm{AC}^{\mathrm{fin}}$, the axiom of choice for (nonempty) finite sets. We also provide an equivalent of the statement $\mathrm{AC}^{\mathrm{fin}}$ in terms of "graphic" matroids. Several open questions stay open in ZF, for example: does sH imply the axiom of choice?

Keywords: axiom of choice; finitary matroid; circuit; hyperplane; graph

Classification: 03E25, 05B99

1. Introduction

A *choice function* for a family $(A_i)_{i\in I}$ of nonempty sets is a family $(x_i)_{i\in I}$ such that for every $i \in I$, $x_i \in A_i$. The axiom of choice (AC) is the following statement: "*Every family of nonempty sets has a choice function.*". We work in set theory without the axiom of choice ZF (Zermelo–Fraenkel set theory). We shall also consider the more general set theory ZFA, see [8, pages 44–45], a modified version of set theory, in which "atoms" (i.e. empty objects which are not sets) are allowed. Consider the statement VB (vector basis): *"Every vector space has a basis."*, see [7, Note 75 page 271]. It is known that in ZFA, VB implies the multiple choice axiom MC, see [7, form 67]), and that in ZF, MC is equivalent to AC, but it is an open question to know whether VB imply AC in ZFA. In this paper, we discuss various statements about "finitary matroids" (which can be seen as generalizations of vector spaces, see Section 2.3.3) and their links with AC. We show that the statement "*Every finitary matroid has a basis.*" is equivalent to AC in ZFA, see Proposition 5. We then consider the three following consequences of AC involving hyperplanes in finitary matroids, possibly satisfying the "binary elimination property", see Section 3.2:

sH: "*Every proper flat in a finitary matroid is the intersection of hyperplanes including it.*"

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$\mathrm{sH_{bep}}$: "*Every proper flat in a finitary matroid with the binary elimination property is the intersection of hyperplanes including it.*"

H: "*Every nonempty finitary matroid has a hyperplane.*"

It is known that AC $\Rightarrow$ sH and of course sH $\Rightarrow$ H and sH $\Rightarrow$ $\mathrm{sH_{bep}}$. In this paper, we shall prove that $\mathrm{sH_{bep}}$ implies the following axiom of choice for finite sets:

$\mathrm{AC^{fin}}$: (form 62 of [7]) *Every nonempty family of finite nonempty sets has a choice function.* It is known, see [7], that $\mathrm{AC^{fin}}$ does not imply AC and that $\mathrm{AC^{fin}}$ is not provable in ZF. We do not know whether H implies sH or $\mathrm{sH_{bep}}$ or $\mathrm{AC^{fin}}$ nor do we know whether H or sH implies AC, see Figure 2 at the end of the paper. For every natural number $k \geq 2$ we consider the following consequence of $\mathrm{AC^{fin}}$:

> AC^k: "*For every nonempty family $(A_i)_{i\in I}$ of finite sets with k-elements, $\prod_{i\in I} A_i$ is nonempty.*"

We also denote by for all k AC^k the following statement, which is form 61 of [7]:

> *For every natural number $k \geq 2$, for every nonempty family $(A_i)_{i\in I}$ of finite sets with k-elements, $\prod_{i\in I} A_i$ is nonempty.*

In ZF, for every natural number $n \geq 2$, AC $\Rightarrow$ $\mathrm{AC^{fin}}$ $\Rightarrow$ for all k AC^k $\Rightarrow$ AC^n, and it is known, see [7], that in ZF, none of these implications is reversible, and that AC^n is not provable.

Using the natural structure of finitary matroid over a vector space, see Example 1, H implies the following statement D: "*Given a commutative field $\mathbb{K}$ and a non null vector space E over $\mathbb{K}$, there exists a non null linear form $f: E \longrightarrow \mathbb{K}$*". For every commutative field $\mathbb{K}$, we denote by $\mathrm{D}_{\mathbb{K}}$ the previous statement restricted to vector spaces over $\mathbb{K}$: "*For every non null $\mathbb{K}$-vector space E, the algebraic dual of E is non null.*" In [10, Corollary 2], we proved that for every prime number p, the statement $\mathrm{D}_{\mathbb{F}_p}$ (where $\mathbb{F}_p$ is the finite field $\mathbb{Z}/p\mathbb{Z}$) implies the statement C(p): "*For every family $(A_i)_{i\in I}$ of nonempty finite sets, there exists a family $(B_i)_{i\in I}$ such that for every $i \in I$, $B_i \subseteq A_i$ and p does not divide the cardinal of B_i*". Denoting by for all p C(p) the statement for all $p \in \mathbb{P}$ C(p) where $\mathbb{P}$ is the set of prime natural numbers, then for all p C(p) implies (and thus is equivalent to) the statement for all k AC^k, see [10, Remarks 3 and 4]). It follows that sH $\Rightarrow$ H $\Rightarrow$ D $\Rightarrow$ for all k AC^k. However, we do not know whether D implies H. Notice that in ZFA, D does not imply $\mathrm{AC^{fin}}$, since the statement for all p MC(p), see [7, form 218], implies the Ingleton statement I (the ultrametric counterpart of the Hahn–Banach statement, see [11]) which implies D, but for all p MC(p) does not imply $\mathrm{AC^{fin}}$, see Figure 2 at the end of the paper.

The paper is organized as follows. In Section 2 we review in set theory ZF some definitions and results about operators on finite or infinite sets in the sense of D. A. Higgs [3] and V. Klee [9]: finitary operators, matroidal operators with particular emphasis on circuits and hyperplanes. We introduce the three notions of "circuit-accessibility", "hyperplane-accessibility" and "symmetric circuits". In Section 3, we formulate an equivalent of AC is terms of hyperplanes in a certain (non finitary) matroid, and we prove that the statement sH restricted to certain binary matroids implies $\mathrm{AC}^{\mathrm{fin}}$. Finally, in the last section, we prove that $\mathrm{AC}^{\mathrm{fin}}$ is equivalent to various statements about "graphic" matroids. We end with several questions about finitary matroids and AC.

2. Operators and the axiom of choice

2.1 Operators on a set.

2.1.1 Operators and their circuits. An *operator* on a set X, see [9, page 138], is a mapping $\varphi\colon \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ which is *isotonic* $\big($for all subsets A, B of X, $(A \subseteq B \Rightarrow \varphi(A) \subseteq \varphi(B))\big)$ and *enlarging* (for every subset A of X, $A \subseteq \varphi(A)$). Given an operator φ on a set X, a subset D of X is said to be *φ-dependent* if there exists $x \in D$ such that $x \in \varphi(D\backslash\{x\})$. A subset I of X is said to be *φ-independent* if I is not φ-dependent i.e. if for every $x \in I$, $x \notin \varphi(I\backslash\{x\})$. Minimal φ-dependent subsets of X are called *φ-circuits.* A *loop* of the operator φ on X is an element x of X such that $\{x\}$ is a φ-circuit i.e. $\{x\}$ is φ-dependent i.e. $x \in \varphi(\emptyset)$. Two distinct elements x, y of X are *parallel* if $\{x, y\}$ is a φ-circuit.

