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FABER POLYNOMIAL COEFFICIENT ESTIMATES OF BI-UNIVALENT FUNCTIONS CONNECTED WITH THE q-CONVOLUTION

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Abstract. We introduce a new class of bi-univalent functions defined in the open unit disc and connected with a q-convolution. We find estimates for the general Taylor-Maclaurin coefficients of the functions in this class by using Faber polynomial expansions and we obtain an estimation for the Fekete-Szegö problem for this class.

Keywords: Faber polynomial; bi-univalent function; convolution; q-derivative operator

MSC 2020: 05A30, 30C45, 11B65, 47B38

1. Introduction, definitions and preliminaries

In his survey-cum-expository review article, Srivastava (see [33]) presented and motivated a brief expository overview of the classical q-analysis versus the so-called (p, q)-analysis with an obviously redundant additional parameter p. We also briefly consider several other families of such extensively and widely-investigated linear convolution operators as (for example) the Dziok-Srivastava, Srivastava-Wright and Srivastava-Attiya linear convolution operators, together with their extended and generalized versions. The theory of (p, q)-analysis plays important role in many areas of mathematics and physics. Our usages here of the q-calculus and the fractional q-calculus in the geometric function theory of complex analysis are believed to encourage significant further developments of these and other related topics (see Srivastava and Karlsson [39], pages 350–351; Srivastava [30], [31], [32]). Our main objective in this survey-cum-expository article is based chiefly upon the fact that the recent and future usages of the classical q-calculus and the fractional q-calculus in the geometric function theory of complex analysis have the potential to motivate further

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research of these and other related subjects. Jackson (see [20], [21]) was the first who gave some application of the q-calculus and introduced the q-analogue of derivative and integral operator (see also [1], [29]). We apply the concept of q-convolution in order to introduce and study the general Taylor-Maclaurin coefficient estimates for functions belonging to a new class of normalized analytic functions in the open unit disk, which we define here.

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) := z + \sum_{m=2}^{\infty} a_m z^m, \quad z \in \Delta := \{z \in \mathbb{C}\colon\ |z| < 1\}, \tag{1.1}$$

and let $\mathcal{S} \subset \mathcal{A}$ consist of functions that are univalent in Δ. Let the function $h \in \mathcal{A}$ be given by

$$h(z) := z + \sum_{m=2}^{\infty} b_m z^m, \quad z \in \Delta. \tag{1.2}$$

The *Hadamard product* (or *convolution*) of f and h, given by (1.1) and (1.2), respectively, is defined by

$$(f * h)(z) := z + \sum_{m=2}^{\infty} a_m b_m z^m, \quad z \in \Delta. \tag{1.3}$$

Srivastava in [33] made use of various operators of q-calculus and fractional q-calculus. Recall the definition and notations. The *q-shifted factorial* is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as

$$(\lambda; q)_m = \begin{cases} 1, & m = 0, \\ (1-\lambda)(1-\lambda q)\ldots(1-\lambda q^{k-1}), & m \in \mathbb{N}. \end{cases}$$

By using the q-gamma function $\Gamma_q(z)$, we get

$$(q^{\lambda}; q)_m = \frac{(1-q)^m \Gamma_q(\lambda+m)}{\Gamma_q(\lambda)}, \quad m \in \mathbb{N}_0,$$

where (see [19])

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q;q)_\infty}{(q^z;q)_\infty}, \quad |q| < 1.$$

Also, we note that

$$(\lambda; q)_\infty = \prod_{m=0}^{\infty} (1 - \lambda q^m), \quad |q| < 1$$

and the q-gamma function $\Gamma_q(z)$ satisfies

$$\Gamma_q(z+1) = [z]_q \Gamma_q(z),$$

where $[m]_q$ denotes the *basic q-number* defined as

$$[m]_q := \begin{cases} \dfrac{1-q^m}{1-q}, & m \in \mathbb{C}, \\ 1 + \sum_{j=1}^{m-1} q^j, & m \in \mathbb{N}. \end{cases} \tag{1.4}$$

Using the definition formula (1.4) we have the next two products:

(i) For any non-negative integer m, the *q-shifted factorial* is given by

$$[m]_q! := \begin{cases} 1, & \text{if } m = 0, \\ \prod_{n=1}^{m} [n]_q, & \text{if } m \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the *q-generalized Pochhammer symbol* is defined by

$$[r]_{q,m} := \begin{cases} 1, & \text{if } m = 0, \\ \prod_{n=r}^{r+m-1} [n]_q, & \text{if } m \in \mathbb{N}. \end{cases}$$

It is known in terms of classical (Euler's) gamma function $\Gamma(z)$, that

$$\Gamma_q(z) \to \Gamma(z) \quad \text{as } q \to 1^-.$$

Also, we observe that

$$\lim_{q \to 1^-} \left\{ \frac{(q^\lambda; q)_m}{(1-q)^m} \right\} = (\lambda)_m,$$

where $(\lambda)_m$ is the familiar Pochhammer symbol defined by

$$(\lambda)_m = \begin{cases} 1, & \text{if } m = 0, \\ \lambda(\lambda+1)\ldots(\lambda+m-1), & \text{if } m \in \mathbb{N}. \end{cases}$$

