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THE MAXIMUM REGULARITY PROPERTY OF THE STEADY STOKES PROBLEM ASSOCIATED WITH A FLOW THROUGH A PROFILE CASCADE IN L^r -FRAMEWORK

TOMÁŠ NEUSTUPA, Praha

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Abstract. We deal with the steady Stokes problem, associated with a flow of a viscous incompressible fluid through a spatially periodic profile cascade. Using the reduction to domain Ω , which represents one spatial period, the problem is formulated by means of boundary conditions of three types: the conditions of periodicity on curves Γ_{-} and Γ_{+} (lower and upper parts of $\partial\Omega$), the Dirichlet boundary conditions on $\Gamma_{\rm in}$ (the inflow) and Γ_{0} (boundary of the profile) and an artificial "do nothing"-type boundary condition on $\Gamma_{\rm out}$ (the outflow). We show that the considered problem has a strong solution with the L^{r} -maximum regularity property for appropriately integrable given data. From this we deduce a series of properties of the corresponding strong Stokes operator.

Keywords: Stokes problem; artificial boundary condition; maximum regularity property *MSC 2020*: 35Q30, 76D03, 76D05

1. INTRODUCTION

One spatial period: domain Ω . Mathematical models of a flow through a threedimensional turbine wheel often use the reduction to two space dimensions, where the flow is studied as a flow through an infinite planar profile cascade. In an appropriately chosen Cartesian coordinate system, the cascade consists of an infinite sequence of profiles P_k (for $k \in \mathbb{Z}$), which periodically repeat with the period τ in the x_2 direction. The profiles are supposed to be pairwise disjoint compact sets inside the stripe $\mathbb{R}^2_{(0,d)} := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2; 0 < x_1 < d\}$. Denote by \mathcal{O} the domain $\mathbb{R}^2_{(0,d)} \setminus \bigcup_{k=-\infty}^{\infty} P_k$. Due to the spatial periodicity of \mathcal{O} , one can naturally assume that the flow $k=-\infty$

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is spatially periodic, too, with the same spatial period $\tau \mathbf{e}_2$. (Here and further on, we denote by \mathbf{e}_2 the unit vector in the x_2 -direction.) This enables one to study the flow through one spatial period, which contains just one profile P_0 , see domain Ω on Fig. 1. This approach is used e.g. in [12], [23], [38], where the authors present the numerical analysis of the models or corresponding numerical simulations, and in papers [13]–[15] and [32]–[34], devoted to theoretical analysis of the mathematical models.



Figure 1. Domain Ω .

We assume that a viscous incompressible fluid flows into domain \mathcal{O} through the straight line γ_{in} (the x_2 -axis, the inflow) and essentially leaves the cascade through the straight line γ_{out} , whose equation is $x_1 = d$ (the outflow). By "essentially" we mean that we do not exclude possible reverse flows on the line γ_{out} . The parts of $\partial\Omega$ (the boundary of Ω), lying on the straight lines γ_{in} and γ_{out} are the line segments $\Gamma_{\text{in}} \equiv A_-A_+$ and $\Gamma_{\text{out}} \equiv B_-B_+$ of length τ , respectively. The other parts of $\partial\Omega$ are denoted by Γ_0 (the boundary of profile P_0), Γ_- and $\Gamma_+ \equiv \Gamma_- + \tau \mathbf{e}_2$, see Fig. 1. We assume that domain Ω is Lipschitzian and the curves Γ_0 , Γ_- and Γ_+ are of the class C^2 .

The Stokes boundary-value problem on one spatial period. The fluid flow is described by the Navier-Stokes equations. An important role in theoretical studies of these equations is played by the properties of solutions to the steady Stokes problem. The steady Stokes equation, which comes from the momentum equation in the Navier-Stokes system if one neglects the derivative with respect to time and the nonlinear "convective" term, has the form

(1.1)
$$-\nu\Delta \mathbf{u} + \nabla p = \mathbf{f}.$$

It is studied together with the equation of continuity (= condition of incompressibility)

 $div \mathbf{u} = 0.$

The unknowns are $\mathbf{u} = (u_1, u_2)$ (the velocity) and p (the pressure). The positive constant ν is the kinematic coefficient of viscosity and \mathbf{f} denotes the external body force. The density of the fluid can be without loss of generality supposed to be equal to one. System (1.1), (1.2) is completed by appropriate boundary conditions on $\partial\Omega$. One can naturally assume that the velocity profile on $\Gamma_{\rm in}$ is known, which leads to the inhomogeneous Dirichlet boundary condition

(1.3)
$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_{\text{in}}.$$

Further, we consider the homogeneous Dirichlet boundary condition

(1.4)
$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_0$$

and the conditions of periodicity on Γ_{-} and Γ_{+}

(1.5)
$$\mathbf{u}(x_1, x_2 + \tau) = \mathbf{u}(x_1, x_2) \qquad \text{for } \mathbf{x} \equiv (x_1, x_2) \in \Gamma_-,$$

(1.6)
$$\frac{\partial \mathbf{u}}{\partial \mathbf{n}}(x_1, x_2 + \tau) = -\frac{\partial \mathbf{u}}{\partial \mathbf{n}}(x_1, x_2) \quad \text{for } \mathbf{x} \equiv (x_1, x_2) \in \Gamma_-,$$

(1.7)
$$p(x_1, x_2 + \tau) = p(x_1, x_2)$$
 for $\mathbf{x} \equiv (x_1, x_2) \in \Gamma_-$.

Finally, we consider the artificial boundary condition

(1.8)
$$-\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} + p\mathbf{n} = \mathbf{h} \quad \text{on } \Gamma_{\text{out}},$$

where **h** is a given vector-function on Γ_{out} and **n** denotes the unit outer normal vector, which is equal to $\mathbf{e}_1 \equiv (1,0)$ on Γ_{out} . The boundary condition (1.8) (with $\mathbf{h} = \mathbf{0}$) is often called the "do nothing" condition, because it naturally follows from an appropriate weak formulation of the boundary-value problem, see [17] and [20].

On some previous related results. In studies of the Navier-Stokes equations in channels or profile cascades with artificial boundary conditions on the outflow, many authors use various modifications of condition (1.8), see, e.g., [8], [13], [14], [15], [32], [33], [34]. The reason is that while condition (1.8) does not enable one to control the amount of kinetic energy in Ω in the case of a reverse flow on Γ_{out} , the modifications are suggested so that one can derive an energy inequality, and consequently prove the existence of weak solutions. In papers [28] and [27], the authors use the boundary condition on an outflow in connection with a flow in a channel, and they prove the existence of weak solutions of the Navier-Stokes equations for "small data". Possible reverse flows (again on an "outflow" of a channel) are controlled by means of additional conditions in [24], [25], [26], where the Navier-Stokes equations are replaced by the Navier-Stokes variational inequalities. The regularity up to the boundary of existing weak solutions (stationary or timedependent) to the Navier-Stokes equations with the boundary condition (1.8) on a part of the boundary has not been studied in literature yet. This is mainly because one at first needs the information on regularity of solutions of the corresponding steady Stokes problem. However, to our best knowledge, there are only two papers which bring this information: (1) paper [7], where the authors studied a flow in a 2D channel D of a special geometry, considering the homogeneous Dirichlet boundary condition on the walls and condition (1.8) on the outflow, and proved that the velocity is in $\mathbf{W}^{2-\beta,2}(D)$ for certain $\beta > 0$ depending on the geometry of D, provided that $\mathbf{f} \in \mathbf{L}^2(D)$, and (2) paper [36], where the inclusion of the solution (\mathbf{u}, p) of the Stokes problem (1.1)–(1.8) to $\mathbf{W}^{2,2}(\Omega) \times W^{1,2}(\Omega)$ has been recently proven under natural assumptions on \mathbf{f} , \mathbf{g} and \mathbf{h} .

Some authors studied the Stokes system with boundary conditions of a similar nature to (1.8). The so-called Neumann condition $\mathbb{T}(\mathbf{u}, p) \cdot \mathbf{n} = \mathbf{h}$, where $\mathbb{T}(\mathbf{u}, p) =$ $-p\mathbb{I} + 2\nu(\nabla \mathbf{u})_{sym}$ is the stress tensor, has been considered in [1] in connection with the Stokes resolvent problem in an infinite layer $\Omega = \{0 < x_n < h\}$ in \mathbb{R}^n with the no slip boundary condition $\mathbf{u} = \mathbf{0}$ on the hyper-plane $\{x_n = 0\}$ and Neumann's condition on the hyper-plane $\{x_n = h\}$. The author obtained estimates of the $W^{2,r}$ norm of **u** and $W^{1,r}$ -norm of the pressure p for the range of resolvent parameters, including a "small" neighborhood of zero. The used procedure is based on the Fourier transform in the whole space \mathbb{R}^n and then an appropriate reduction to Ω . The Stokes problem in a bounded simply connected domain Ω in \mathbb{R}^2 with three types of boundary conditions on $\partial \Omega$, prescribing the normal component of velocity plus the tangential component of vorticity, or the tangential component of $\partial \mathbf{u}/\partial \mathbf{n} - p\mathbf{n} + b\mathbf{u}$, or the tangential component of $\mathbb{T}(\mathbf{u}, p) \cdot \mathbf{n} + b\mathbf{u}$, where b is a given function, was studied in [31]. Applying the theory of hydrodynamical potentials and writing the studied problem in the form of appropriate integral equations, the author derived necessary and sufficient conditions for the existence of a solution in $\mathbf{W}^{s,r}(\Omega) \times W^{s-1,r}(\Omega)$ or in Besov spaces or in the class of classical solutions. A similar method in a planar bounded simply connected domain has also been applied to the Stokes problem with the boundary conditions, prescribing p and the tangential component of \mathbf{u} , in [30].