Remark 1. Given an operator φ on a set X, we have:

(1) The collection $\mathcal{I}_\varphi$ of φ-independent subsets of X contains $\emptyset$ and is *initial*: for all subsets A, B of X, if $A \subseteq B$ and $B \in \mathcal{I}_\varphi$, then $A \in \mathcal{I}_\varphi$.
(2) The collection $\mathcal{D}_\varphi$ of φ-dependent subsets of X does not contain $\emptyset$ and is *final*: for all subsets A, B of X, if $A \subseteq B$ and $A \in \mathcal{D}_\varphi$, then $B \in \mathcal{D}_\varphi$.
(3) The collection $\mathcal{C}_\varphi$ of φ-circuits is an *antichain* of nonempty sets: no member of $\mathcal{C}_\varphi$ includes another one.

2.1.2 Finitary operators. An operator φ on X is said to be *finitary* if for every subset Y of X and every $x \in \varphi(Y)$, there exists a finite subset F of Y satisfying $x \in \varphi(F)$. If the operator φ is finitary, then every φ-dependent set includes a (finite) φ-circuit.

Definition 1. Given two *finitary* operators φ_1 and φ_2 on sets X_1 and X_2 and given a bijection $f\colon X_1 \longrightarrow X_2$, the following statements are equivalent:

(1) for every subset I of X_1, I is φ_1-independent if and only if $f[I]$ is φ_2-independent;
(2) for every subset C of X_1, C is a φ_1-circuit if and only if $f[C]$ is a φ_2-circuit.

Every bijection $f\colon X_1 \longrightarrow X_2$ satisfying one of the two previous statements is called an *isomorphism* of finitary operators.

2.1.3 Hyperplanes of an operator. A subset A of X is said to be *φ-spanning* if $\varphi(A) = X$. Subsets of X which are both φ-independent and φ-spanning are called *bases* of the operator φ (or φ-bases). Maximal non-spanning subsets of X are called *φ-hyperplanes*. Subsets of X which are fixed points of φ are called *flats* or *closed subsets* of the operator φ.

Remark 2. Given an operator φ on a set X for every nonempty family $(F_i)_{i\in I}$ of φ-closed subsets of X, $\bigcap_{i\in I} F_i$ is φ-closed, and thus, the *poset* $\mathcal{L}_\varphi$ of φ-closed subsets of X endowed with the inclusion relation is a complete lattice (but it is not an induced sub-lattice of the lattice $(\mathcal{P}(X), \subseteq)$ in general).

2.2 Minors of an operator.

2.2.1 Suboperators. Given an operator φ on a set X, and a subset Y of X, the mapping $\varphi_Y\colon \mathcal{P}(Y) \longrightarrow \mathcal{P}(Y)$ such that for every subset Z of Y, $\varphi_Y(Z) = \varphi(Z) \cap Y$ is an operator on Y, called the *suboperator* induced by φ on Y, or the *restriction operator* of φ to Y, see [13, page 263]. If the operator φ on X is finitary, then the suboperator φ_Y is also finitary.

Remark 3. Given an operator φ on a set X, and a subset Y of X, then:

(1) The φ_Y-dependent subsets of Y are the φ-dependent sets that are included in Y.
(2) The φ_Y-independent subsets of Y are the φ-independent sets that are included in Y.
(3) The φ_Y-circuits are the φ-circuits that are included in Y.

2.2.2 Quotient operators. Given an operator φ on a set X, and a subset Y of X, the mapping $\varphi^Y\colon \mathcal{P}(Y) \longrightarrow \mathcal{P}(Y)$ associating to every subset A of Y the set $Y \cap \varphi(A \cup (X \backslash Y))$ is an operator on Y. The operator φ^Y on Y is called the *quotient operator* φ^Y, or the *contraction* operator φ^Y, see [13, page 263]. If the operator φ on X is finitary, then the operator φ^Y is also finitary.

Proposition 1. *Given an operator φ on a set X and a proper flat F of φ, then:*

(1) *φ-flats including F are subsets $F \cup Z$ where Z is a flat of the quotient operator $\varphi^{X\backslash F}$ on $X\backslash F$.*
(2) *φ-hyperplanes including F are subsets $F \cup Z$ where Z is a hyperplane of the operator $\varphi^{X\backslash F}$.*

Proof: (1) Given a subset Z of $X\backslash F$, the following sentences are equivalent: $F \cup Z$ is a φ-flat; $\varphi(F \cup Z) \subseteq F \cup Z$; $\varphi(F \cup Z)\backslash F \subseteq Z$; $\varphi^{X\backslash F}(Z) \subseteq Z$; Z is a $\varphi^{X\backslash F}$-flat subset of $X\backslash F$.

(2) Given a subset Z of $X\backslash F$, the following sentences are equivalent: $F \cup Z$ is a φ-hyperplane; $(F \cup Z)$ is a proper φ-flat but for every $x \in X\backslash(F \cup Z)$, $\varphi((F \cup Z) \cup \{x\}) = X$; Z is a proper $\varphi^{X\backslash F}$-flat but for every $x \in X\backslash(F \cup Z)$, $\varphi^{X\backslash F}(Z \cup \{x\}) = X\backslash F$; the subset Z of $X\backslash F$ is a $\varphi^{X\backslash F}$-hyperplane. □

Remark 4. Proposition 1 implies that given a class $\mathcal{O}$ of operators which is closed by quotient operators, if every $\varphi \in \mathcal{O}$ has a hyperplane, then for every $\varphi \in \mathcal{O}$, every proper flat of φ is included in a φ-hyperplane.

Definition 2. Given an operator φ on a set X, a *minor* of an operator φ on a set X is an operator ψ on a subset Y of X such that there exists a sequence of operators $(\varphi_i)_{0\leq i\leq n}$ such that $\varphi_0 = \varphi$, $\varphi_n = \psi$ and for each $i \in \{1, \dots, n\}$, φ_i is a suboperator or a quotient operator of φ_{i-1}.

2.3 Finitary matroidal operators.

2.3.1 Idempotency properties. A *closure operator* on X is an operator φ on X which is *idempotent*, see [9, page 140]: for every subset A of X, $\varphi(\varphi(A)) = \varphi(A)$.

If the operator φ on X is idempotent, then for every subset Y of X, the operators φ_Y and φ^Y are also idempotent.

Proposition 2. *Given an idempotent operator φ on a set X, a subset H of X is a φ-hyperplane if and only if H is a maximal proper φ-closed subset of X.*

Proof: Given an operator φ on a set X for every φ-hyperplane H, then either $\varphi(H) = H$, and thus H is a maximal proper φ-closed subset of X, or $\varphi(H)$ is spanning (else $H \subsetneq \varphi(H) \subseteq \varphi(\varphi(H)) \subsetneq X$ and H would not be a φ-hyperplane since $\varphi(H)$ would be a non spanning subset of X strictly including H). It follows that if φ is idempotent, then every φ-hyperplane is a maximal proper φ-closed subset of X (else, $\varphi(H)$ would be spanning i.e. $X = \varphi(\varphi(H)) = \varphi(H)$ by idempotency, and thus H would be spanning). Reciprocally, if H is a maximal proper φ-closed subset of X, then for every $x \in X\backslash H$, $\varphi(H \cup \{x\})$ is closed and thus $\varphi(H \cup \{x\}) = X$ whence H is a φ-hyperplane. □

Definition 3. An operator φ on X is *circuit-accessible* if for every subset Y of X and every $x \in \varphi(Y)\backslash Y$, there exists a φ-circuit C such that $x \in C \subseteq Y \cup \{x\}$.

Remark 5. Every finitary idempotent operator is circuit-accessible.