For $0 < q < 1$, El-Deeb et al. in [16] defined the *q-derivative operator* (or, equivalently, the *q-difference operator*) D_q for $f * h$ given by (1.3) as (see [20], [21])

$$\begin{aligned} D_q(f*h)(z) : D_q\left(z + \sum_{m=2}^{\infty} a_m b_m z^m\right) &= \frac{(f*h)(z) - (f*h)(qz)}{z(1-q)} \\ &= 1 + \sum_{m=2}^{\infty} [m]_q a_m b_m z^{m-1}, \quad z \in \Delta, \end{aligned}$$

where, like in the definition (1.4),

$$[m]_q := \begin{cases} \dfrac{1-q^m}{1-q} = 1 + \sum_{j=1}^{m-1} q^j, & m \in \mathbb{N}, \\ 0, & m = 0. \end{cases} \tag{1.5}$$

For $\lambda > -1$ and $0 < q < 1$, El-Deeb et al. (see [16]) defined the linear operator $\mathcal{H}_h^{\lambda,q}\colon \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{H}_h^{\lambda,q} f(z) * \mathcal{M}_{q,\lambda+1}(z) = z\, D_q(f * h)(z), \quad z \in \Delta,$$

where the function $\mathcal{M}_{q,\lambda+1}$ is given by

$$\mathcal{M}_{q,\lambda+1}(z) := z + \sum_{m=2}^{\infty} \frac{[\lambda+1]_{q,m-1}}{[m-1]_q!} z^m, \quad z \in \Delta.$$

A simple computation shows that

$$\mathcal{H}_h^{\lambda,q} f(z) := z + \sum_{m=2}^{\infty} \frac{[m]_q!}{[\lambda+1]_{q,m-1}} a_m b_m z^m, \quad \lambda > -1,\ 0 < q < 1,\ z \in \Delta. \tag{1.6}$$

From the definition relation (1.6), we can easily verify that the next relations hold for all $f \in \mathcal{A}$:

$$[\lambda+1]_q \mathcal{H}_h^{\lambda,q} f(z) = [\lambda]_q \mathcal{H}_h^{\lambda+1,q} f(z) + q^{\lambda}\, z\, D_q(\mathcal{H}_h^{\lambda+1,q} f(z)), \quad z \in \Delta;$$

$$\mathcal{I}_h^{\lambda} f(z) := \lim_{q \to 1^-} \mathcal{H}_h^{\lambda,q} f(z) = z + \sum_{m=2}^{\infty} \frac{m!}{(\lambda+1)_{m-1}} a_m b_m z^m, \quad z \in \Delta. \tag{1.7}$$

R e m a r k 1.1. Taking different particular values of the coefficients b_m we obtain the next special cases for the operator $\mathcal{H}_h^{\lambda,q}$:

(i) For $b_m = 1$, we obtain the operator $\mathcal{I}_q^{\lambda}$ defined by Srivastava (see [40]) and Arif et al. (see [3]) as

$$\mathcal{I}_q^{\lambda} f(z) := z + \sum_{m=2}^{\infty} \frac{[m]_q!}{[\lambda+1]_{q,m-1}} a_m z^m, \quad \lambda > -1,\ 0 < q < 1,\ z \in \Delta. \tag{1.8}$$

(ii) For $b_m = (-1)^{m-1}\Gamma(\upsilon+1)/(4^{m-1}(m-1)!\Gamma(m+\upsilon))$, $\upsilon > 0$, we obtain the operator $\mathcal{N}_{\upsilon,q}^{\lambda}$ defined by El-Deeb and Bulboacă in [14], and El-Deeb in [12] as

$$\begin{aligned} \mathcal{N}_{\upsilon,q}^{\lambda} f(z) &:= z + \sum_{m=2}^{\infty} \frac{(-1)^{m-1}\Gamma(\upsilon+1)}{4^{m-1}(m-1)!\Gamma(m+\upsilon)} \frac{[m]_q!}{[\lambda+1]_{q,m-1}} a_m z^m \\ &= z + \sum_{m=2}^{\infty} \frac{[m]_q!}{[\lambda+1]_{q,m-1}} \varphi_m a_m z^m, \quad \upsilon > 0,\ \lambda > -1,\ 0 < q < 1,\ z \in \Delta, \end{aligned} \tag{1.9}$$

where

$$\psi_m := \frac{(-1)^{m-1}\Gamma(\upsilon+1)}{4^{m-1}(m-1)!\,\Gamma(m+\upsilon)}. \tag{1.10}$$

(iii) For $b_m = (n+1)^\alpha/(n+m)^\alpha$, $\alpha > 0$, $n \geqslant 0$, we obtain the operator $\mathcal{M}^{\lambda,\alpha}_{n,q}$ defined by El-Deeb and Bulboacă in [13], and Srivastava and El-Deeb in [37] as

$$\mathcal{M}^{\lambda,\alpha}_{n,q} f(z) := z + \sum_{m=2}^{\infty} \Big(\frac{n+1}{n+m}\Big)^{\alpha} \frac{[m]_q!}{[\lambda+1]_{q,m-1}} a_m z^m, \quad z \in \Delta. \tag{1.11}$$