One usually says that the Stokes problem has the maximum regularity property if the solution **u**, or p, has respectively two or one, spatial derivatives more than function **f**, integrable with the same power as **f**. It should be noted that the maximum regularity property of solutions of the steady Stokes problem is well known if domain Ω is sufficiently smooth, see, e.g., [39], Proposition I.2.2, [29], Theorem III.3, [16], Theorem IV.6.1 and [37], Theorem III.2.1.1 for problems with inhomogeneous Dirichlet boundary conditions, [4], [9] for problems with the Navier-type boundary condition and [2], [5], [10] for problems with Navier's boundary condition on the whole boundary. Concerning non-smooth domains, we can cite [18], [22] and [11], where the authors considered the Stokes problem in a 2D polygonal domain with the Dirichlet boundary conditions, and the aforementioned paper [36], where the maximum regularity property of the Stokes problem (1.1)-(1.8) has been proven in the L^2 -framework.

On the results of this paper. The main purpose of this paper is to prove the L^r -maximum regularity property of the steady Stokes problem (1.1)–(1.8) for $r \in (1, \infty)$ (see Theorem 3.1). In contrast to [1], [31] and [30], we use the advantage of having the weak solution $\mathbf{u} \in \mathbf{W}^{1,r}(\Omega)$ due to [35], and we can therefore focus just on the higher regularity and $W^{2,r}$ -estimates (or $W^{1,r}$ -estimates) of **u** (or p). In the case $r \ge 2$, we apply Theorem 2 from [36] in order to obtain the inclusion (\mathbf{u}, p) in $\mathbf{W}^{2,2}(\Omega) \times W^{1,2}(\Omega)$ and then, using Lemma 2.1 (on properties of the associated Stokes-type operator \mathcal{A}_r), Lemma 2.3 (on a weak solution of the considered Stokes problem in $\mathbf{W}^{1,r}(\Omega) \times L^{r}(\Omega)$ and Lemmas 3.2–3.4 (on local regularity properties of the solution (\mathbf{u}, p) in various parts of Ω), we show that (\mathbf{u}, p) belongs to $\mathbf{W}^{2,r}(\Omega) \times$ $W^{1,r}(\Omega)$. In the case 1 < r < 2, we approximate the given data by sequences of data which yield solutions in $\mathbf{W}^{2,2}(\Omega) \times W^{1,2}(\Omega)$, and afterwards we obtain a solution in $\mathbf{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$ by means of an appropriate limit procedure. We also sketch how Theorem 3.1 can be generalized so that it yields $(\mathbf{u}, p) \in \mathbf{W}^{n+2,r}(\Omega) \times W^{n+1,r}(\Omega)$ for $n \in \{0\} \cup \mathbb{N}$. The results do not directly follow from the previously cited papers on the Stokes problem due to the variety of used boundary conditions. In comparison to [36] (L^2 -maximum regularity property), the proof of Theorem 3.1 requires a series of different and subtler tools and arguments. Theorem 3.1 directly implies a series of properties of the so-called strong Stokes operator A_r , see Remark 3.4.

Finally, note that the presented results on the L^r -maximum regularity property of the considered Stokes-type problem and the strong Stokes operator A_r play a fundamental role in further studies of regularity and the structure of the set of weak and strong solutions to the corresponding Navier-Stokes problem. (Here, the L^2 maximum regularity property is not sufficient.)

2. NOTATION AND AUXILIARY RESULTS

Notation. We assume that $1 < r < \infty$ throughout the paper.

▷ Recall that Ω is a Lipschitzian domain in \mathbb{R}^2 which represents one spatial period of an unbounded spatially periodic domain \mathcal{O} . (See Section 1 and Fig. 1.) The boundary of Ω consists of the line segments Γ_{in} , Γ_{out} and of the C^2 -curves Γ_- , Γ_+ and Γ_0 . We denote by $\mathbf{n} = (n_1, n_2)$ the outer normal vector field on $\partial\Omega$.

- $\triangleright \ \Gamma_{in}^{0}$ and Γ_{out}^{0} denote the open line segment with the end points A_{-}, A_{+} and $B_{-},$ B_+ , respectively. Similarly, Γ^0_- and Γ^0_+ denote the curve Γ_- and Γ_+ without the end points A_{-} , B_{-} and A_{+} , B_{+} , respectively.
- \triangleright We denote by $\|\cdot\|_r$ the norm in $L^r(\Omega)$ or in $\mathbf{L}^r(\Omega)$ or in $L^r(\Omega)^{2\times 2}$. Similarly, $\|\cdot\|_{s,r}$ is the norm in $W^{s,r}(\Omega)$ or in $\mathbf{W}^{s,r}(\Omega)$ or in $W^{s,r}(\Omega)^{2\times 2}$.
- \triangleright For $k \in \mathbb{N}$ we denote by $W^{k,r}_{\text{per}}(\mathcal{O})$ the space of functions from $W^{k,r}_{\text{loc}}(\mathcal{O}), \tau$ periodic in variable x_2 , whose restriction to Ω is in $W^{k,r}(\Omega)$.
- $\triangleright W^{k,r}_{per}(\Omega)$ is the space of functions that can be extended from Ω to \mathcal{O} as functions in $W^{k,r}_{\text{per}}(\mathcal{O})$. (The traces of these functions on Γ_{-} and Γ_{+} satisfy the condition of periodicity, analogous to (1.5).)
- $> W_{\text{per}}^{k-1/r,r}(\gamma_{\text{out}}) \text{ (for } k \in \mathbb{N} \text{) denotes the space of } \tau \text{-periodic functions in } W_{\text{loc}}^{k-1/r,r}(\gamma_{\text{out}}).$ $> W_{\text{per}}^{k-1/r,r}(\Gamma_{\text{out}}) \text{ is the space of functions from } W^{k-1/r,r}(\Gamma_{\text{out}}) \text{ that can be ex-}$
- tended from Γ_{out} to γ_{out} as functions in $W_{\text{per}}^{k-1/r,r}(\gamma_{\text{out}})$. \triangleright The space $W_{\text{per}}^{k-1/r,r}(\Gamma_{\text{in}})$ is defined by analogy with $W_{\text{per}}^{k-1/r,r}(\Gamma_{\text{out}})$.
- ▷ Vector functions and spaces of vector functions are denoted by boldface letters. Spaces of 2nd-order tensor functions are denoted by the superscript 2×2 .
- $\triangleright \ \mathcal{C}^{\infty}_{\sigma}(\Omega)$ denotes the linear space of all infinitely differentiable divergence-free vector functions in $\overline{\Omega}$, whose support is disjoint with $\Gamma_{in} \cup \Gamma_0$ and that satisfy, together with all their derivatives (of all orders), the condition of periodicity (1.5). Note that each $\mathbf{w} \in \mathcal{C}^{\infty}_{\sigma}(\Omega)$ automatically satisfies the outflow condition $\int_{\Gamma_{\text{out}}} \mathbf{w} \cdot \mathbf{n} \, \mathrm{d}l = 0.$
- $\triangleright \mathbf{V}^{1,r}_{\sigma}(\Omega)$ is the closure of $\mathcal{C}^{\infty}_{\sigma}(\Omega)$ in $\mathbf{W}^{1,r}(\Omega)$. It is a space of divergence-free vector functions from $\mathbf{W}^{1,r}(\Omega)$, whose traces on $\Gamma_{\rm in} \cup \Gamma_0$ are equal to zero and the traces on Γ_{-} and Γ_{+} satisfy the condition of periodicity (1.5). Since functions from $\mathbf{V}_{\sigma}^{1,r}(\Omega)$ are equal to zero on $\Gamma_{\mathrm{in}} \cup \Gamma_0$ (in the sense of traces) and domain Ω is bounded, the norm in $\mathbf{V}^{1,r}_{\sigma}(\Omega)$ is equivalent to $\|\nabla \cdot\|_r$.
- \triangleright The conjugate exponent to r is denoted by r', the dual space to $\mathbf{W}_{0}^{1,r'}(\Omega)$ is denoted by $\mathbf{W}^{-1,r}(\Omega)$ and the dual space to $\mathbf{W}^{1,r'}(\Omega)$ is denoted by $\mathbf{W}_{0}^{-1,r}(\Omega)$. The corresponding norms are denoted by $\|\cdot\|_{\mathbf{W}^{-1,r}}$ and $\|\cdot\|_{\mathbf{W}^{-1,r}_{o}}$.
- $\triangleright \mathbf{V}_{\sigma}^{-1,r}(\Omega)$ denotes the dual space to $\mathbf{V}_{\sigma}^{1,r'}(\Omega)$. The duality pairing between $\mathbf{V}_{\sigma}^{-1,r}(\Omega) \text{ and } \mathbf{V}_{\sigma}^{1,r'}(\Omega) \text{ is denoted by } \langle \cdot, \cdot \rangle_{(\mathbf{V}_{\sigma}^{-1,r},\mathbf{V}_{\sigma}^{1,r'})}. \text{ The norm in } \mathbf{V}_{\sigma}^{-1,r}(\Omega)$ is denoted by $\|\cdot\|_{\mathbf{V}^{-1,r}}$.
- \triangleright Denote by \mathcal{A}_r the linear mapping $\mathbf{V}^{1,r}_{\sigma}(\Omega) \to \mathbf{V}^{-1,r}_{\sigma}(\Omega)$ defined by the equation

(2.1)
$$\langle \mathcal{A}_{r}\mathbf{v},\mathbf{w}\rangle_{(\mathbf{V}_{\sigma}^{-1,r},\mathbf{V}_{\sigma}^{1,r'})} = (\nabla\mathbf{v},\nabla\mathbf{w}) \text{ for } \mathbf{v}\in\mathbf{V}_{\sigma}^{1,r}(\Omega) \text{ and } \mathbf{w}\in\mathbf{V}_{\sigma}^{1,r'}(\Omega),$$

where $(\nabla \mathbf{v}, \nabla \mathbf{w})$ represents the integral $\int_{\Omega} \nabla \mathbf{v} : \nabla \mathbf{w} \, \mathrm{d} \mathbf{x}$. $\triangleright \mathbb{R}^2_{d-}$ denotes the half-plane $\{(x_1, x_2) \in \mathbb{R}^2; x_1 < d\}.$

 \triangleright We use c as a generic constant, i.e., a constant whose values may change throughout the text.