Proof: Let φ be a finitary idempotent operator on a set X. Given some subset A of X, and some $x \in \varphi(A)\backslash A$, let I be a minimal finite subset of A such that $x \in \varphi(I)$. Then I is independent, else there exists $y \in I$ such that $y \in \varphi(I\backslash\{y\})$, whence, denoting by G the set $I\backslash\{y\}$, $x \in \varphi(G \cup \{y\})$ and thus, by idempotency of φ and since $y \in \varphi(G)$, $x \in \varphi(G)$ which contradicts the minimality of I. Since $I \cup \{x\}$ is finite and dependent, there exists a φ-circuit C such that $C \subseteq I \cup \{x\}$. Since I is independent, $x \in C$ and finally, $x \in C \subseteq A \cup \{x\}$. It follows that φ is circuit-accessible. $\square$

2.3.2 Exchange properties. An operator φ on a set X is said to satisfy the *exchange property*, see property (E) in [9, page 140], if for all subsets Y, Z of X and every $x \in X$, if $x \in \varphi(Y \cup Z)$ and $x \notin \varphi(Y)$, then there exists $y \in Z$ such that $y \in \varphi\big(((Y \cup Z)\backslash\{y\}) \cup \{x\}\big)$.

Definition 4. Given an operator φ on a set X, a φ-circuit C is *symmetric* if for every $x \in C$, $x \in \varphi(C \backslash \{x\})$.

Remark 6. If an operator φ on a set X satisfies the exchange property, then every φ-circuit is symmetric.

2.3.3 Matroidal operators (or matroids). We say that an operator φ on a set X is *matroidal* if φ is idempotent and satisfies the exchange property. In the following, the term "matroid operator" is also abbreviated to "matroid".

Example 1 (The operator span_X associated to a vector space X)**.** Given a vector space X over a commutative field $\mathbb{K}$, the operator span on X, associating to every subset Y of X the vector subspace generated by Y in X is a finitary matroidal operator on X. The span-independent subsets of X are the $\mathbb{K}$-linearly independent subsets of X; the span-bases of X are the bases of the $\mathbb{K}$-vector space X; the span-flats are the vector subspaces of X, and the span-hyperplanes of X are the kernels of non null linear forms $f\colon X \longrightarrow \mathbb{K}$. The only loop of this operator is $\{0_X\}$.

Example 2 (The matroidal operator associated to a family of vectors)**.** Given a $\mathbb{K}$-vector space X and a mapping $f\colon I \longrightarrow X$, the mapping $\varphi\colon \mathcal{P}(I) \longrightarrow \mathcal{P}(I)$ associating to every subset J of I the set $\{i \in I\colon f(i) \in \mathrm{span}(f[J])\}$ is a finitary matroidal operator. Loops of this operator are elements $i \in I$ such that $f(i) = 0_X$. Two elements i, j of I are parallel if and only if i, j are not loops and if $f(i)$ and $f(j)$ are colinear.

Given a (commutative) field $\mathbb{K}$, a finitary matroidal operator φ on a set X is said to be $\mathbb{K}$-*representable* if there exist a $\mathbb{K}$-vector space E and a mapping $f\colon I \longrightarrow E$ such that the matroidal operator φ is isomorphic with the finitary matroidal operator associated to f.

Remark 7. There are many equivalent definitions for the notion of *matroid* on a finite set, see [16, Chapter 1] or [12, Chapter 2]. Given an infinite set X, the notion of finitary matroidal operator on X is equivalent to the notion of "transitive dependence relation" on X, see for example [17, page 97], [1, Proposition 2.1 page 253], [16, Chapter 20.5], [2, page 2]). In ZFC, finitary matroids (i.e. finitary matroidal operators) have bases, but infinite matroids do not have bases in general.

2.3.4 Hyperplane-accessibility.

Definition 5. An operator φ on a set X is *hyperplane-accessible* if every proper flat of φ is the intersection of the set of the φ-hyperplanes including it.

Given a commutative field $\mathbb{K}$, the statement $\mathrm{D}_{\mathbb{K}}$: "*Every non null vector space has a non null linear form.*" is equivalent to the statement "*For every $\mathbb{K}$-vector space E, the finitary matroidal operator* span_E *is hyperplane-accessible.*" Indeed, we have shown in [10, Theorem 2] that $\mathrm{D}_{\mathbb{K}}$ implies that every proper subspace of a $\mathbb{K}$-vector space is the intersection of the hyperplanes including it.

2.4 Finitary operators and the axiom of choice.

2.4.1 Axiom of choice and finitary operators.

Proposition 3 ([15, pager 95] and [4])**.** AC *is equivalent to each of the following statements:*

(1) *AL'_3 ([15, page 95]): "For every finitary closure operator φ on a set X, for every collection $\mathcal{F}$ of subsets of X which has finite character (i.e. for every subset Z of X, $Z \in \mathcal{F}$ if and only if for every finite subset Y of Z, $Y \in \mathcal{F}$), and for every proper φ-flat F of X such that $F \in \mathcal{F}$, there exists a maximal φ-flat G such that $F \subseteq G$ and $G \in \mathcal{F}$."*

(2) *AL''_3: "For every finitary closure operator φ on a set X, for every proper φ-flat F of X and every $x \in X \backslash F$, there exists a maximal φ-flat G such that $F \subseteq G$ and $x \notin G$."*

(3) *K (W. Krull): "Every proper ideal of commutative unitary ring is contained in a maximal proper ideal."*

It follows that AC *implies the statement* sH*: "Every finitary matroid is hyperplane-accessible."*

Proof: AC $\Rightarrow AL'_3$: The set $P := \{Z \in \mathcal{F} : F \subseteq Z \text{ and } \varphi(Z) = Z\}$ endowed with the order induced by "$\subseteq$" is inductive (for every chain C of P, $\bigcup C \in P$) and thus, Zorn's lemma implies a maximal element G of P.

$AL'_3 \Rightarrow AL''_3$: Given a proper φ-flat F and $x \in X \backslash F$, the collection $\mathcal{F}$ of subsets of X which do not contain x has the finite character, and thus AL'_3 implies a maximal φ-flat including F and not containing x.

$AL_3'' \Rightarrow K$: Given a proper ideal I of a commutative unitary ring A, consider the closure operator φ on A associating to each subset Z of A the ideal of A generated by Z. Then φ is finitary, and thus AL_3'' implies a maximal φ-closed subset M of A including I such that $1 \notin M$.

$K \Rightarrow$ AC: This implication is due to W. Hodges, see [4].

In the conditions of statement AL_3'', if moreover φ satisfies the exchange property, then G is a φ-hyperplane, so the statement sH is the restriction of statement AL_3'' to finitary matroids. It follows that AC $\Rightarrow AL_3'' \Rightarrow$ sH. □

2.4.2 Axiom of choice and finitary matroids.

Definition 6. An operator φ on a set X is said to satisfy the *interpolation property (for bases)* if for every φ-independent subset I of X and every φ-generating subset G of X such that $I \subseteq G$, there exists a φ-basis B such that $I \subseteq B \subseteq G$.

A *B-matroidal* operator on a set X, see [3, page 217], [13, page 264]) is a matroidal operator φ on X such that for every subset Y of X, the suboperator φ_Y satisfies the interpolation property. Of course, every suboperator of a B-matroidal operator is B-matroidal.