(iv) For $b_m = \varrho^{m-1}\mathrm{e}^{-\varrho}/(m-1)!$, $\varrho > 0$, we obtain the q-analogue of the Poisson operator defined by El-Deeb et al. in [16] (see also [27]) as

$$\mathcal{I}^{\lambda,\varrho}_{q} f(z) := z + \sum_{m=2}^{\infty} \frac{\varrho^{m-1}}{(m-1)!}\mathrm{e}^{-\varrho} \frac{[m]_q!}{[\lambda+1]_{q,m-1}} a_m z^m, \quad z \in \Delta. \tag{1.12}$$

(v) For $b_m = (1+l+\mu(m-1))^n/(1+l)^n$, $n \in \mathbb{Z}$, $l \geqslant 0$, $\mu \geqslant 0$, we obtain the q-analogue of the Prajapat operator defined by El-Deeb et al. in [16] (see also [28]) as

$$\mathcal{J}^{\lambda,n}_{q,l,\mu} f(z) := z + \sum_{m=2}^{\infty} \Big(\frac{1+l+\mu(m-1)}{1+l}\Big)^{n} \frac{[m,q]!}{[\lambda+1,q]_{m-1}} a_m z^m, \quad z \in \Delta. \tag{1.13}$$

(vi) For $b_m = \begin{pmatrix} n+m-2 \\ m-1 \end{pmatrix} \theta^{m-1}(1-\theta)^n$, $n \in \mathbb{N}$, $0 \leqslant \theta \leqslant 1$, we obtain the q-analogue of the Pascal distribution operator defined by Srivastava and El-Deeb in [38] (see also [16], [15]) as

$$\Theta^{\lambda,n}_{q,\theta} f(z) := z + \sum_{m=2}^{\infty} \begin{pmatrix} n+m-2 \\ m-1 \end{pmatrix} \theta^{m-1}(1-\theta)^n \frac{[m,q]!}{[\lambda+1,q]_{m-1}} a_m z^m, \quad z \in \Delta. \tag{1.14}$$

If f and F are analytic functions in Δ, we say that f *is subordinate to* F, written as $f(z) \prec F(z)$, if there exists a *Schwarz function* s, which is analytic in Δ, with $s(0) = 0$ and $|s(z)| < 1$ for all $z \in \Delta$, such that $f(z) = F(s(z))$, $z \in \Delta$. Furthermore, if the function F is univalent in Δ, then we have the equivalence (see [7] and [24])

$$f(z) \prec F(z) \rightarrow f(0) = F(0) \quad \text{and} \quad f(\Delta) \subset F(\Delta).$$

The Koebe one-quarter theorem (see [11]) proves that the image of Δ under every univalent function $f \in \mathcal{S}$ contains the disk of radius $\frac{1}{4}$. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1} that satisfies

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f),\ r_0(f) \geqslant \tfrac{1}{4},$$

where

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots \\ &= w + \sum_{m=2}^{\infty} A_m w^m. \end{aligned}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions in Δ given by (1.1). The class of analytic bi-univalent functions was first introduced by Lewin (see [23]), who proved that $|a_2| < 1.51$. Brannan and Clunie in [4] improved Lewin's result to $|a_2| < \sqrt{2}$ and later Netanyahu in [26] proved that $|a_2| < \frac{4}{3}$.

Note that the functions

$$f_1(z) = \frac{z}{1-z}, \quad f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \quad f_3(z) = -\log(1-z)$$

with their corresponding inverses

$$f_1^{-1}(w) = \frac{w}{1+w}, \quad f_2^{-1}(w) = \frac{\mathrm{e}^{2w}-1}{\mathrm{e}^{2w}+1}, \quad f_3^{-1}(w) = \frac{\mathrm{e}^{w}-1}{\mathrm{e}^{w}}$$

are elements of Σ (see [16], [41], [35]). For a brief history and interesting examples in the class Σ, see [5]. Brannan and Taha in [6] (see also [41]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order α, $0 \leqslant \alpha < 1$, respectively (see [5], [10], [34]). Following Brannan and Taha, a function $f \in \mathcal{A}$ is said to be in the class $S^*_\Sigma(\alpha)$ of bi-starlike functions of order α, $0 < \alpha \leqslant 1$, if each of the following conditions is satisfied (see [6]):

$$f \in \Sigma \quad \text{with} \quad \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, \quad z \in \Delta \quad \text{and} \quad \left|\arg\frac{wg'(w)}{g(w)}\right| < \frac{\alpha\pi}{2}, \quad w \in \Delta,$$

where the function g is the analytic extension of f^{-1} to Δ, given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots, \quad w \in \Delta. \tag{1.15}$$

A function $f \in \mathcal{A}$ is said to be in the class $K_\Sigma(\alpha)$ of bi-convex functions of order α, $0 < \alpha \leqslant 1$, if each of the following conditions is satisfied:

$$f \in \Sigma \quad \text{with} \quad \left|\arg\Big(1 + \frac{zf''(z)}{f'(z)}\Big)\right| < \frac{\alpha\pi}{2}, \quad z \in \Delta$$

and

$$\left|\arg\Big(1 + \frac{wg''(w)}{g'(w)}\Big)\right| < \frac{\alpha\pi}{2}, \quad w \in \Delta.$$

The classes $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, $0 < \alpha \leqslant 1$, corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$, non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ have been found (see [6] and [41]).