Further, we cite some auxiliary results from previous papers. They all concern in a certain sense the equation $\mathcal{A}_r \mathbf{v} = \mathbf{f}$, which can be interpreted as the weak Stokes problem.

Lemma 2.1. \mathcal{A}_r is a bounded, closed and one-to-one operator from $\mathbf{V}^{1,r}_{\sigma}(\Omega)$ to $\mathbf{V}^{-1,r}_{\sigma}(\Omega)$ with $D(\mathcal{A}_r) = \mathbf{V}^{1,r}_{\sigma}(\Omega)$ and $R(\mathcal{A}_r) = \mathbf{V}^{-1,r}_{\sigma}(\Omega)$. The adjoint operator to \mathcal{A}_r is $\mathcal{A}_{r'}$.

Lemma 2.1 comes from [35], Theorem 1.

For $\mathbb{F} \in L^r(\Omega)^{2 \times 2}$, define $\mathbf{F} \in \mathbf{V}_{\sigma}^{-1,r}(\Omega)$ by the formula

(2.2)
$$\langle \mathbf{F}, \mathbf{w} \rangle_{(\mathbf{V}_{\sigma}^{-1,r}, \mathbf{V}_{\sigma}^{1,r'})} := -\int_{\Omega} \mathbb{F} : \nabla \mathbf{w} \, \mathrm{d} \mathbf{x}$$

for all $\mathbf{w} \in \mathbf{V}_{\sigma}^{1,r'}(\Omega)$. Obviously, div $\mathbb{F} = \mathbf{F}$ in the sense of distributions and $\|\mathbf{F}\|_{\mathbf{V}_{\sigma}^{-1,r}} \leq c \|\mathbb{F}\|_{r}$.

Lemma 2.2. Let $\mathbf{g} \in \mathbf{W}_{per}^{1-1/r,r}(\Gamma_{in})$ be a given function on Γ_{in} . There exists a divergence-free extension $\mathbf{g}_* \in \mathbf{W}_{per}^{1,r}(\Omega)$ of \mathbf{g} from Γ_{in} to Ω and a constant $c_1 > 0$, independent of \mathbf{g} , such that

- (a) $\|\mathbf{g}_*\|_{1,r} \leq c_1 \|\mathbf{g}\|_{1-1/r,r;\Gamma_{\text{in}}},$
- (b) $\mathbf{g}_* = (\Phi/\tau)\mathbf{e}_1$ in a neighborhood of Γ_{out} , where $\Phi = -\int_{\Gamma_{\text{in}}} \mathbf{g} \cdot \mathbf{n} \, \mathrm{d}l$,
- (c) $\mathbf{g}_* = \mathbf{0}$ in the sense of traces on Γ_0 .

Lemma 2.2 is a slight modification of Lemma 2 in [35] in the sense that the function **g** is supposed to be in $\mathbf{W}_{per}^{1-1/r,r}(\Gamma_{in})$ instead of $\mathbf{W}^{s,r}(\Gamma_{in})$ (for s > 1/r if $1 < r \leq 2$ and s = 1 - 1/r if r > 2), as in [35]. It can be proven by means of the same arguments as Lemma 2 in [35].

Define $\mathbf{G} \in \mathbf{V}_{\sigma}^{-1,r}(\Omega)$ by the formula

(2.3)
$$\langle \mathbf{G}, \mathbf{w} \rangle_{(\mathbf{V}_{\sigma}^{-1,r}, \mathbf{V}_{\sigma}^{1,r'})} := \int_{\Omega} \nabla \mathbf{g}_{*} : \nabla \mathbf{w} \, \mathrm{d} \mathbf{x}$$

The norm of **G** in $\mathbf{V}_{\sigma}^{-1,r}(\Omega)$ satisfies $\|\mathbf{G}\|_{\mathbf{V}_{\sigma}^{-1,r}} \leq c \|\nabla \mathbf{g}_{*}\|_{r}$. Finally, the next Lemma 2.3 follows from [35], Theorem 2.

Lemma 2.3. Let the elements \mathbf{F} and \mathbf{G} of $\mathbf{V}_{\sigma}^{-1,r}(\Omega)$ be defined by formulas (2.2) and (2.3), respectively. Then there exists a unique solution $\mathbf{v} \in \mathbf{V}_{\sigma}^{1,r}(\Omega)$ of the equation $\nu \mathcal{A}_r \mathbf{v} = \mathbf{F} + \nu \mathbf{G}$. Moreover, there exists an associated pressure $p \in L^r(\Omega)$ such that $\mathbf{u} := \mathbf{g}_* + \mathbf{v}$ and p satisfy the equation

(2.4)
$$-\nu\Delta \mathbf{u} + \nabla p = \operatorname{div} \mathbb{F}$$

in the sense of distributions in Ω ,

(2.5)
$$(-\nu\nabla\mathbf{u} - p\mathbb{I} - \mathbb{F}) \cdot \mathbf{n} = \mathbf{0}$$

holds as an equality in $\mathbf{W}_{per}^{-1/r,r}(\Gamma_{out})$ (the dual to $\mathbf{W}_{per}^{1-1/r',r'}(\Gamma_{out})$) and

(2.6)
$$\|\mathbf{u}\|_{1,r} + \|p\|_r \leqslant c_2(\|\mathbb{F}\|_r + \|\nabla \mathbf{g}_*\|_r).$$

where $c_2 = c_2(\Omega, \nu)$.

3. The strong Stokes problem

Lemma 3.1. Let $\mathbf{f} \in \mathbf{L}^r(\Omega)$ and $\mathbf{h} \in \mathbf{W}_{per}^{1-1/r,r}(\Gamma_{out})$ be given. Then there exists $\mathbb{F} \in W_{per}^{1,r}(\Omega)^{2\times 2}$ such that div $\mathbb{F} = \mathbf{f}$ a.e. in Ω , $\mathbb{F} = \mathbb{O}$ on Γ_0 , $\mathbb{F} \cdot \mathbf{n} = \mathbf{h}$ a.e. on Γ_{out} in the sense of traces and

(3.1)
$$\|\mathbb{F}\|_{1,r} \leq c(\|\mathbf{f}\|_r + \|\mathbf{h}\|_{1-1/r,r;\Gamma_{\text{out}}}),$$

where $c = c(\Omega, r)$.

We present only main ideas of the proof of Lemma 3.1, because there exists a series of analogous results in literature. The authors typically show that to given functions f in Ω and \mathbf{g} on $\partial\Omega$ (satisfying $\int_{\Omega} \mathbf{g} \cdot \mathbf{n} \, d\mathbf{x} = 0$), there exists a vector function \mathbf{v} in Ω , satisfying div $\mathbf{v} = f$ in Ω and $\mathbf{v} = \mathbf{g}$ on $\partial\Omega$. See, e.g., [6], Lemma 3.3 or [16], Theorem III.3.3, where f and \mathbf{g} are supposed to be in $L^r(\Omega)$ and $\mathbf{W}^{1-1/r,r}(\partial\Omega)$, respectively, and \mathbf{v} is constructed in $\mathbf{W}^{1,r}(\Omega)$, satisfying appropriate estimates.

Principles of the proof. (1) Denote by Ω_{-} the mirror image of Ω in the halfplane $x_1 < 0$ with respect to the line $x_1 = 0$. Hence,

$$\Omega_{-} = \{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2; \ (-x_1, x_2) \in \Omega \}.$$

Put $\widetilde{\Omega} := \Omega \cup \Gamma_{in}^0 \cup \Omega_-$.

(2) Extend **f** from Ω to $\widetilde{\Omega}$ as an odd function in variable x_1 . Then $\int_{\widetilde{\Omega}} \mathbf{f} \, \mathrm{d} \mathbf{x} = 0$. Hence, there exists $\mathbb{H}_0 \in W_0^{1,r}(\widetilde{\Omega})^{2\times 2}$ such that $\operatorname{div} \mathbb{H}_0 = \mathbf{f}$ in $\widetilde{\Omega}$ and $\mathbb{H}_0|_{1,r;\widetilde{\Omega}} \leq c \|\mathbf{f}\|_r$.

(3) Put $\overline{\mathbf{h}} \equiv (\overline{h}_1, \overline{h}_2) := \tau^{-1} \int_{B_-}^{B_+} \mathbf{h}(d, \vartheta) \, d\vartheta$. Let $\overline{\mathbb{H}}$ be the 2 × 2 matrix whose 1st column is $(\overline{h}_1, \overline{h}_2)^T$ and the second column is $(0, 0)^T$. Then div $\overline{\mathbb{H}} = \mathbf{0}$ and $\overline{\mathbb{H}} \cdot \mathbf{n} = \overline{\mathbf{h}}$ on Γ_{out} .