Proposition 4 ([3, page 219]). *Every B-matroidal operator is hyperplane-accessible and circuit-accessible.*

Proof: Higgs defines a "C-matroid" as a matroidal operator which is both hyperplane-accessible and circuit-accessible. He proves that every B-matroid is a "C-matroid". □

Proposition 5. (1) AC *is equivalent to each of the following statements:*

FB_0: *"Every finitary matroid satisfies the interpolation property."*

FB_1: *"Every finitary matroid is a B-matroid."*

FB_2: *"Every finitary matroid has a basis."*

FB_3 *(form* [1A] *of* [7]*): "Given a vector space E, every generating subset of E includes a basis of E."*

FB_4 *"Every connected graph has a spanning tree."*

(2) *The statement* H*: "Every nonempty finitary matroid has a hyperplane." is equivalent to the statement "Every proper flat of a finitary matroid is included in a hyperplane."*

Proof: (1) AC $\Rightarrow FB_0$. Given a finitary matroidal operator φ on a set X, a φ-independent subset I of X and a φ-generating subset G of X such that $I \subseteq G$, consider the set $\mathcal{J}$ of φ-independent subsets J such that $I \subseteq J \subseteq G$. Then the *poset* $(\mathcal{J}, \subseteq)$ is inductive (every chain $(J_t)_{t \in T}$ of this poset is dominated by $\bigcup_{t \in T} J_t$), so with Zorn's lemma, one gets a maximal element B of the *poset*

$(\mathcal{J}, \subseteq)$, and B is a φ-basis such that $I \subseteq B \subseteq G$. $FB_0 \Rightarrow FB_1$ follows from the previous point and the fact that every submatroid of a finitary matroid is finitary. $FB_1 \Rightarrow FB_2$ is trivial. $FB_2 \Rightarrow FB_3$: Consider a vector space E and a generating subset G of E. The operator φ induced by span on G is finitary and matroidal, and thus FB_2 implies a φ-basis, which is a basis of the vector space E included in G. $FB_3 \Rightarrow FB_4$: See [6]. $FB_4 \Rightarrow$ AC: See [5].

(2) Given a finitary matroidal operator φ on a set X, and a proper flat F of φ, the statement H applied to the finitary operator φ^F provides a hyperplane Z of φ^F, and then $F \cup Z$ is a φ-hyperplane using Proposition 1. □

3. Hyperplanes in matroids and the axiom of choice

3.1 The operator on X associated to an antichain of nonempty subsets of X.

Definition 7. Given an antichain $\mathcal{C}$ of nonempty subsets of a set X, the mapping $\varphi\colon \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ associating to each subset Y of X the set $Y \cup B$ where B is the set of elements $x \in X$ such that there exists $C \in \mathcal{C}$ satisfying $x \in C \subseteq Y \cup \{x\}$ is an operator on X. We call it the operator *associated to the antichain* $\mathcal{C}$.

Proof: By definition of φ, the mapping φ is expansive; moreover φ is isotonic since if $Y_1 \subseteq Y_2 \subseteq X$ for every $x \in X$ and every $C \in \mathcal{C}$ such that $x \in C \subseteq Y_1 \cup \{x\}$, then $x \in C \subseteq Y_2 \cup \{x\}$, thus $\varphi(Y_1) \subseteq \varphi(Y_2)$. □

Proposition 6. *Every circuit-accessible operator φ on a set X such that φ-circuits are symmetric satisfies the exchange property.*

Proof: Assume that Y, Z are two subsets of X and that for some $x \in X$, $x \in \varphi(Y \cup Z)$ but $x \notin \varphi(Y)$. Since φ is circuit-accessible, let C be a φ-circuit such that $x \in C \subseteq (Y \cup Z) \cup \{x\}$. Since the circuit C is symmetric, $x \in \varphi(C \setminus \{x\})$, and thus $C \setminus \{x\}$ meets Z (else $C \setminus \{x\} \subseteq Y$ so $\varphi(C \setminus \{x\}) \subseteq \varphi(Y)$ whence $x \in \varphi(Y)$, which is contradictory!). Let $z \in (C \setminus \{x\}) \cap Z$; then, since the circuit C is symmetric, $z \in \varphi(C \setminus \{z\}) \subseteq \varphi\big(((Y \cup Z) \cup \{x\}) \setminus \{z\}\big)$. □

Lemma 1. *Given an antichain $\mathcal{C}$ of nonempty subsets of a set X, denote by φ the operator on X associated to the antichain $\mathcal{C}$, see Definition 7.*

(1) *Each element of $\mathcal{C}$ is a symmetric φ-circuit.*
(2) *$\mathcal{C}$ is the set of φ-circuits, and the operator φ on X is circuit-accessible.*
(3) *The operator φ satisfies the exchange property.*
(4) *If elements of $\mathcal{C}$ are finite sets, then the operator φ is finitary.*

Proof: (1) If $C \in \mathcal{C}$, then, by definition of φ for every $x \in \mathcal{C}$, $x \in \varphi(C \setminus \{x\})$, thus C is φ-dependent; moreover, the set $I := C \setminus \{x\}$ is φ-independent, else let $y \in I$ such that $y \in \varphi(I \backslash \{y\})$; then there would exist $C' \in \mathcal{C}$ such that $y \in C' \subseteq I \subsetneq C$ which is contradictory since $\mathcal{C}$ is an antichain.

(2) Let C be a φ-circuit. Then there exists $x \in C$ such that $x \in \varphi(C \backslash \{x\})$. By definition of φ, let $C' \in \mathcal{C}$ such that $x \in C' \subseteq (C \backslash \{x\}) \cup \{x\} = C$; using Point (1), C' is a φ-circuit, and since the set of φ-circuits is an antichain, $C' = C$, and thus $C \in \mathcal{C}$. Since $\mathcal{C}$ is the set of φ-circuits, it follows by definition of φ that the operator φ is circuit-accessible.

(3) This follows from Proposition 6 using the fact that φ is circuit-accessible and has symmetric circuits.

(4) Trivial since φ is circuit-accessible. □

3.2 Binary matroids. A family $\mathcal{C}$ of subsets of a set X is said to satisfy the *binary elimination property* if for all distinct elements C_1, C_2 of $\mathcal{C}$, the symmetric difference $C_1 \Delta C_2$ is a union of pairwise disjoint elements of $\mathcal{C}$.

Theorem 1 ([14, Theorem 9.1.2 page 344]). *Given a matroidal operator φ on a finite set X and denoting by $\mathcal{C}$ the set of φ-circuits, the following statements are equivalent:*

(1) *The operator φ is representable over the two-element field $\mathbb{F}_2$.*
(2) *The symmetric difference of any set of circuits is either empty or contains a circuit.*
(3) *$\mathcal{C}$ satisfies the binary elimination property.*
(4) *For all distinct circuits $C_1, C_2 \in \mathcal{C}$, $C_1 \Delta C_2$ is a (finite) union of circuits.*
(5) *For all distinct circuits $C_1, C_2 \in \mathcal{C}$, $C_1 \Delta C_2$ includes a circuit.*

The following corollary holds in ZF for *infinite* finitary matroids.