The object of the paper is to introduce a new subclass of functions $\mathcal{L}_\Sigma^{q,\lambda}(\eta; h; \Phi)$ of the class Σ, that generalizes the previous defined classes. This subclass is defined with the aid of a general $\mathcal{H}_h^{\lambda,q}$ linear operator defined by convolution products together with the aid of the q-derivative operator. This new class extends and generalizes many previous operators as it was presented in Remark 1.1, and the main goal of the paper is to find estimates on the coefficients $|a_2|$, $|a_3|$ for the Fekete-Szegö functional for functions in these new subclasses.

These classes are introduced by using the subordination and the results are obtained by employing the techniques used earlier by Srivastava et al. in [41]. This last work represents one of the most important studies of the bi-univalent functions and has inspired many investigations in this area including the present paper, while many other recent papers deal with the problems initiated in this work, see [9], [22], [2], [17], and many others.

Bulut in [8] defined and studied the class $\mathcal{N}_\Sigma(\alpha,\lambda,\delta)$, $\lambda \geqslant 1$, $\delta \geqslant 0$, $0 \leqslant \alpha < 1$. In the same way, we define the following subclass of bi-univalent functions $\mathcal{M}_\Sigma^{q,\lambda}(\gamma,\eta,\beta,h)$ as follows.

Definition 1.1. For $\gamma \geqslant 1$ and $\eta \geqslant 0$, let a function $f \in \Sigma$ have the form (1.1) and h be given by (1.2), then the function f is said to be *in the class* $\mathcal{M}_\Sigma^{q,\lambda}(\gamma,\eta,\beta,h)$ if the following conditions are satisfied:

$$\Re\Big\{(1-\gamma)\frac{\mathcal{H}_h^{\lambda,q}f(z)}{z} + \gamma(\mathcal{H}_h^{\lambda,q}f(z))' + \eta z(\mathcal{H}_h^{\lambda,q}f(z))''\Big\} > \beta \tag{1.16}$$

and

$$\Re\Big\{(1-\gamma)\frac{\mathcal{H}_h^{\lambda,q}g(w)}{w} + \gamma(\mathcal{H}_h^{\lambda,q}g(w))' + \eta w(\mathcal{H}_h^{\lambda,q}g(w))''\Big\} > \beta \tag{1.17}$$

with $\lambda > -1$, $0 < q < 1$, $0 \leqslant \beta < 1$ and $z, w \in \Delta$, where the function g is the analytic extension of f^{-1} to Δ and is given by (1.15).

Remark 1.2. (i) Putting $q \to 1^-$ we obtain that $\lim\limits_{q\to 1^-} \mathcal{M}_\Sigma^{q,\lambda}(\gamma,\eta,\beta;h) =: \mathcal{G}_\Sigma^{\lambda}(\gamma,\eta,\beta;h)$, where $\mathcal{G}_\Sigma^{\lambda}(\gamma,\eta,\beta;h)$ represents the functions $f \in \Sigma$ that satisfy (1.16) and (1.17) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{I}_h^{\lambda}$ (1.7).

(ii) Putting $b_m = (-1)^{m-1}\Gamma(\upsilon+1)/(4^{m-1}(m-1)!\,\Gamma(m+\upsilon))$, $\upsilon > 0$, we obtain the class $\mathcal{B}_\Sigma^{q,\lambda}(\gamma,\eta,\beta,\upsilon)$, that represents the functions $f \in \Sigma$ that satisfy (1.16) and (1.17) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{N}_{\upsilon,q}^{\lambda}$ (1.9).

(iii) Putting $b_m = (n+1)^\alpha/(n+m)^\alpha$, $\alpha > 0$, $n \geqslant 0$, we obtain the class $\mathcal{L}_\Sigma^{q,\lambda}(\gamma,\eta,\beta,n,\alpha)$, that represents the functions $f \in \Sigma$ that satisfy (1.16) and (1.17) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{M}_{n,q}^{\lambda,\alpha}$ (1.11).

(iv) Putting $b_m = \varrho^{m-1}\mathrm{e}^{-\varrho}/(m-1)!$, $\varrho > 0$, we obtain the class $\mathcal{M}_\Sigma^{q,\lambda}(\gamma,\eta,\beta,\varrho)$, that represents the functions $f \in \Sigma$ that satisfy (1.16) and (1.17) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{I}_q^{\lambda,\varrho}$ (1.12).