(4) Let $\zeta = \zeta(x_1)$ be an infinitely differentiable even function in [-d, d], supported in $[-d, -d + \delta] \cup [d - \delta, d]$, satisfying $\zeta(-d) = \zeta(d) = 1$. (Suppose that $\delta > 0$ is chosen so small that the profile P_0 lies on the left from the line $x_1 = d - \delta$.) Put $\mathbb{H}_1 := \zeta(x_1)\overline{\mathbb{H}}$. Then div $\mathbb{H}_1 = \zeta'(x_1)\overline{\mathbf{h}}$ in $\widetilde{\Omega}$, $\mathbb{H}_1 = \mathbb{O}$ in $\Gamma_{\mathrm{in}} \cup \Gamma_0$ and $\mathbb{H}_1 \cdot \mathbf{n} = \overline{\mathbf{h}}$ on Γ_{out} . (5) As $\int_{\widetilde{\Omega}} \zeta'(x_1) \overline{\mathbf{h}} \, \mathrm{d} \mathbf{x} = 0$, we deduce that there exists $\mathbb{H}_2 \in W_0^{1,r}(\widetilde{\Omega})^{2 \times 2}$, satisfying $\operatorname{div} \mathbb{H}_2 = -\zeta'(x_1) \overline{\mathbf{h}}$ in $\widetilde{\Omega}$ and $\|\mathbb{H}_2\|_{1,r;\widetilde{\Omega}} \leq c \|\zeta'(x_1) \overline{\mathbf{h}}\|_{r;\widetilde{\Omega}} \leq c \|\mathbf{h}\|_{1-1/r,r;\Gamma_{\mathrm{out}}}$.

(6) We construct a vector function $= (\psi_1, \psi_2)$ in $\mathbf{W}_{per}^{2,r}(\Omega)$, supported just in the neighborhood of Γ_{out} , satisfying the condition $\partial_2 = \mathbf{h} - \overline{\mathbf{h}}$ on Γ_{out} , and we put $\mathbb{H}_3 := \nabla^{\perp}$. (Hence, the *i*th row in \mathbb{H}_3 is $\nabla^{\perp}\psi_i = (\partial_2\psi_i, -\partial_1\psi_i)$, i = 1, 2.) Then $\mathbb{H}_3 \in W_{per}^{1,r}(\Omega)^{2\times 2}$, $\mathbb{H}_3 = \mathbb{O}$ on $\Gamma_{in} \cup \Gamma_0$, div $\mathbb{H}_3 = \mathbf{0}$ in Ω and $\mathbb{H}_3 \cdot \mathbf{n} = \mathbf{h} - \overline{\mathbf{h}}$ on Γ_{out} . (7) The sum $\mathbb{F} := \mathbb{H}_0 + \mathbb{H}_1 + \mathbb{H}_2 + \mathbb{H}_3$ has all the required properties.

Remark 3.1. Let the function \mathbf{g} in the assumptions of Lemma 2.2 be in $\mathbf{W}_{\text{per}}^{2-1/r,r}(\Gamma_{\text{in}})$. Then the extension \mathbf{g}_* of \mathbf{g} from Γ_{in} to Ω can be constructed so that in addition to the properties (a)–(c) listed in Lemma 2.2, it is in $\mathbf{W}^{2,r}(\Omega)$ and

- (d) $\|\mathbf{g}_*\|_{2,r} \leq c \|\mathbf{g}\|_{2-1/r,r;\Gamma_{\text{in}}}$, where *c* is independent of \mathbf{g} ,
- (e) \mathbf{g}_* satisfies the condition of periodicity (1.6) on $\Gamma_- \cup \Gamma_+$.

The possibility of the construction of \mathbf{g}_* with the properties (d) and (e) follows, similarly as Lemma 2.2, from an appropriate modification of the proof of Lemma 2 in [35]. Using the higher regularity of \mathbf{g} , one can in principle apply the same arguments so that one obtains the extension \mathbf{g}_* with all the properties (a)–(e).

Theorem 3.1 (On a strong solution of the Stokes problem (1.1)–(1.8)). Let the closed curve Γ_0 (which is the boundary of the profile) be of the class C^2 , $\mathbf{f} \in \mathbf{L}^r(\Omega)$, $\mathbf{h} \in \mathbf{W}_{per}^{1-1/r,r}(\Gamma_{out})$, $\mathbf{g} \in \mathbf{W}_{per}^{2-1/r,r}(\Gamma_{in})$ be given. Let \mathbb{F} and \mathbf{g}_* be the functions given by Lemma 3.1 and Remark 3.1. Let the functionals \mathbf{F} and \mathbf{G} be defined by formulas (2.2) and (2.3), respectively. Then

- (1) the unique solution \mathbf{v} of the equation $\nu \mathcal{A}_r \mathbf{v} = \mathbf{F} + \nu \mathbf{G}$ belongs to $\mathbf{V}^{1,r}_{\sigma}(\Omega) \cap \mathbf{W}^{2,r}(\Omega)$,
- (2) there exists an associated pressure $p \in W^{1,r}(\Omega)$ so that the functions $\mathbf{u} := \mathbf{g}_* + \mathbf{v}$ and p satisfy equations (1.1) (with $\mathbf{f} = \operatorname{div} \mathbb{F}$) and (1.2) a.e. in Ω ,
- (3) **u**, *p* satisfy boundary conditions (1.3), (1.4) and (1.8) in the sense of traces on $\Gamma_{\rm in}$, Γ_0 and $\Gamma_{\rm out}$, respectively,
- (4) u, p satisfy the conditions of periodicity (1.5)–(1.7) in the sense of traces on Γ₋ and Γ₊,
- (5) there exists a constant $c_3 = c_3(\nu, \Omega)$ such that

(3.2)
$$\|\mathbf{u}\|_{2,r} + \|\nabla p\|_r \leq c_3(\|\mathbf{f}\|_r + \|\mathbf{g}\|_{2-1/r,r;\Gamma_{\text{in}}} + \|\mathbf{h}\|_{1-1/r,r;\Gamma_{\text{out}}})$$

R e m a r k 3.2. The conclusions $\mathbf{u} \in \mathbf{W}^{2,r}(\Omega)$ and $p \in W^{1,r}(\Omega)$, following from Theorem 3.1, together with inequality (3.2), represent the maximum regularity property of the Stokes problem (1.1)–(1.8).

Proof of Theorem 3.1. The existence and uniqueness of the solution $\mathbf{v} \in \mathbf{V}_{\sigma}^{1,r}(\Omega)$ of the equation $\nu \mathcal{A}_r \mathbf{v} = \mathbf{F} + \nu \mathbf{G}$ and an associated pressure $p \in L^r(\Omega)$ are guaranteed by Lemma 2.3. It also follows from Lemma 4 that the functions $\mathbf{u} := \mathbf{g}_* + \mathbf{v}$ and psatisfy (2.4)–(2.6). The proof can be split in two cases with respect to value of r.

Case 1: $r \ge 2$. Then, due to [36], Theorem 2, $\mathbf{v} \in \mathbf{W}_{per}^{2,2}(\Omega)$ and $p \in W_{per}^{1,2}(\Omega)$, **u** and p satisfy equations (2.4), (1.2) a.e. in Ω and the boundary conditions (1.3), (1.4), (1.8) in the sense of traces on Γ_{in} , Γ_0 and Γ_{out} , respectively. Moreover,

(3.3)
$$\|\mathbf{u}\|_{2,2} + \|\nabla p\|_2 \leqslant c(\|\mathbf{f}\|_2 + \|\mathbf{g}\|_{3/2,2;\Gamma_{\text{in}}} + \|\mathbf{h}\|_{1/2,2;\Gamma_{\text{out}}}).$$

This implies the validity of statements (2) and (3). The validity of statement (4) follows from the fact that $\mathbf{v} \in \mathbf{W}_{per}^{2,2}(\Omega)$ and the extended function \mathbf{g}_* satisfies the conditions of periodicity (1.5), (1.6). Thus, we only need to prove items (1) and (5).

We split the proof into three lemmas, where we successively show that $\mathbf{v} \in \mathbf{W}^{2,r}$ and $p \in W^{1,r}$ in the interior of Ω plus the neighborhood of Γ_0 and the neighborhood of any closed subset of Γ_{in}^0 (Lemma 3.2), in the neighborhood of Γ_{out}^0 (Lemma 3.3) and in the neighborhoods of Γ_- and Γ_+ (Lemma 3.4). Lemmas 3.2–3.4 also provide estimates which finally imply (3.2).

Lemma 3.2. Let Ω' be a sub-domain of Ω such that $\overline{\Omega'} \subset \Omega \cup \Gamma_{in}^0 \cup \Gamma_0$. Then $\mathbf{v} \in \mathbf{W}^{2,r}(\Omega'), \ p \in W^{1,r}(\Omega')$ and

(3.4)
$$\|\mathbf{v}\|_{2,r;\Omega'} + \|\nabla p\|_{r;\Omega'} \leqslant c(\|\operatorname{div} \mathbb{F}\|_r + \|\mathbf{g}_*\|_{2,r} + \|\mathbf{v}\|_{1,r}),$$

where $c = c(\nu, \Omega, \Omega')$.