Corollary 1. *Given a finitary matroidal operator φ on a (non necessarily finite) set X and denoting by $\mathcal{C}$ the set of φ-circuits, the following statements are equivalent:*

(1) *φ is $\mathbb{F}_2$-representable.*
(2) *Every finite submatroid of φ is $\mathbb{F}_2$-representable.*
(3) *$\mathcal{C}$ satisfies the binary elimination property.*
(4) *For all distinct φ-circuits $C_1, C_2 \in \mathcal{C}$, $C_1 \Delta C_2$ is a (finite) union of circuits.*
(5) *For all distinct φ-circuits $C_1, C_2 \in \mathcal{C}$, $C_1 \Delta C_2$ includes a circuit.*
(6) *The symmetric difference of any set of φ-circuits is either empty or contains a circuit.*

Proof: (1) $\Rightarrow$ (2) is easy and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are consequences of Theorem 1. We prove (6) $\Rightarrow$ (1). We consider the vector space ${\mathbb{F}_2}^{(X)}$ and its canonical basis $(e_x)_{x \in X}$ where for every $x \in X$, $e_x \colon X \longrightarrow \mathbb{F}_2$ is the indicator function of the singleton $\{x\}$. Let V be the vector subspace of ${\mathbb{F}_2}^{(X)}$ generated by the set $\{v_C := \sum_{x\in C} x \colon C\ \varphi\text{-circuit}\}$. Let Q be the quotient vector space $\mathbb{F}_2{}^{(X)}/V$ and let $f\colon X \longrightarrow Q$ be the quotient mapping $x \mapsto e_x + V$. The (finitary) matroidal operator ψ associated to f is isomorphic with φ since φ and ψ have the same circuits: given a subset C of X, C is a ψ-circuit if and only if $\sum_{x\in C}(e_x + V) = 0_Q$ and for every proper subset I of C, $\sum_{x\in I}(e_x + V) \neq 0_Q$; equivalently, $\sum_{x\in C} e_x \in V$ and for every proper subset I of C, $\sum_{x\in I} e_x \notin V$; this means that there exist φ-circuits $C_1, \dots, C_m$ such that $C = C_1\Delta \dots \Delta C_m$ and that no proper subset I of C is the symmetric difference of a nonempty sequence of φ-circuits; using (2) it means that C is a φ-circuit. □

Definition 8. A finitary matroid is said to be *binary* if it satisfies one of the previous equivalent statements.

3.3 The matroidal operator associated to a family of pairwise disjoint nonempty sets.

Definition 9. Given an integer $n \geq 2$, a family $\mathcal{C}$ of subsets of a set X is said to satisfy the *n-binary elimination property* if for all distinct elements C_1, C_2 of $\mathcal{C}$, the symmetric difference $C_1 \Delta C_2$ is a union of at most n elements of $\mathcal{C}$.

Theorem 2. *Given a nonempty family $(A_i)_{i\in I}$ of pairwise disjoint nonempty sets, consider the set $X = \bigcup_{i\in I} A_i \cup \{O\}$ where O is some set such that $O \notin \bigcup_{i\in I} A_i$. For every $i \in I$, let $C_i^1 := A_i \cup \{O\}$, and for all distinct elements $i, j \in I$, let $C_{i,j}^2 = A_i \cup A_j$. Let $\mathcal{C} := \{C_i^1 \colon i \in I\} \cup \{C_{i,j}^2 \colon i, j \in I,\ i \neq j\}$. Then*

1. *$\mathcal{C}$ is an antichain of nonempty subsets of X.*
2. *$\mathcal{C}$ satisfies the 2-binary elimination property.*
3. *Let φ be the operator associated to the antichain $\mathcal{C}$. Then φ is finitary if and only if for every $i \in I$, the set A_i is finite.*
4. *The operator φ is idempotent (and thus matroidal).*

Proof: Points (1), (2) and (3) are easy to check.

(4) Let Z be a subset of X. Let I_1 be the set of elements $i \in I$ such that $A_i \backslash Z$ has at least two elements. Let $I_2 = I \backslash I_1$. If $O \in Z$ then $\varphi(Z) = Z \cup \bigcup_{i\in I_2} A_i$ and thus, $\varphi(\varphi(Z)) = \varphi(Z)$. If $O \notin Z$ and if there exists $i_0 \in I_2$ such that $A_{i_0} \subseteq Z$, then $\varphi(Z) = Z \cup \{O\} \cup \bigcup_{i\in I_2} A_i$ and thus, $\varphi(\varphi(Z)) = \varphi(Z)$; if $O \notin Z$ and if for every $i \in I_2$, $A_i \backslash Z$ has exactly one element, then $\varphi(Z) = Z$ and thus $\varphi(\varphi(Z)) = \varphi(Z)$. □

Definition 10. In the conditions of the previous theorem, we call φ the *matroidal operator associated to O and the family $(A_i)_{i\in I}$*.

Definition 11. Given a nonempty family $(A_i)_{i\in I}$ of pairwise disjoint nonempty sets, a *selector* for this family is a subset S of $\bigcup_{i\in I} A_i$ such that for every $i \in I$,

$S \cap A_i$ has at most one element; the selector S is said to be *total* if for every $i \in I$, $S \cap A_i$ has exactly one element.

Theorem 3. *Given a nonempty family $(A_i)_{i\in I}$ of pairwise disjoint nonempty sets, consider the set $X = \bigcup_{i\in I} A_i \cup \{O\}$ where O is some set such that $O \notin \bigcup_{i\in I} A_i$. Let φ be the matroidal operator associated to O and the family $(A_i)_{i\in I}$.*

1. *A subset L of X is φ-independent if and only if either ($O \in L$ and for all $i \in I A_i \not\subseteq L$), or ($O \notin L$ and there exists at most one element $i_0 \in I$ such that $A_{i_0} \subseteq L$).*
2. *A subset G of X is φ-generating if and only if $S := \left(\bigcup_{i\in I} A_i\right)\backslash G$ is a selector for the family $(A_i)_{i\in I}$, which is not total if $O \notin G$.*
3. *A subset B of X is a φ-basis if and only if there exists a total selector S for the family $(A_i)_{i\in I}$ such that $B = \left(\left(\bigcup_{i\in I} A_i\right)\backslash S\right) \cup \{a\}$ where a is some element of $\{O\} \cup S$.*
4. *A proper subset F of X is a φ-flat if and only if ($O \in F$ or exists $i_0 \in I \;\; A_{i_0} \subseteq F$) $\Rightarrow$ for all $i \in I$ $A_i\backslash F$ is not a singleton.*
5. *A subset H of X is a φ-hyperplane if and only if $H = \left(\bigcup_{i\in I} A_i\right)\backslash S$ where S is a total selector for the family $(A_i)_{i\in I}$, or $H = X\backslash\{x, y\}$ where $i_0 \in I$ and $x, y \in A_{i_0}$ with $x \neq y$.*
6. *The following statements are equivalent:*
   (a) *The operator φ is hyperplane-accessible.*
   (b) *Every family $(B_i)_{i\in I}$ such that for every $i \in I$, $\emptyset \subsetneq B_i \subseteq A_i$ has a total selector.*
   (c) *The operator φ is B-matroidal.*
   (d) *The operator φ satisfies the interpolation property for bases.*

Proof: Points (1), (2), (3), (4) and (5) are consequences of the definitions. We prove Point (6). (a) $\Rightarrow$ (b): Given a family $(B_i)_{i\in I}$ such that for every $i \in I$, $\emptyset \subsetneq B_i \subseteq A_i$, consider the proper φ-flat subset $F := \bigcup_{i\in I}(A_i\backslash B_i)$ of X; since φ is hyperplane-accessible, let H be a φ-hyperplane such that $F \subseteq H$ and $O \notin H$; then $\bigcup_{i\in I}(A_i\backslash H)$ is a total selector for the family $(B_i)_{i\in I}$.