(v) Putting $b_m = (1+l+\mu(m-1))^n/(1+l)^n$, $n \in \mathbb{Z}$, $l \geqslant 0$, $\mu \geqslant 0$, we obtain the class $\mathcal{M}_{\Sigma}^{q,\lambda}(\gamma,\eta,\beta,n,l,\mu)$, that represents the functions $f \in \Sigma$ that satisfy (1.16) and (1.17) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{J}_{q,l,\mu}^{\lambda,n}$ (1.13).

Using that the Faber polynomial expansion of functions $f \in \mathcal{A}$ has the form (1.1), the coefficients of its inverse map may be expressed as (see [18], [25], [38], [43])

$$g(w) = f^{-1}(w) = w + \sum_{m=2}^{\infty} \frac{1}{m} K_{m-1}^{-m}(a_2, a_3, \ldots) w^m, \tag{1.18}$$

where

$$\begin{aligned} \mathcal{K}_{m-1}^{-m}&(a_2,a_3,\ldots) \\ &= \frac{(-m)!}{(-2m+1)!\,(m-1)!} a_2^{m-1} + \frac{(-m)!}{(2(-m+1))!\,(m-3)!} a_2^{m-3} a_3 \\ &+ \frac{(-m)!}{(-2m+3)!\,(m-4)!} a_2^{m-4} a_4 \\ &+ \frac{(-m)!}{(2(-m+2))!\,(m-5)!} a_2^{m-5}(a_5 + (-m+2)a_3^2) \\ &+ \frac{(-m)!}{(-2m+5)!(m-6)!} a_2^{m-6}(a_6 + (-2m+5)a_3a_4) + \sum_{i \geqslant 7} a_2^{m-i} U_i \end{aligned} \tag{1.19}$$

is such that U_i with $7 \leqslant i \leqslant m$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_m$. In particular, the first three terms of $\mathcal{K}_{m-1}^{-m}$ are

$$\mathcal{K}_1^{-2} = -2a_2, \quad \mathcal{K}_2^{-3} = 3(2a_2^2 - a_3), \quad \mathcal{K}_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

In general, an expansion of $\mathcal{K}_m^{-n}$, $n \in \mathbb{N}$, is (see [2], [8], [36], [40], [42], [44])

$$\mathcal{K}_m^{-n} = na_m + \frac{n(n-1)}{2}\mathcal{D}_m^2 + \frac{n!}{3!(n-3)!}\mathcal{D}_m^3 + \ldots + \frac{n!}{m!(n-m)!}\mathcal{D}_m^m,$$

where $\mathcal{D}_m^n = \mathcal{D}_m^n(a_2, a_3, \ldots)$ and

$$\mathcal{D}_m^p(a_1, a_2, \ldots, a_m) = \sum_{m=1}^{\infty} \frac{p!}{i_1! \ldots i_m!} a_1^{i_1} \ldots a_m^{i_m},$$

while $a_1 = 1$ and the sum is taken over all non-negative integers $i_1, \ldots, i_m$ satisfying

$$i_1 + i_2 + \ldots + i_m = p, \quad i_1 + 2i_2 + \ldots + mi_m = m.$$

Evidently

$$\mathcal{D}_m^m(a_1, a_2, \ldots, a_m) = a_1^m.$$

The following lemma is needed to prove our results.

Lemma 1.1 (Carathéodory lemma [11]). *If $\varphi \in \mathcal{P}$ and $\varphi(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$ then $|c_n| \leqslant 2$ for each n. This inequality is sharp for all n where $\mathcal{P}$ is the family of all functions φ analytic and having a positive real part in Δ with $\varphi(0) = 1$.*

2. Main results

Throughout this paper, we assume that $\gamma \geqslant 1$, $\eta \geqslant 0$, $\lambda > -1$, $0 \leqslant \beta < 1$, $0 < q < 1$. We firstly introduce a bound for the general coefficients of functions belonging to the class $\mathcal{M}_{\Sigma}^{q,\lambda}(\gamma, \eta, \beta; h)$.

Theorem 2.1. *Let the function f given by equation* (1.1) *belong to the class $\mathcal{M}_{\Sigma}^{q,\lambda}(\gamma, \eta, \beta; h)$. If $a_k = 0$ for $2 \leqslant k \leqslant m-1$, then*

$$|a_m| \leqslant \frac{2(1-\beta)[\lambda+1, q]_{m-1}}{(1+(\gamma+\eta m)(m-1))[m,q]!\, b_m}.$$