Proof. Let Ω'' be a smooth sub-domain of Ω such that $\Omega' \subset \Omega'' \subset \Omega$, $\overline{\Omega''} \subset \Omega \cup \Gamma_{\text{in}}^0 \cup \Gamma_0$ and $\operatorname{dist}(\partial \Omega'' \cap \Omega; \partial \Omega' \cap \Omega) > 0$. Let η be an infinitely differentiable cut-off function in Ω such that $\operatorname{supp} \eta \subset \overline{\Omega''}$ and $\eta = 1$ in Ω' . Put $\widetilde{\mathbf{v}} := \eta \mathbf{v}$ and $\widetilde{p} := \eta p$. The functions $\widetilde{\mathbf{v}}, \widetilde{p}$ represent a strong solution of the problem

$$(3.5) \qquad \quad -\nu\Delta\widetilde{\mathbf{v}} + \nabla\widetilde{p} = \widetilde{\mathbf{f}} \quad \text{in } \Omega''$$

(3.6)
$$\operatorname{div} \widetilde{\mathbf{v}} = h \quad \text{in } \Omega'',$$

(3.7)
$$\widetilde{\mathbf{v}} = \mathbf{0} \quad \text{on } \partial \Omega''$$

where

$$\tilde{\mathbf{f}} := \eta \mathrm{div} \, \mathbb{F} - 2\nu \nabla \eta \cdot \nabla \mathbf{v} - \nu (\Delta \eta) \mathbf{v} - (\nabla \eta) p + \nu \eta \Delta \mathbf{g}_* \quad \mathrm{and} \quad \widetilde{h} := \nabla \eta \cdot \mathbf{v}.$$

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As div $\mathbb{F} \in \mathbf{L}^{r}(\Omega)$ and $\mathbf{u} \in \mathbf{W}^{1,r}(\Omega)$, $\mathbf{v} \in \mathbf{V}^{1,r}_{\sigma}(\Omega)$ and $p \in L^{r}(\Omega)$ (satisfying (2.6)), we have $\tilde{\mathbf{f}} \in \mathbf{L}^{r}(\Omega)$, $\tilde{h} \in W^{1,r}(\Omega)$ and

(3.8)
$$\|\widetilde{\mathbf{f}}\|_{r} \leq c(\|\operatorname{div} \mathbb{F}\|_{r} + \|\mathbf{g}_{*}\|_{2,r} + \|\mathbf{v}\|_{1,r}),$$

$$\|\tilde{h}\|_{1,r} \leqslant c \|\mathbf{v}\|_r \leqslant c \|\mathbf{v}\|_{1,r},$$

where $c = c(\nu, \eta)$. Due to [39], Proposition I.2.3, p. 35, $\tilde{\mathbf{v}} \in \mathbf{W}^{2,r}(\Omega'')$, $\tilde{p} \in W^{1,r}(\Omega'')$ and

$$\|\widetilde{\mathbf{v}}\|_{2,r;\Omega^{\prime\prime}} + \|\nabla\widetilde{p}\|_{r;\Omega^{\prime\prime}} \leqslant c(\|\widetilde{\mathbf{f}}\|_{r;\Omega^{\prime\prime}} + \|\widetilde{h}\|_{1,r;\Omega^{\prime\prime}})$$

where $c = c(\Omega'')$. Consequently, $\mathbf{v} \in \mathbf{W}^{2,r}(\Omega')$, $p \in W^{1,r}(\Omega')$ and (3.4) holds. \Box

Recall that Γ_{out}^0 is the open line segment with the end points B_- and B_+ .

Lemma 3.3. Let Ω' be a sub-domain of Ω such that $\overline{\Omega'} \subset \Omega \cup \Gamma_{out}^0$. Then $\mathbf{v} \in \mathbf{W}^{2,r}(\Omega'), p \in W^{1,r}(\Omega')$ and inequality (3.4) holds.

Proof. Here, we must use a different method than in the proof of Lemma 3.2. The reason is that we cannot apply Proposition I.2.3 from [39], because it concerns the Stokes problem with the Dirichlet boundary condition, which we do not have on Γ_{out} .



Figure 2. The sets Ω' , Γ'_{out} , U_1 and U_2 .

Denote by Γ'_{out} the intersection of $\overline{\Omega'}$ with Γ^0_{out} . We may assume, without loss of generality, that $\Gamma'_{out} \neq \emptyset$ and it is a line segment.

Let $\varrho_2 > \varrho_1 > 0$. Denote $U_1 := \{ \mathbf{x} \in \mathbb{R}^2 ; \operatorname{dist}(\mathbf{x}, \Gamma'_{\operatorname{out}}) < \varrho_1 \}$ and $U_2 := \{ \mathbf{x} \in \mathbb{R}^2 ; \operatorname{dist}(\mathbf{x}, \Gamma'_{\operatorname{out}}) < \varrho_2 \}$. Suppose that ϱ_2 is so small that $\overline{U_2} \cap \mathbb{R}^2_{d-1} \subset \Omega$. (See Fig. 2.)

Step 1. We will construct a divergence-free function $\tilde{\mathbf{v}}$ in $\mathbf{W}_0^{1,r}(U_2)$ that coincides with \mathbf{v} in $U_1 \cap \mathbb{R}^2_{d-}$.

Let η be a C^{∞} -function in \mathbb{R}^2 , supported in $\overline{U_2}$, such that $\eta = 1$ in U_1 and η is symmetric with respect to the line $x_1 = d$. (It means that $\eta(d + \vartheta, x_2) = \eta(d - \vartheta, x_2)$ for all $\vartheta, x_2 \in \mathbb{R}$.)

Due to [21], there exists a divergence-free extension \mathbf{v}_2 of function \mathbf{v} from $U_2 \cap \mathbb{R}^2_{d-}$ to the whole set U_2 , such that $\mathbf{v}_2 \in \mathbf{W}^{1,r}(U_2)$ and $\|\mathbf{v}_2\|_{1,r;U_2} \leq c \|\mathbf{v}\|_{1,r}$, where c is independent of \mathbf{v} .

Since $\nabla \eta \cdot \mathbf{v}_2 \in W_0^{1,r}(U_2)$ and $\int_{U_2} \nabla \eta \cdot \mathbf{v}_2 \, \mathrm{d}\mathbf{x} = 0$, there exists (by [16], Theorem III.3.3) $\mathbf{v}_* \in \mathbf{W}_0^{2,r}(U_2)$ such that div $\mathbf{v}_* = \nabla \eta \cdot \mathbf{v}_2$ in U_2 and

$$\|\mathbf{v}_*\|_{2,r;U_2} \leqslant c \|\nabla \eta \cdot \mathbf{v}_2\|_{1,r;U_2} \leqslant c \|\mathbf{v}_2\|_{1,r;U_2} \leqslant c \|\mathbf{v}\|_{1,r},$$

where c is independent of **v**. Extending \mathbf{v}_* by zero to $\mathbb{R}^2 \setminus U_2$, we have $\|\mathbf{v}_*\|_{2,r} \leq c \|\mathbf{v}\|_{1,r}$. Put

(3.10)
$$\widetilde{\mathbf{v}} := \eta \mathbf{v}_2 - \mathbf{v}_*, \quad \widetilde{p} := \eta p.$$

Function $\widetilde{\mathbf{v}}$ is divergence-free, belongs to $\mathbf{W}_0^{1,r}(U_2)$ and satisfies the estimates

$$\|\widetilde{\mathbf{v}}\|_{1,r;U_2} \leqslant c(\|\mathbf{v}_2\|_{1,r;U_2} + \|\mathbf{v}_*\|_{1,r;U_2}) \leqslant c\|\mathbf{v}_2\|_{1,r;U_2} \leqslant c\|\mathbf{v}\|_{1,r},$$

where c is independent of **v**. Moreover, as $\mathbf{v} \in \mathbf{W}_{per}^{2,2}(\Omega)$ and $p \in W_{per}^{1,2}(\Omega)$, the functions $\tilde{\mathbf{v}}$ and \tilde{p} (defined by (3.10) in $U_2 \cap \mathbb{R}^2_{d_-}$ and extended by zero to $\mathbb{R}^2_{d_-}$), are in $\mathbf{W}^{2,2}(\mathbb{R}^2_{d_-})$ and $W^{2,2}(\mathbb{R}^2_{d_-})$, respectively. They satisfy equation (3.5) a.e. in the half-plane $\mathbb{R}^2_{d_-}$, where function $\tilde{\mathbf{f}}$ is now given by the formula

$$\tilde{\mathbf{f}} := \eta \mathrm{div} \,\mathbb{F} - 2\nu \nabla \eta \cdot \nabla \mathbf{v} - \nu(\Delta \eta) \mathbf{v} - (\nabla \eta) p + \nu \eta \Delta \mathbf{g}_* + \nu \Delta \mathbf{v}_* \quad \mathrm{in} \ U_2 \cap \mathbb{R}^2_{d-1}$$

and $\tilde{\mathbf{f}} := \mathbf{0}$ in $\mathbb{R}^2_{d-} \setminus U_2$. This function, although it is different from function $\tilde{\mathbf{f}}$ from the proof of Lemma 3.2, satisfies the estimate (3.8).

Note that

(3.11)
$$\nu \frac{\partial \widetilde{\mathbf{v}}}{\partial \mathbf{n}} - \widetilde{p} \widetilde{\mathbf{n}} = \nu \eta \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - \nu \frac{\partial \mathbf{v}_*}{\partial \mathbf{n}} - \eta p \mathbf{n} = \eta \left(\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p \mathbf{n} \right) - \nu \frac{\partial \mathbf{v}_*}{\partial \mathbf{n}}$$
$$= -\eta \mathbb{F} \cdot \mathbf{n} - \nu \frac{\partial \mathbf{v}_*}{\partial \mathbf{n}} = -\widetilde{\mathbf{h}}$$

a.e. on Γ_{out} , where

 $\widetilde{\mathbf{h}} := \eta \mathbb{F} \cdot \mathbf{n} + \nu \frac{\partial \mathbf{v}_*}{\partial \mathbf{n}} \quad (\text{in the sense of traces on } \Gamma_{\text{out}}).$

We have used the identities $\mathbf{u} = \mathbf{v} + \mathbf{g}_*$ (in Ω) and $\partial \mathbf{g}_* / \partial \mathbf{n} = \mathbf{0}$ (a.e. on Γ_{out}).