(b) $\Rightarrow$ (c): Let Y be a subset of X. Let L be a φ-independent subset of Y and let G be a φ_Y-generating subset of Y such that $L \subseteq G$. Let $J := \{i \in I\colon Y \cap A_i \neq \emptyset\}$. Let $J_1 := \{i \in J\colon A_i \not\subseteq G\}$. Let $J_2 = \{i \in J\colon A_i \subseteq G \text{ and } A_i \not\subseteq L\}$. Let $J_3 = \{i \in J\colon A_i \subseteq L\}$: notice that $J = J_1 \cup J_2 \cup J_3$ and that J_1, J_2 and J_3 are pairwise disjoint. For each $i \in J_1$, let x_i be the element of $A_i\backslash G$. Using (b), consider a choice function $(x_i)_{i\in J_2}$ for the family $(A_i\backslash L)_{i\in J_2}$. If J_3 is nonempty, then J_3 has a unique element i_0 and let $x_{i_0} = O$ if $O \in Y$. If $O \in Y$, let $B := Y\backslash\{x_i\colon i \in J\}$, and if $O \notin Y$, let $B := Y\backslash\{x_i\colon i \in J_1 \cup J_2\}$. Then B is a φ_Y-basis such that $L \subseteq B \subseteq G$.

(c) $\Rightarrow$ (d) follows from the definitions.

(d) $\Rightarrow$ (a): Let F be a proper subset of X which is a φ-flat and let $x \in X \backslash F$. If $x = O$, then for every $i \in I$, $A_i \backslash F$ has at least one element (else O would belong to F), and thus F is φ-independent; then $G = \bigcup_{i \in I} A_i$ is φ-spanning and $F \subseteq G$: using the interpolation property, there exists a φ-basis B such that $F \subseteq B \subseteq G$; it follows that there exists a total selector S for $(A_i)_i$ and an element $i_0 \in I$ such that $B = A_{i_0} \cup \left(\bigcup_{i \neq i_0} A_i\right) \backslash S$; let $x_{i_0} \in A_{i_0} \backslash F$; then $H = B \backslash \{x_{i_0}\}$ is a φ-hyperplane including F such that $O \notin H$. If $x \neq O$, then let i_0 be the element of I such that $x \in A_{i_0}$. If $A_{i_0} \backslash F$ contains an element y distinct from x, then $H := X \backslash \{x, y\}$ is a φ-hyperplane including F and not containing x. If $A_{i_0} \backslash F = \{x\}$, then for every $i \in I \backslash \{i_0\}$, $A_i \backslash F \neq \emptyset$ and $O \notin F$ (else x would belong to F); using the independent set $L = F \backslash \{O\}$ and the generating set $G = \bigcup_i A_i$, consider a φ-basis B such that $L \subseteq B \subseteq G$; then B yields a selector S for the family $(A_i \backslash F)_{i \in I}$ (and thus $x \in S$). It follows that $H := \left(\bigcup_{i \in I} A_i\right) \backslash S$ is a φ-hyperplane including F. □

Corollary 2. AC *is equivalent to the following statement: "For every nonempty family $(A_i)_{i \in I}$ of pairwise disjoint nonempty sets, and for every set O such that $O \notin \bigcup_{i \in I} A_i$, the matroidal operator associated to O and the family $(A_i)_{i \in I}$ has a hyperplane not containing O."*

3.4 The axiom sH implies $\mathrm{AC}^{\mathrm{fin}}$. We denote by $\mathrm{sH}_{\mathrm{bep}}$ the axiom sH restricted to finitary matroids satisfying the binary elimination property (i.e. such that the set of circuits of the matroid satisfies the binary elimination property). For every natural number $n \geq 2$, we denote by $\mathrm{sH}_{\mathrm{bep}_n}$ the axiom sH restricted to finitary matroids with the n-binary elimination property. We denote by $\mathrm{H}_{\mathrm{bep}}$ (or $\mathrm{H}_{\mathrm{bep}_n}$) the axiom H restricted to finitary matroids with the binary elimination property (n-binary elimination property, respectively).

Remark 8. The matroidal operator associated to a family $(A_i)_{i \in I}$ of pairwise finite disjoint nonempty sets satisfies the 2-binary elimination property (i.e. the set of its circuits satisfies the 2-binary elimination property) and hence is binary.

Theorem 4. *In* ZF, $\mathrm{sH} \Rightarrow \mathrm{sH}_{\mathrm{bep}} \Rightarrow \mathrm{sH}_{\mathrm{bep}_2} \Rightarrow \mathrm{AC}^{\mathrm{fin}}$.

PROOF: Notice that $\mathrm{AC}^{\mathrm{fin}}$ is equivalent to the statement "For every nonempty family $(A_i)_{i \in I}$ of *pairwise disjoint* finite nonempty sets, $\prod_{i \in I} A_i$ is nonempty.": given a family $(A_i)_{i \in I}$ of nonempy sets, consider the family $(A_i \times \{i\})_{i \in I}$. Given a nonempty family $(A_i)_{i \in I}$ of pairwise disjoint finite nonempty sets, consider the set $X = \bigcup_{i \in I} A_i \cup \{O\}$ where $O \notin \bigcup_{i \in I} A_i$, and consider the finitary matroidal operator φ on X associated to the family $(A_i)_{i \in I}$ (see Theorem 2). Since φ has no loops, $\varphi(\emptyset) = \emptyset$, so $\emptyset$ is a proper flat of φ and thus, $\mathrm{sH}_{\mathrm{bep}_2}$ implies a φ-hyperplane H not containing O. It follows from Theorem 3 that for each

$i \in I$, $A_i \backslash H$ is a singleton $\{x_i\}$ where $(x_i)_{i \in I}$ is a choice function for the family $(A_i)_{i \in I}$. □

Question 1. We have shown that $\mathrm{AC} \Rightarrow \mathrm{sH}_1 \Rightarrow \mathrm{sH} \Rightarrow \mathrm{sH}_{\mathrm{bep}} \Rightarrow \mathrm{sH}_{\mathrm{bep}_2} \Rightarrow \mathrm{AC}^{\mathrm{fin}}$ and of course $\mathrm{sH} \Rightarrow \mathrm{H} \Rightarrow \mathrm{H}_{\mathrm{bep}} \Rightarrow \mathrm{H}_{\mathrm{bep}_2}$. Does $\mathrm{sH}_{\mathrm{bep}}$ imply sH? Does H imply $\mathrm{AC}^{\mathrm{fin}}$? Does H imply sH?

4. Graphic matroids and the finite axiom of choice

4.1 Elimination properties.

Definition 12. A family $\mathcal{C}$ of subsets of a set X is said to satisfy the *elimination property* if for all distinct elements $C_1, C_2 \in \mathcal{C}$ for every $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq C_1 \cup C_2$ and $x \notin C_3$. The family $\mathcal{C}$ is said to satisfy the *strong elimination property* if for all elements $C_1, C_2 \in \mathcal{C}$ for every $x \in C_1 \cap C_2$ and every $y \in C_1 \backslash C_2$, there exists $C_3 \in \mathcal{C}$ such that $y \in C_3 \subseteq C_1 \cup C_2$ and $x \notin C_3$.

Notice that the binary elimination property implies the strong elimination property, which in turn implies the elimination property.

Notation 1. For every finite set F, we denote by $|F|$ the cardinal of F.