Proof. If $f \in \mathcal{M}_{\Sigma}^{q,\lambda}(\gamma, \eta, \beta; h)$, from (1.16), (1.17), we have

$$\begin{aligned}(2.1)\quad &(1-\gamma)\frac{\mathcal{H}_h^{\lambda,q} f(z)}{z} + \gamma(\mathcal{H}_h^{\lambda,q} f(z))' + \eta z(\mathcal{H}_h^{\lambda,q} f(z))'' \\ &= 1 + \sum_{m=2}^{\infty} (1+(\gamma+\eta m)(m-1)) \frac{[m,q]!}{[\lambda+1,q]_{m-1}} b_m a_m z^{m-1}, \quad z \in \Delta\end{aligned}$$

and

$$\begin{aligned}(2.2)\quad &(1-\gamma)\frac{\mathcal{H}_h^{\lambda,q} g(w)}{w} + \gamma(\mathcal{H}_h^{\lambda,q} g(w))' + \eta w(\mathcal{H}_h^{\lambda,q} g(w))'' \\ &= 1 + \sum_{m=2}^{\infty} (1+(\gamma+\eta m)(m-1)) \frac{[m,q]!}{[\lambda+1,q]_{m-1}} b_m\, A_m w^{m-1} \\ &= 1 + \sum_{m=2}^{\infty} (1+(\gamma+\eta m)(m-1)) \frac{[m,q]!}{[\lambda+1,q]_{m-1}} \\ &\quad \times b_m \frac{1}{m} \mathcal{K}_{m-1}^{-m}(a_2, \ldots, a_m) w^{m-1}, \quad w \in \Delta.\end{aligned}$$

Since

$$f \in \mathcal{M}_{\Sigma}^{q,\lambda}(\gamma, \eta, \beta; h) \quad \text{and} \quad g = f^{-1} \in \mathcal{M}_{\Sigma}^{q,\lambda}(\gamma, \eta, \beta; h),$$

we know that there are two functions with positive real parts,

$$U(z) = 1 + \sum_{m=1}^{\infty} c_m z^m \quad \text{and} \quad V(w) = 1 + \sum_{m=1}^{\infty} d_m w^m,$$

where

$$\Re(U(z)) > 0 \quad \text{and} \quad \Re(V(w)) > 0, \quad z, w \in \Delta,$$

so that

$$(2.3) \qquad (1-\gamma)\frac{\mathcal{H}_h^{\lambda,q} f(z)}{z} + \gamma(\mathcal{H}_h^{\lambda,q} f(z))' + \eta z(\mathcal{H}_h^{\lambda,q} f(z))'' = \beta + (1-\beta)U(z) = 1 + (1-\beta)\sum_{m=1}^{\infty} c_m z^m,$$

and

$$(2.4) \qquad (1-\gamma)\frac{\mathcal{H}_h^{\lambda,q} g(w)}{w} + \gamma(\mathcal{H}_h^{\lambda,q} g(w))' + \eta w(\mathcal{H}_h^{\lambda,q} g(w))'' = \beta + (1-\beta)V(w) = 1 + (1-\beta)\sum_{m=1}^{\infty} d_m w^m.$$

Using (2.1) and comparing the corresponding coefficients in (2.3), we obtain

$$(2.5) \qquad (1 + (\gamma + \eta m)(m-1))\frac{[m,q]!}{[\lambda+1,q]_{m-1}} b_m a_m = (1-\beta)c_{m-1}$$

and similarly, by using (2.2) in the equality (2.4), we have

$$(2.6) \quad (1 + (\gamma + \eta m)(m-1))\frac{[m,q]!}{[\lambda+1,q]_{m-1}} b_m \frac{1}{m}\mathcal{K}_{m-1}^{-m}(a_2, a_3, \ldots, a_m) = (1-\beta)d_{m-1}.$$

Under the assumption $a_k = 0$ for $0 \leqslant k \leqslant m-1$, we obtain $A_m = -a_m$, and so

$$(2.7) \qquad (1 + (\gamma + \eta m)(m-1))\frac{[m,q]!}{[\lambda+1,q]_{m-1}} b_m a_m = (1-\beta)c_{m-1}$$

and

$$(2.8) \qquad -(1 + \gamma(m-1) + \eta m(m-1))\frac{[m,q]!}{[\lambda+1,q]_{m-1}} b_m a_m = (1-\beta)d_{m-1}.$$

Taking the absolute values of (2.7) and (2.8), we conclude that

$$|a_m| = \left|\frac{(1-\beta)[\lambda+1,q]_{m-1}c_{m-1}}{(1+(\gamma+\eta m)(m-1))[m,q]!\, b_m}\right| = \left|\frac{-(1-\beta)[\lambda+1,q]_{m-1}d_{m-1}}{(1+(\gamma+\eta m)(m-1))[m,q]!\, b_m}\right|.$$

Applying Carathéodory lemma 1.1, we obtain

$$|a_m| \leqslant \frac{2(1-\beta)[\lambda+1,q]_{m-1}}{(1+(\gamma+\eta m)(m-1))[m,q]!\, b_m},$$

which completes the proof of the theorem. □

Taking $b_m = (-1)^{m-1}\Gamma(\upsilon+1)/(4^{m-1}(m-1)!\Gamma(m+\upsilon))$, $\upsilon > 0$, in Theorem 2.1, we obtain the following special case.

Corollary 2.1. *Let the function f given by equation* (1.1) *belong to the class* $\mathcal{B}_{\Sigma}^{q,\lambda}(\gamma,\eta,\beta,\upsilon)$. *If* $a_k = 0$ *for* $2 \leqslant k \leqslant m-1$, *then*

$$|a_m| \leqslant \frac{2(1-\beta)[\lambda+1,q]_{m-1}}{(1+(\gamma+\eta m)(m-1))[m,q]!\,\psi_m},$$

where ψ_m *is given by* (1.10).