Step 2. Function $\widetilde{\mathbf{h}}$ satisfies the estimates

(3.12)
$$\|\widetilde{\mathbf{h}}\|_{1-1/r,r;\Gamma_{\text{out}}} \leq c \|\mathbb{F}\|_{1-1/r,r;\Gamma_{\text{out}}} + \nu \left\|\frac{\partial \mathbf{v}_*}{\partial \mathbf{n}}\right\|_{1-1/r,r;\Gamma_{\text{out}}}$$
$$\leq c \|\mathbb{F}\|_{1,r} + c \|\mathbf{v}_*\|_{2,r;U_2}$$
$$\leq c \|\mathbb{F}\|_{1,r} + c \|\mathbf{v}\|_{1,r}.$$

Let $\widetilde{\mathbb{F}}$ be a function in $W^{1,r}_{\text{per}}(\Omega)^{2\times 2}$, provided by Lemma 3.1, where we consider $\tilde{\mathbf{f}}$ instead of \mathbf{f} and $\tilde{\mathbf{h}}$ instead of \mathbf{h} . Then div $\widetilde{\mathbb{F}} = \tilde{\mathbf{f}}$ a.e. in Ω and $\widetilde{\mathbb{F}} \cdot \mathbf{n} = \tilde{\mathbf{h}}$ a.e. on Γ_{out} . Moreover, due to (3.1), (3.8) and (3.12),

(3.13)
$$\|\widetilde{\mathbb{F}}\|_{1,r} \leq c \|\widetilde{\mathbf{f}}\|_{r} + c \|\widetilde{\mathbf{h}}\|_{1-1/r,r;\Gamma_{\text{out}}} \leq c(\|\mathbb{F}\|_{1,r} + \|\mathbf{g}_{*}\|_{2,r} + \|\mathbf{v}\|_{1,r}).$$

Let the functional $\widetilde{\mathbf{F}} \in \mathbf{V}_{\sigma}^{-1,r}(\Omega)$ be defined by the same formula as (2.2), where we only consider $\widetilde{\mathbb{F}}$ instead of \mathbb{F} . We claim that $\nu \mathcal{A}_r \widetilde{\mathbf{v}} = \widetilde{\mathbf{F}}$. Indeed, applying (3.11), we obtain for any $\mathbf{w} \in \mathcal{C}_{\sigma}^{\infty}(\Omega)$:

$$\begin{split} \nu \langle \mathcal{A}_{r} \widetilde{\mathbf{v}}, \mathbf{w} \rangle_{(\mathbf{V}_{\sigma}^{-1, r}, \mathbf{V}_{\sigma}^{1, r'})} &= \nu \int_{\Omega} \nabla \widetilde{\mathbf{v}} : \nabla \mathbf{w} \, \mathrm{d} \mathbf{x} \\ &= \int_{\Gamma_{\mathrm{out}}} \nu \frac{\partial \widetilde{\mathbf{v}}}{\partial \mathbf{n}} \cdot \mathbf{w} \, \mathrm{d} l - \nu \int_{\Omega} \Delta \widetilde{\mathbf{v}} \cdot \mathbf{w} \, \mathrm{d} \mathbf{x} \\ &= \int_{\Gamma_{\mathrm{out}}} \nu \frac{\partial \widetilde{\mathbf{v}}}{\partial \mathbf{n}} \cdot \mathbf{w} \, \mathrm{d} l + \int_{\Omega} (-\nabla \widetilde{p} + \widetilde{\mathbf{f}}) \cdot \mathbf{w} \, \mathrm{d} \mathbf{x} \\ &= \int_{\Gamma_{\mathrm{out}}} \left[\nu \frac{\partial \widetilde{\mathbf{v}}}{\partial \mathbf{n}} - \widetilde{p} \mathbf{n} \right] \cdot \mathbf{w} \, \mathrm{d} l + \int_{\Omega} \mathrm{div} \, \widetilde{\mathbb{F}} \cdot \mathbf{w} \, \mathrm{d} \mathbf{x} \\ &= -\int_{\Gamma_{\mathrm{out}}} \widetilde{\mathbf{h}} \cdot \mathbf{w} \, \mathrm{d} l + \int_{\Gamma_{\mathrm{out}}} (\widetilde{\mathbb{F}} \cdot \mathbf{n}) \cdot \mathbf{w} \, \mathrm{d} l - \int_{\Omega} \widetilde{\mathbb{F}} : \nabla \mathbf{w} \, \mathrm{d} \mathbf{x} \\ &= -\int_{\Omega} \widetilde{\mathbb{F}} : \nabla \mathbf{w} \, \mathrm{d} \mathbf{x} = \langle \widetilde{\mathbf{F}}, \mathbf{w} \rangle_{(\mathbf{V}_{\sigma}^{-1, r}, \mathbf{V}_{\sigma}^{1, r'})}. \end{split}$$

(The integrals containing $\partial \tilde{\mathbf{v}} / \partial \mathbf{n}$, $\Delta \tilde{\mathbf{v}}$ and $\nabla \tilde{p}$, make sense, because $\tilde{\mathbf{v}} \in \mathbf{W}^{2,2}(\Omega)$ and $\tilde{p} \in W^{1,2}(\Omega)$.) As $\mathcal{C}^{\infty}_{\sigma}(\Omega)$ is dense in $\mathbf{V}^{1,r'}_{\sigma}(\Omega)$, the equation $\nu \langle \mathcal{A}_r \tilde{\mathbf{v}}, \mathbf{w} \rangle_{(\mathbf{V}^{-1,r}_{\sigma}, \mathbf{V}^{1,r'}_{\sigma})} = \langle \tilde{\mathbf{F}}, \mathbf{w} \rangle_{(\mathbf{V}^{-1,r}_{\sigma}, \mathbf{V}^{1,r'}_{\sigma})}$ holds for all $\mathbf{w} \in \mathbf{V}^{1,r'}_{\sigma}(\Omega)$.

Step 3. In this part, we apply the method of difference quotients (see [3], [19] and [37]) in order to derive estimate (3.4).

Recall that $\tilde{\mathbf{f}}$ is defined in \mathbb{R}^2_{d-} and supported in the closure of $U_2 \cap \mathbb{R}^2_{d-}$, and $\tilde{\mathbf{h}}$ is defined in Γ_{out} and supported in the closure of $U_2 \cap \gamma_{\text{out}}$. For $\delta \in \mathbb{R}$, denote

$$D_2^{\delta} \tilde{\mathbf{f}}(x_1, x_2) := \frac{\tilde{\mathbf{f}}(x_1, x_2 + \delta) - \tilde{\mathbf{f}}(x_1, x_2)}{\delta}, \quad D_2^{\delta} \tilde{\mathbf{h}}(d, x_2) := \frac{\tilde{\mathbf{h}}(d, x_2 + \delta) - \tilde{\mathbf{h}}(d, x_2)}{\delta}.$$

 $D_2^{\delta} \widetilde{\mathbf{f}}$ and $D_2^{\delta} \widetilde{\mathbf{h}}$ are the so-called *difference quotients*.

Recall that $\widetilde{\mathbb{F}} \in W^{1,r}_{\text{per}}(\Omega)^{2\times 2}$ and $\widetilde{\mathbb{F}} = \mathbb{O}$ in the sense of traces on Γ_0 , which follows from Lemma 3.1. Thus, if $\widetilde{\mathbb{F}}$ is extended by \mathbb{O} from Ω to $\Omega \cup P_0$, then the extended function is in $W^{1,r}_{\text{per}}(\Omega \cup P_0)^{2\times 2}$, div $\widetilde{\mathbb{F}} = \mathbb{O}$ in P_0 and $\|\widetilde{\mathbb{F}}\|_{1,r;\Omega \cup P_0} =$ $\|\widetilde{\mathbb{F}}\|_{1,r}$. Furthermore, $\widetilde{\mathbb{F}}$ can be extended from $\Omega \cup P_0$ to the stripe $\mathbb{R}^2_{(0,d)} :=$ $\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2; 0 < x_1 < d\}$ as τ -periodic function in variable x_2 , lying in $W^{1,r}_{\text{loc}}(\mathbb{R}^2_{(0,d)})$. Let us denote the extension again by $\widetilde{\mathbb{F}}$ and define

$$D_2^{\delta} \widetilde{\mathbb{F}}(x_1, x_2) := \frac{\widetilde{\mathbb{F}}(x_1, x_2 + \delta) - \widetilde{\mathbb{F}}(x_1, x_2)}{\delta} \quad \text{for } (x_1, x_2) \in \mathbb{R}^2_{(0,d)}$$

Denote $\widetilde{\mathbb{F}}_{\delta}(x_1, x_2) := \delta^{-1} \int_0^{\delta} \widetilde{\mathbb{F}}(x_1, x_2 + \vartheta) \, \mathrm{d}\vartheta$. Then

$$D_2^{\delta} \widetilde{\mathbb{F}}(x_1, x_2) = \frac{1}{\delta} \int_0^{\delta} \partial_2 \widetilde{\mathbb{F}}(x_1, x_2 + \vartheta) \, \mathrm{d}\vartheta = \partial_2 \widetilde{\mathbb{F}}_{\delta}(x_1, x_2).$$

Using the τ -periodicity of the function $\widetilde{\mathbb{F}}_{\delta}$ in variable x_2 in $\mathbb{R}^2_{(0,d)}$ and applying Hölder's inequality, we get

$$\begin{split} \|\widetilde{\mathbb{F}}_{\delta}\|_{r;\Omega\cup P_{0}}^{r} &= \int_{\Omega\cup P_{0}} \left|\frac{1}{\delta} \int_{0}^{\delta} \widetilde{\mathbb{F}}(x_{1}, x_{2} + \vartheta) \,\mathrm{d}\vartheta \right|^{r} \,\mathrm{d}\mathbf{x} \\ &= \int_{0}^{d} \int_{0}^{\tau} \left|\frac{1}{\delta} \int_{0}^{\delta} \widetilde{\mathbb{F}}(x_{1}, x_{2} + \vartheta) \,\mathrm{d}\vartheta \right|^{r} \,\mathrm{d}x_{2} \,\mathrm{d}x_{1} \\ &\leqslant \int_{0}^{d} \int_{0}^{\tau} \frac{1}{\delta} \int_{0}^{\delta} |\widetilde{\mathbb{F}}(x_{1}, x_{2} + \vartheta)|^{r} \,\mathrm{d}\vartheta \,\mathrm{d}x_{2} \,\mathrm{d}x_{1} \\ &= \int_{0}^{d} \frac{1}{\delta} \int_{0}^{\delta} \int_{0}^{\tau} |\widetilde{\mathbb{F}}(x_{1}, y_{2})|^{r} \,\mathrm{d}y_{2} \,\mathrm{d}\vartheta \,\mathrm{d}x_{1} \\ &= \int_{0}^{d} \int_{0}^{\tau} |\widetilde{\mathbb{F}}(x_{1}, y_{2})|^{r} \,\mathrm{d}y_{2} \,\mathrm{d}x_{1} \\ &= \int_{\Omega\cup P_{0}} |\widetilde{\mathbb{F}}(\mathbf{x})|^{r} \,\mathrm{d}\mathbf{x} = \|\widetilde{\mathbb{F}}\|_{r;\Omega\cup P_{0}}^{r}. \end{split}$$