The following result is classical:

Proposition 7 ([16], [2])**.** *Let $\mathcal{C}$ be an antichain of nonempty finite subsets of a set X, and let φ be the (finitary) operator associated to $\mathcal{C}$. If $\mathcal{C}$ satisfies the elimination property, then:*

(1) *$\mathcal{C}$ satisfies the strong elimination property.*
(2) *The operator φ is a closure operator.*
(3) *The operator φ is matroidal.*

Proof: (1) See [16, Theorem 2 page 24] or [2, Lemme 4 page 17].

(2) See [2, Théorème 8 page 18]. We sketch the proof. Let A be a subset of X and let $x \in \varphi(\varphi(A))$. Let us show that $x \in \varphi(A)$. Let $C \in \mathcal{C}$ such that $x \in C \subseteq \varphi(A) \cup \{x\}$, and such that $C \cap (\varphi(A) \backslash A)$ is minimal. If $(C \backslash \{x\}) \cap (\varphi(A) \backslash A)$ is nonempty, let $y \in (C \backslash \{x\}) \cap (\varphi(A) \backslash A)$; since $y \in \varphi(A)$, let $C_1 \in \mathcal{C}$ such that $y \in C_1 \subseteq A \cup \{y\}$. Using the strong elimination property, let $C_2 \in \mathcal{C}$ such that $x \in C_2 \subseteq (C \cup C_1) \backslash \{y\}$: then $|C_2 \cap (\varphi(A) \backslash A)| < |C \cap (\varphi(A) \backslash A)|$, which contradicts the minimality of $C \cap (\varphi(A) \backslash A)$. It follows that $(C \backslash \{x\}) \cap (\varphi(A) \backslash A) = \emptyset$ and thus, $(C \backslash \{x\}) \subseteq A$ so $x \in \varphi(A)$.

(3) Using Lemma 1, $\mathcal{C}$ is the set of φ-circuits and φ satisfies the exchange property, whence the closure operator φ is a matroidal operator on X. □

4.2 The binary matroid associated to a multigraph.

4.2.1 Multigraphs. A *multigraph* on a set V is given by a mapping $f : X \longrightarrow [V]^1 \cup [V]^2$, where for each natural number $n \geq 1$, $[V]^n$ is the set of n-element subsets of V. Elements of X such that $f(x) \in [V]^1$ are called *loops* of the multigraph.

Denoting by $\mathbb{F}_2^{(V)}$ the vector space of all functions from V to $\mathbb{F}_2$ which are zero outside a finite set, and denoting by $(e_v)_{v \in V}$ the canonical basis of $\mathbb{F}_2^{(V)}$, the *incidence matrix* of the multigraph f is the mapping $\tilde{f} \colon X \longrightarrow \mathbb{F}_2^{(V)}$ such that for every $x \in X$, $\tilde{f}(x)$ is $0_{\mathbb{F}_2^{(V)}}$ if $f(x) \in [V]^1$, and $\tilde{f}(x) = e_{v_1} + e_{v_2}$ if $f(x)$ is the two-element sets $\{v_1, v_2\}$. The *matroid associated to the multigraph* f is the (binary) matroidal operator on X associated with the incidence matrix $\tilde{f}$, see Example 2. Loops of this matroid correspond to loops of the multigraph. A matroidal operator which is isomorphic with the (binary hence finitary) matroid associated to a multigraph is said to be *graphic*.

4.2.2 Simple graphs. A simple graph on a set V is a binary relation R on V which is irreflexive (for every $x \in V$, xRx) and symmetric (for every $x, y \in V$, $xRy \Rightarrow yRx$). Elements of V are called the *vertices* of the graph, and pairs $\{x, y\}$ of vertices such that xRy are the edges of the simple graph. A simple graph on a set V with set E of edges is also denoted by (V, E). A (partial) subgraph of a simple graph G on a set X with set of edges E is a simple graph (X', E') such that $X' \subseteq X$ and $E' \subseteq E$. Two graphs (V_1, E_1) and (V_2, E_2) are *isomorphic* when there exists a bijection $f \colon V_1 \longrightarrow V_2$ which respects the edges.

Notation 2. Given some integer $n \geq 3$, we denote by C_n the simple graph on $\mathbb{Z}/n\mathbb{Z} = \{0, \ldots, n-1\}$ with set of edges $E_n = \{\{i, i +_n 1\} \colon i \in \mathbb{Z}/n\mathbb{Z}\}$, where "$+_n$" is the additive law on $\mathbb{Z}/n\mathbb{Z}$;

Given some integer $n \geq 3$, a simple graph is a *n-cycle* if it is isomorphic with the simple graph C_n. Given a simple graph $G = (V, E)$, a *cycle of the graph G* is a (partial) subgraph of G which is isomorphic with a n-cycle for some natural number $n \geq 3$.

4.2.3 Graphic matroids. Given a set V and a multigraph $f \colon X \longrightarrow [V]^1 \cup [V]^2$, if $E = f[X] \cap [V]^2$, then (V, E) is called the simple graph underlying the multigraph f. Reciprocally, every simple graph (V, E) underlies the multigraph $\mathrm{id}_E \colon E \longrightarrow E$ on V.

Proposition 8 ([14, Proposition 1.1.7])**.** *Let $G = (V, E)$ be a simple graph. Let $\mathcal{C}_G$ be the set of (finite) subsets F of E such that F is the set of edges of a cycle of G. Then $\mathcal{C}_G$ is the set of circuits of the (binary) matroidal operator $\mathcal{M}_G$ associated to the multigraph $\mathrm{id}_E \colon E \longrightarrow E$.*

Proof: Let W be the $\mathbb{F}_2$-vector space $\mathbb{F}_2^{(V)}$. For every $v \in V$ we denote by e_v the vth vector of the canonical basis of W. We identify each edge $\{a, b\}$ of G with the vector $e_a + e_b$ of W. A subset F of E is a circuit of the matroid $\mathcal{M}_G$ if and only if F is nonempty, $\sum_{e \in F} e = 0_W$ and for every nonempty proper subset G of F, $\sum_{e \in G} e \neq 0_W$; replacing each element $e = \{a, b\}$ of E by $e_a + e_b$, this means that $F \neq \emptyset$, every vertex of the subgraph $\left(\bigcup F, F\right)$ has even degree, but for every proper subset G of F, some vertex of the subgraph $\left(\bigcup G, G\right)$ has an odd degree; this means that F is a nonempty finite union of cycles of G, and that no proper subset of F is a cycle of G; equivalently, F is a cycle of the graph G. □

Remark 9. If $f \colon X \longrightarrow [V]^1 \cup [V]^2$ is a multigraph on a set V, if $\mathcal{M}_f$ is the matroid associated to the multigraph f, loops of $\mathcal{M}_f$ are the singletons $\{x\}$ such that $x \in X$ and $f(x)$ is a singleton; circuits of cardinal two of $\mathcal{M}_f$ are the pairs $\{x, y\}$ of distinct elements of X such that $f(x) = f(y)$. Given some natural number $n \geq 3$, then the n-circuits of $\mathcal{M}_f$ are the n-element subsets $\{x_1, \dots, x_n\}$ of X such that $\{f(x_1), \dots, f(x_n)\}$ is the set of edges of an n-cycle of the underlying simple graph of f.

4.3 An equivalent of $\mathrm{AC}^{\mathrm{fin}}$ in terms of graphic matroids.

Theorem 5. *The following statements are equivalent:*

(1) $\mathrm{AC}^{\mathrm{fin}}$.