Taking $b_m = (n+1)^{\alpha}/(n+m)^{\alpha}$, $\alpha > 0$, $n \geqslant 0$, in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let the function f given by equation* (1.1) *belong to the class* $\mathcal{L}_{\Sigma}^{q,\lambda}(\gamma,\eta,\beta,n,\alpha)$. *If* $a_k = 0$ *for* $2 \leqslant k \leqslant m-1$, *then*

$$|a_m| \leqslant \frac{2(1-\beta)(n+m)^{\alpha}[\lambda+1,q]_{m-1}}{(1+(\gamma+\eta m)(m-1))[m,q]!\,(n+1)^{\alpha}}.$$

Putting $b_m = \varrho^{m-1}\mathrm{e}^{-\varrho}/(m-1)!$, $\varrho > 0$, in Theorem 2.1, we obtain the following special case.

Corollary 2.3. *Let the function f given by equation* (1.1) *belong to the class* $\mathcal{M}_{\Sigma}^{q,\lambda}(\gamma,\eta,\beta,\varrho)$. *If* $a_k = 0$ *for* $2 \leqslant k \leqslant m-1$, *then*

$$|a_m| \leqslant \frac{2(1-\beta)[\lambda+1,q]_{m-1}(m-1)!}{(1+(\gamma+\eta m)(m-1))[m,q]!\,\varrho^{m-1}\mathrm{e}^{-\varrho}}.$$

Theorem 2.2. *Let the function f given by equation* (1.1) *belong to the class* $\mathcal{M}_{\Sigma}^{q,\lambda}(\gamma,\eta,\beta;h)$, *then*

$$(2.9)\qquad |a_2| \leqslant \begin{cases} \dfrac{2(1-\beta)[\lambda+1,q]}{(1+\gamma+2\eta)[2,q]!\,b_2}, \\ \qquad 0 \leqslant \beta < 1 - \dfrac{(1+\gamma+2\eta)^2([2,q]!)^2\,[\lambda+2,q]b_2^2}{2(1+2\gamma+6\eta)[3,q]!\,[\lambda+1,q]b_3}, \\ \sqrt{\dfrac{2(1-\beta)[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,b_3}}, \\ \qquad 1 - \dfrac{(1+\gamma+2\eta)^2([2,q]!)^2\,[\lambda+2,q]b_2^2}{2(1+2\gamma+6\eta)[3,q]!\,[\lambda+1,q]b_3} \leqslant \beta < 1, \end{cases}$$

$$(2.10)\qquad |a_3| \leqslant \frac{2(1-\beta)[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,b_3},$$

and

$$(2.11)\qquad |a_3 - 2a_2^2| \leqslant \frac{2(1-\beta)[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,b_3}.$$

P r o o f. Putting $n = 2$ and $n = 3$ in (2.5), (2.6), we have

$$(1+\gamma+2\eta)\frac{[2,q]!}{[\lambda+1,q]}\,b_2a_2 = (1-\beta)c_1, \tag{2.12}$$

$$(1+2\gamma+6\eta)\frac{[3,q]!}{[\lambda+1,q]_2}\,b_3a_3 = (1-\beta)c_2, \tag{2.13}$$

$$-(1+\gamma+2\eta)\frac{[2,q]!}{[\lambda+1,q]}\,b_2a_2 = (1-\beta)d_1, \tag{2.14}$$

and

$$(1+2\gamma+6\eta)\frac{[3,q]!}{[\lambda+1,q]_2}\,b_3(2a_2^2-a_3) = (1-\beta)d_2. \tag{2.15}$$

From (2.12) and (2.14), by using Carathéodory lemma 1.1, we obtain

$$|a_2| = \frac{(1-\beta)[\lambda+1,q]|c_1|}{(1+\gamma+2\eta)[2,q]!\,b_2} = \frac{(1-\beta)[\lambda+1,q]|d_1|}{(1+\gamma+2\eta)[2,q]!\,b_2} \leqslant \frac{2(1-\beta)[\lambda+1,q]}{(1+\gamma+2\eta)[2,q]!\,b_2}. \tag{2.16}$$

Also, from (2.13) and (2.15), we have

$$2(1+2\gamma+6\eta)\frac{[3,q]!}{[\lambda+1,q]_2}\,b_3a_2^2 = (1-\beta)(c_2+d_2)$$

and by using Carathéodory lemma 1.1, we obtain

$$|a_2| \leqslant \sqrt{\frac{2(1-\beta)[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,b_3}}. \tag{2.17}$$

From (2.16) and (2.17), we have the desired estimate on the coefficient as asserted in (2.9).