We can similarly show that $\|\nabla \widetilde{\mathbb{F}}_{\delta}\|_{r;\Omega \cup P_0}^r \leq \|\nabla \widetilde{\mathbb{F}}\|_{r;\Omega \cup P_0}^r$. Consequently, $\|\widetilde{\mathbb{F}}_{\delta}\|_{1,r} \leq \|\widetilde{\mathbb{F}}_{\delta}\|_{1,r;\Omega \cup P_0} \leq \|\widetilde{\mathbb{F}}\|_{1,r;\Omega \cup P_0} \leq c \|\widetilde{\mathbb{F}}\|_{1,r}$. Thus,

$$(3.14) \|D_2^{\delta}\widetilde{\mathbb{F}}\|_r = \|\partial_2\widetilde{\mathbb{F}}_{\delta}\|_r \leqslant \|\widetilde{\mathbb{F}}_{\delta}\|_{1,r} \leqslant c \|\widetilde{\mathbb{F}}\|_{1,r} \leqslant c (\|\widetilde{\mathbf{f}}\|_r + \|\widetilde{\mathbf{h}}\|_{1-1/r,r;\Gamma_{\text{out}}}).$$

Let $D_2^{\delta} \widetilde{\mathbf{v}}$ and $D_2^{\delta} \widetilde{p}$ be defined by analogy with $D_2^{\delta} \widetilde{\mathbf{f}}$ and $D_2^{\delta} \widetilde{\mathbb{F}}$. The functions $D_2^{\delta} \widetilde{\mathbf{v}}$, $D_2^{\delta} \widetilde{p}$ satisfy the equations

$$-\nu\Delta D_2^{\delta}\widetilde{\mathbf{v}} + \nabla D_2^{\delta}\widetilde{p} = \operatorname{div} D_2^{\delta}\widetilde{\mathbb{F}}, \quad \operatorname{div} D_2^{\delta}\widetilde{\mathbf{v}} = 0$$

a.e. in Ω . (The validity of these equations a.e. in Ω follows from the inclusions $\widetilde{\mathbf{v}} \in \mathbf{W}^{2,2}(\Omega)$ and $\widetilde{p} \in W^{1,2}(\Omega)$.) Since $\widetilde{\mathbf{v}}$ and \widetilde{p} satisfy (3.11) not only on Γ_{out} , but

also on γ_{out} , the functions $D_2^{\delta} \widetilde{\mathbf{v}}$ and $D_2^{\delta} \widetilde{p}$ satisfy the boundary condition

$$-
u rac{\partial D_2^\delta \widetilde{\mathbf{v}}}{\partial \mathbf{n}} + D_2^\delta \widetilde{p} \mathbf{n} = D_2^\delta \widetilde{\mathbf{h}}$$

on $\Gamma_{\text{out.}}$ From this, one can deduce that $\nu \mathcal{A}_r D_2^{\delta} \widetilde{\mathbf{v}} = \widetilde{\mathbf{F}}_{\delta}$, where the functional $\widetilde{\mathbf{F}}_{\delta}$ in $\mathbf{V}_{\sigma}^{-1,r}(\Omega)$ is defined by the same formula as (2.2), where we only consider $D_2^{\delta} \widetilde{\mathbb{F}}$ instead of \mathbb{F} . It follows from Lemma 2.1 that

$$\|\nabla D_2^{\delta} \widetilde{\mathbf{v}}\|_r \leqslant \|\widetilde{\mathbf{F}}_{\delta}\|_{\mathbf{V}_{\sigma}^{-1,r}}.$$

Since $\|\widetilde{\mathbf{F}}_{\delta}\|_{\mathbf{V}_{\sigma}^{-1,r}} \leq \|D_{2}^{\delta}\widetilde{\mathbb{F}}\|_{r} \leq c(\|\widetilde{\mathbf{f}}\|_{r} + \|\widetilde{\mathbf{h}}\|_{1-1/r,r;\Gamma_{\mathrm{out}}})$, we obtain

(3.15)
$$\|\nabla D_2^{\delta} \widetilde{\mathbf{v}}\|_r \leqslant c(\|\widetilde{\mathbf{f}}\|_r + \|\widetilde{\mathbf{h}}\|_{1-1/r,r;\Gamma_{\text{out}}}).$$

Applying further Lemma 2.3 (with $\mathbf{g}_* = \mathbf{0}$), (3.14) and (3.15), we obtain the estimate of $D_2^{\delta} \widetilde{p}$:

(3.16)
$$\|D_2^{\delta}\widetilde{p}\|_r \leqslant c(\|\nabla D_2^{\delta}\widetilde{\mathbf{v}}\|_r + \|D_2^{\delta}\widetilde{\mathbb{F}}\|_r) \leqslant c(\|\widetilde{\mathbf{f}}\|_r + \|\widetilde{\mathbf{h}}\|_{1-1/r,r;\Gamma_{\text{out}}}).$$

As the right-hand sides of (3.15) and (3.16) are independent of δ , we may let δ tend to 0 and we obtain

(3.17)
$$\|\nabla \partial_r \widetilde{\mathbf{v}}\|_r + \|\partial_2 \widetilde{p}\|_r \leqslant c(\|\widetilde{\mathbf{f}}\|_r + \|\widetilde{\mathbf{h}}\|_{1-1/r,r;\Gamma_{\text{out}}}).$$

This shows that $\partial_1 \partial_2 \tilde{v}_1$, $\partial_2^2 \tilde{v}_1$, $\partial_1 \partial_2 \tilde{v}_2$, $\partial_2^2 \tilde{v}_2$ and $\partial_2 \tilde{p}$ are all in $L^r(\Omega)$ and their norms are less than or equal to the right-hand side of (3.17). Consequently, as $\tilde{\mathbf{v}}$ is divergence-free, the same statement also holds for $\partial_1^2 \tilde{v}_1$. Now, from (3.5) (considering just the first scalar component of this vectorial equation), we deduce that $\partial_1 \tilde{p} \in L^r(\Omega)$. Finally, considering the second scalar component in equation (3.5), we obtain $\partial_1^2 \tilde{v}_2 \in L^r(\Omega)$, too. Thus, applying also (3.8) and (3.9), we get

$$\|\widetilde{\mathbf{v}}\|_{2,r} + \|\nabla\widetilde{p}\|_r \leq c(\|\mathbb{F}\|_{1,r} + \|\mathbf{g}_*\|_{2,r} + \|\mathbf{v}\|_{1,r}).$$

This inequality, formulas (3.9), the estimate of $\|\mathbf{v}_*\|_{2,r}$ and the fact that $\eta = 1$ on U_1 yield (3.4).

The next corollary is an immediate consequence of Lemmas 3.2 and 3.3.

Corollary 3.1. Let Ω' be a sub-domain of Ω such that $\overline{\Omega'} \subset \Omega \cup \Gamma_{in}^0 \cup \Gamma_0 \cup \Gamma_{out}^0$. Then $\mathbf{v} \in \mathbf{W}^{2,r}(\Omega')$, $p \in W^{1,r}(\Omega')$ and estimate (3.3) holds. **Lemma 3.4.** Let Ω' be a sub-domain of Ω such that $\partial \Omega' \cap \Gamma_0 = \emptyset$ and $\Gamma_+ \subset \partial \Omega'$. Then $\mathbf{v} \in \mathbf{W}^{2,r}(\Omega')$, $p \in W^{1,r}(\Omega')$ and

(3.18)
$$\|\mathbf{v}\|_{2,r;\Omega'} + \|\nabla p\|_{r;\Omega'} \leqslant c(\|\operatorname{div} \mathbb{F}\|_r + \|\mathbf{g}_*\|_{2,r} + \|\mathbf{v}\|_{1,r}),$$

where $c = c(\nu, \Omega, \Omega')$.

Lemma 3.4 is quite obvious because Γ_+ is in fact an artificial boundary of Ω and all relevant quantities are τ -periodic in variable x_2 . Nevertheless, we sketch the principle of the proof: consider $\delta > 0$ and denote

$$\begin{split} A^{\delta}_{-} &:= A_{-} + \delta \mathbf{e}_{2}, \qquad A^{\delta}_{+} = A_{+} + \delta \mathbf{e}_{2}, \\ B^{\delta}_{-} &:= B_{-} + \delta \mathbf{e}_{2}, \qquad B^{\delta}_{+} = B_{+} + \delta \mathbf{e}_{2}, \\ \Gamma^{\delta}_{\mathrm{in}} &:= \Gamma_{\mathrm{in}} + \delta \mathbf{e}_{2}, \qquad \Gamma^{\delta}_{-} = \Gamma_{-} + \delta \mathbf{e}_{2}, \\ \Gamma^{\delta}_{+} &:= \Gamma_{+} + \delta \mathbf{e}_{2}, \qquad \Gamma^{\delta}_{\mathrm{out}} = \Gamma_{\mathrm{out}} + \delta \mathbf{e}_{2}. \end{split}$$

where \mathbf{e}_2 is the unit vector in the direction of the x_2 -axis. Suppose that $\delta > 0$ is so small that the curve Γ_{δ}^{δ} still lies below the profile P_0 . Denote by Ω^{δ} the domain bounded by the curves $\Gamma_{\mathrm{in}}^{\delta}$, Γ_{-}^{δ} , $\Gamma_{\mathrm{out}}^{\delta}$, Γ_{+}^{δ} and Γ_0 . The "open" curve Γ_{+}^{0} is a subset of Ω^{δ} . Thus, applying appropriately Corollary 3.1 in domain Ω^{δ} instead of Ω , we deduce that \mathbf{v} and p (extended periodically in the direction of x_2) are regular in a neighborhood of Γ_{+}^{0} in Ω^{δ} . This implies the statement of the lemma.