(2) *For every family $(A_i)_{i \in I}$ of pairwise disjoint nonempty finite sets with at least two elements, the (binary hence finitary) matroid associated to this family is graphic.*

Proof: (1) ⇒ (2) Let $(A_i)_{i \in I}$ be an infinite family of pairwise disjoint nonempty finite sets, such that for every $i \in I$, $n_i := |A_i| \geq 2$. Let $\mathcal{M}$ be the matroid associated to the family $(A_i)_{i \in I}$: the underlying set of $\mathcal{M}$ is $M := \bigcup_{i \in I} A_i \cup \{O\}$ where $O \notin \bigcup_{i \in I} A_i$. We consider a family $(V_i)_{i \in I}$ of pairwise disjoint linearly ordered finite sets such that for each $i \in I$, $|V_i| = n_i - 1$. We also consider two distinct elements a and b not belonging to $\bigcup_{i \in I} V_i$, and we define the set $V := \{a, b\} \cup \bigcup_{i \in I} V_i$. Since each V_i is linearly ordered for each $i \in I$, we consider a graph G_i on $V_i \cup \{a, b\}$ which is an $(n_i + 1)$-cycle and such that $\{a, b\}$ is an edge of this graph: we denote by E_i the set of the n_i edges of G_i which are not equal to the edge $\{a, b\}$ of G_i. We consider the simple graph G on V which admits $E := \bigcup_{i \in I} E_i \cup \{\{a, b\}\}$ as set of edges, see Figure 1. Notice that every finite subgraph of G is planar. We denote by $\mathcal{G}$ the matroid on E associated to the graph G. Using $\mathrm{AC}^{\mathrm{fin}}$, we consider a family $(f_i)_{i \in I}$ such that for every $i \in I$, $f_i \colon E_i \longrightarrow A_i$ is a bijection. It follows that $f := \bigcup_{i \in I} f_i$ is a bijection from $\bigcup_{i \in I} E_i$ to $\bigcup_{i \in I} A_i$ and we extend it into a bijection from E to M. Then the bijection f respects circuits of $\mathcal{M}_G$ and $\mathcal{M}$ and thus the matroid $\mathcal{M}$ is graphic.

(2) ⇒ (1) Let $(A_i)_{i\in I}$ be a family of pairwise disjoint nonempty finite sets with at least two elements. Let $\mathcal{M}$ be the finitary matroid on $\{O\}\cup\bigcup_{i\in I} A_i$ associated to this family. Let $G=(V,E)$ be a graph such that $\mathcal{M}$ is the graphic matroid associated to G. Let a, b be the two extremities of the edge O of G. Then, for every $i\in I$, $A_i\cup\{O\}$ is the set of edges of a cycle of the graph G: let e_i be the unique edge of A_i which is incident to the vertex a. Then $(e_i)_{i\in I}$ is a choice function for the family $(A_i)_{i\in I}$. □

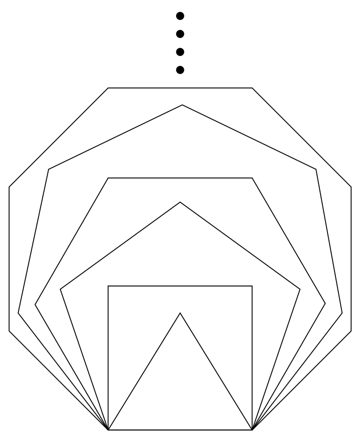

FIGURE 1. The graph G associated to the matroid $\mathcal{M}$.

Consider the following well known consequences of AC imply AC^{fin}:

MG_1: "For every binary matroid $\mathcal{M}$, if every finite minor of $\mathcal{M}$ is graphic then $\mathcal{M}$ is graphic".

MG_2: "For every binary matroid $\mathcal{M}$, if every finite submatroid of $\mathcal{M}$ is graphic and planar then $\mathcal{M}$ is graphic".

MG_3: "For every binary matroid $\mathcal{M}$, if every finite minor of $\mathcal{M}$ is graphic and planar then $\mathcal{M}$ is graphic".

Notice that both statements MG_1 and MG_2 imply MG_3. Moreover, every finite minor of the binary matroid used in the proof of Theorem 5 is graphic and planar, and thus, MG_3 implies AC^{fin}.

Question 2. Does AC^{fin} or sH imply one of the statements MG_1, MG_2 or MG_3?

Question 3. Is the following statement provable in ZF: "*Every (infinite) graphic matroid is hyperplane-accessible.*"?

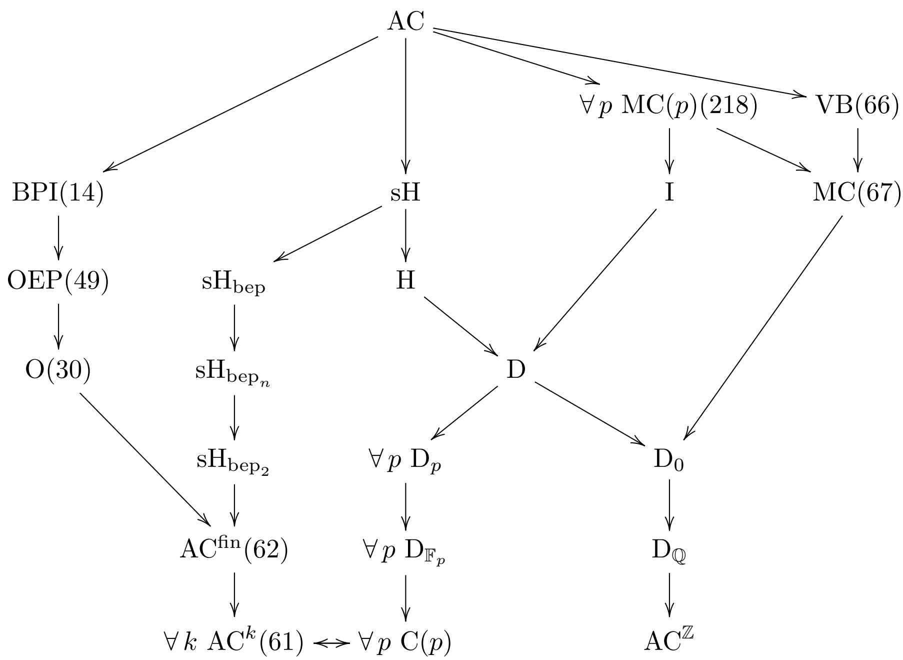

FIGURE 2. Summary diagram of the axioms.

In the diagram in Figure 2, we add the statement $\mathrm{D}_{\mathbb{Q}}$ which implies the statement $\mathrm{AC}^{\mathbb{Z}}$: "*Every family of posets isomorphic with the linear order $\mathbb{Z}$ has a nonempty product.*", see [10, Theorem 4]. We also add the statement D_0 (or D_p) which is D restricted to vector spaces over a commutative field $\mathbb{K}$ of characteristic 0 (p, respectively).

Question 4. The statements BPI ("Every non trivial Boolean algebra has a maximal ideal"), OEP ("Every partial order on a set X can be extended into a linear order on X") and O ("On every set X there exists a linear order"), see forms 14, 49 and 30 of [7], are well known consequences of AC which are stronger than $\mathrm{AC}^{\mathrm{fin}}$. Are there implications between one of them and H or sH or $\mathrm{sH}_{\mathrm{bep}}$ or $\mathrm{sH}_{\mathrm{bep}_n}$ for some integer $n \geq 2$?

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