Completion of the proof of Theorem 3.1 in the case $r \ge 2$. An analogue of Lemma 3.4 also holds if one considers Ω' , satisfying the condition $\Gamma_{-} \subset \partial \Omega'$ instead of $\Gamma_{+} \subset \partial \Omega'$. This and Lemmas 3.2–3.4 complete the proof of statements (1) and (5) of Theorem 3.1.

Case 2: 1 < r < 2. There exist sequences $\{\mathbf{f}^n\}, \{\mathbf{h}^n\}$ and $\{\mathbf{g}^n\}$ in $\mathbf{L}^2(\Omega)$, $\mathbf{W}_{\text{per}}^{1/2,2}(\Gamma_{\text{out}})$ and $\mathbf{W}^{3/2,2}(\Gamma_{\text{in}})$, respectively, such that $\mathbf{f}^n \to \mathbf{f}$ in $\mathbf{L}^r(\Omega), \mathbf{h}^n \to \mathbf{h}$ in $\mathbf{W}_{\text{per}}^{1-1/r,r}(\Gamma_{\text{out}})$ and $\mathbf{g}^n \to \mathbf{g}$ in $\mathbf{W}_{\text{per}}^{2-1/r,r}(\Gamma_{\text{in}})$ for $n \to \infty$. Let \mathbb{F}^n and \mathbf{g}^n_* be the functions given by Lemma 3.1, Lemma 2.2 and Remark 3.1 in the case that we consider $\mathbf{f}^n, \mathbf{h}^n$ and \mathbf{g}^n instead of \mathbf{f}, \mathbf{h} and \mathbf{g} , respectively. Let the functionals \mathbf{F}^n and \mathbf{G}^n (corresponding to \mathbb{F}^n and \mathbf{g}^n_*) be defined by formulas (2.2) and (2.3), respectively. Then it follows from [36], Theorem 2, and also from the first part of this proof (where we assumed that $r \ge 2$), that the unique solution \mathbf{v}^n of the equation $\nu \mathcal{A}_2 \mathbf{v}^n = \mathbf{F}^n + \nu \mathbf{G}^n$ belongs to $\mathbf{V}_{\sigma}^{1,r}(\Omega) \cap \mathbf{W}^{2,r}(\Omega)$ and the associated pressure p^n lies in $W^{1,r}(\Omega)$, the functions $\mathbf{u}^n := \mathbf{g}^n_* + \mathbf{v}^n$ and p^n satisfy equations (1.1) (with $\mathbf{f}^n = \operatorname{div} \mathbb{F}^n$) and (1.2) a.e. in Ω , \mathbf{u}^n , p^n satisfy boundary conditions (1.3), (1.4) and (1.8) in the sense of traces on $\Gamma_{\text{in}}, \Gamma_0$ and Γ_{out} , respectively, \mathbf{u}^n, p^n satisfy the conditions of periodicity (1.5)–(1.7) in the sense of traces on Γ_{-} and Γ_{+} and

(3.19)
$$\|\mathbf{u}^n\|_{2,2} + \|\nabla p^n\|_2 \leqslant c(\|\mathbf{f}^n\|_2 + \|\mathbf{g}^n\|_{3/2,2;\Gamma_{\text{in}}} + \|\mathbf{h}^n\|_{1/2,2;\Gamma_{\text{out}}}),$$

where $c = c(\nu, \Omega)$. However, estimate (3.2) does not follow from (3.19) by the limit transition for $n \to \infty$, because the norms $\|\mathbf{f}^n\|_2$, $\|\mathbf{g}^n\|_{3/2,2;\Gamma_{\text{in}}}$ and $\|\mathbf{h}^n\|_{1/2,2;\Gamma_{\text{out}}}$ may tend to infinity if $n \to \infty$. Nevertheless, repeating the procedures from the proofs of Lemmas 3.2–3.4, we also derive that

(3.20)
$$\|\mathbf{u}^n\|_{2,r} + \|\nabla p^n\|_r \leqslant c(\|\mathbf{f}^n\|_r + \|\mathbf{g}^n\|_{2-1/r,r;\Gamma_{\text{in}}} + \|\mathbf{h}^n\|_{1-1/r,r;\Gamma_{\text{out}}}),$$

where $c = c(\nu, \Omega, r)$. The limit transition for $n \to \infty$ yields (3.2).

Remark 3.3. Theorem 3.1 can be generalized so that it yields $(\mathbf{u}, p) \in \mathbf{W}^{n+2,r}(\Omega) \times W^{n+1,r}(\Omega)$ for $n \in \{0\} \cup \mathbb{N}$. The generalization, however, requires the boundary Γ_0 of profile P_0 to be of the class C^{2+n} , the function \mathbb{F} in $W^{n+1,r}_{\text{per}}(\Omega)^{2\times 2}$ and the given velocity profile \mathbf{g} on Γ_{in} to be in $\mathbf{W}^{n+2-1/r,r}_{\text{per}}(\Gamma_{\text{in}})$. One also needs a modification of Lemma 2.2 and Remark 3.1, which provides \mathbf{g}_* in $\mathbf{W}^{n+2,r}_{\text{per}}(\Omega)$. We do not include the proof of the generalization here, because it would be more or less just a technical modification of the proof of Theorem 3.1.

Remark 3.4 (On the strong Stokes operator A_r). Denote (for $1 < r < \infty$) by $\mathbf{L}_{\sigma}^r(\Omega)$ the closure of $\mathcal{C}_{\sigma}^{\infty}(\Omega)$ in $\mathbf{L}^r(\Omega)$. Functions \mathbf{v} from $\mathbf{L}_{\sigma}^r(\Omega)$ are divergencefree in the sense of distributions in Ω , their normal components (in the sense of traces) belong to the space $W^{-1/r,r}(\partial\Omega)$ (the dual to $W^{1-1/r',r'}(\partial\Omega)$, see [16], Theorem III.2.2), satisfy $\mathbf{v} \cdot \mathbf{n} = \mathbf{0}$ as an equality in $W^{-1/r,r}(\Gamma_{\rm in} \cup \Gamma_0)$ and the condition of periodicity $\mathbf{v} \cdot \mathbf{n}|_{\Gamma_-} = -\mathbf{v} \cdot \mathbf{n}|_{\Gamma_+}$ in the sense that

$$\langle \mathbf{v} \cdot \mathbf{n}, \varphi(\cdot) \rangle_{(\mathbf{W}^{-1/r, r}(\Gamma_{-}), \mathbf{W}^{1-1/r', r'}(\Gamma_{-}))} = - \langle \mathbf{v} \cdot \mathbf{n}, \varphi(\cdot - \tau \mathbf{e}_{2}) \rangle_{(\mathbf{W}^{-1/r, r}(\Gamma_{+}), \mathbf{W}^{1-1/r', r'}(\Gamma_{+}))}$$

for each function $\varphi \in W^{1-1/r',r'}(\Gamma_{-})$. Moreover,

$$\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{(\mathbf{W}^{-1/r, r}(\Gamma_{\text{out}}), \mathbf{W}^{1-1/r', r'}(\Gamma_{\text{out}}))} = 0.$$

Obviously, $\mathbf{V}^{1,r}_{\sigma}(\Omega) \hookrightarrow \mathbf{L}^{r}_{\sigma}(\Omega) \hookrightarrow \mathbf{V}^{-1,r}_{\sigma}(\Omega)$. Denote by $D(A_{r})$ the set $\{\mathbf{v} \in \mathbf{V}^{1,r}_{\sigma}(\Omega); \mathcal{A}_{r}\mathbf{v} \in \mathbf{L}^{r}_{\sigma}(\Omega)\}$ and define

$$A_r := \mathcal{A}_r \big|_{D(A_r)}.$$

(Recall that operator \mathcal{A}_r is defined in Section 1 and its main properties are given by Lemma 2.1.) In contrast to the "weak Stokes operator" \mathcal{A}_r , it is logical to call \mathcal{A}_r

the strong Stokes operator. Considering $\mathbf{g} = \mathbf{0}$ and $\mathbf{h} = \mathbf{0}$ in Theorem 3.1, we deduce that $D(A_r) = \mathbf{V}_{\sigma}^{1,r}(\Omega) \cap \mathbf{W}^{2,r}(\Omega)$ and

$$\|\mathbf{v}\|_{2,r} \leqslant c_3 \|A_r \mathbf{v}\|_r$$

for $\mathbf{v} \in D(A_r)$. Moreover, using (3.21), it is easy to show that A_r is a one-to-one, densely defined and closed operator in $\mathbf{L}_{\sigma}^r(\Omega)$ with a compact resolvent. It follows directly from Lemma 2.1 and the definition of A_r that $R(A_r)$ (the range of A_r) is the whole space $\mathbf{L}_{\sigma}^r(\Omega)$.

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Author's address: Tomáš Neustupa, Czech Technical University, Faculty of Mechanical Engineering, Department of Technical Mathematics, Karlovo nám. 13, 121 35 Praha 2, Czech Republic, e-mail: tomas.neustupa@fs.cvut.cz.