To find the bound on the coefficient $|a_3|$, we subtract (2.15) from (2.13) and get

$$2(1+2\gamma+6\eta)\frac{[3,q]!}{[\lambda+1,q]_2}\,b_3(a_3-a_2^2) = (1-\beta)(c_2-d_2)$$

or

$$a_3 = a_2^2 + \frac{(1-\beta)(c_2-d_2)[\lambda+1,q]_2}{2(1+2\gamma+6\eta)[3,q]!\,b_3}. \tag{2.18}$$

Substituting the value of a_2^2 from (2.12) into (2.18), we obtain

$$a_3 = \frac{(1-\beta)^2[\lambda+1,q]^2c_1^2}{(1+\gamma+2\eta)^2([2,q]!)^2b_2^2} + \frac{(1-\beta)(c_2-d_2)[\lambda+1,q]_2}{2(1+2\gamma+6\eta)[3,q]!\,b_3}.$$

Using Carathéodory lemma 1.1, we find that

$$|a_3| \leqslant \frac{4(1-\beta)^2[\lambda+1,q]^2}{(1+\gamma+2\eta)^2([2,q]!)^2 b_2^2} + \frac{2(1-\beta)[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,b_3} \tag{2.19}$$

and substituting the value of a_2^2 from (2.12) into (2.18), we have

$$a_3 = \frac{(1-\beta)[\lambda+1,q]_2 c_2}{(1+2\gamma+6\eta)[3,q]!\,b_3}.$$

Appling Carathéodory lemma 1.1, we obtain

$$|a_3| \leqslant \frac{2(1-\beta)[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,b_3}. \tag{2.20}$$

Combining (2.19) and (2.20), we have the desired estimate on the coefficient $|a_3|$ as asserted in (2.10).

Finally, from (2.15), we deduce that

$$|a_3 - 2a_2^2| = \frac{(1-\beta)[\lambda+1,q]_2|d_2|}{(1+2\gamma+6\eta)[3,q]!\,b_3} \leqslant \frac{2(1-\beta)[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,b_3}.$$

Thus the proof of Theorem 2.2 was completed. □

Taking $b_m = (-1)^{m-1}\Gamma(\upsilon+1)/(4^{m-1}(m-1)!\Gamma(m+\upsilon))$, $\upsilon > 0$, in Theorem 2.2, we obtain the following special case.

Corollary 2.4. *Let the function f given by equation* (1.1) *belong to the class $\mathcal{B}_{\Sigma}^{q,\lambda}(\gamma,\eta,\beta,\upsilon)$, then*

$$|a_2| \leqslant \begin{cases} \dfrac{2(1-\beta)[\lambda+1,q]}{(1+\gamma+2\eta)[2,q]!\,\psi_2}, & 0 \leqslant \beta < 1 - \dfrac{(1+\gamma+2\eta)^2([2,q]!)^2[\lambda+2,q]\psi_2^2}{2(1+2\gamma+6\eta)[3,q]!\,[\lambda+1,q]\psi_3}, \\ \sqrt{\dfrac{2(1-\beta)[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,\psi_3}}, & 1 - \dfrac{(1+\gamma+2\eta)^2([2,q]!)^2\,[\lambda+2,q]\psi_2^2}{2(1+2\gamma+6\eta)[3,q]!\,[\lambda+1,q]\psi_3} \leqslant \beta < 1, \end{cases}$$

$$|a_3| \leqslant \frac{2(1-\beta)[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,\psi_3}, \quad \textit{and} \quad |a_3 - 2a_2^2| \leqslant \frac{2(1-\beta)[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,\psi_3},$$

where ψ_m is given by (1.10).

Considering $b_m = (n+1)^\alpha/(n+m)^\alpha$, $\alpha > 0$, $n \geqslant 0$, in Theorem 2.2, we obtain the following result.

Corollary 2.5. *Let the function f given by equation* (1.1) *belong to the class* $\mathcal{L}_{\Sigma}^{q,\lambda}(\gamma,\eta,\beta,n,\alpha)$*, then*

$$|a_2| \leqslant \begin{cases} \dfrac{2(1-\beta)(n+2)^{\alpha}[\lambda+1,q]}{(1+\gamma+2\eta)(n+1)^{\alpha}[2,q]!}, \\ \qquad 0 \leqslant \beta < 1 - \dfrac{(1+\gamma+2\eta)^2(n+1)^{\alpha}(n+3)^{\alpha}([2,q]!)^2[\lambda+2,q]}{2(1+2\gamma+6\eta)(n+2)^{2\alpha}[3,q]!\,[\lambda+1,q]}, \\ \sqrt{\dfrac{2(1-\beta)(n+3)^{\alpha}[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,(n+1)^{\alpha}}}, \\ \qquad 1 - \dfrac{(1+\gamma+2\eta)^2(n+1)^{\alpha}(n+3)^{\alpha}([2,q]!)^2[\lambda+2,q]}{2(1+2\gamma+6\eta)(n+2)^{2\alpha}[3,q]!\,[\lambda+1,q]} \leqslant \beta < 1, \end{cases}$$

$$|a_3| \leqslant \frac{2(1-\beta)(n+3)^{\alpha}[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,(n+1)^{\alpha}}, \quad \text{and} \quad |a_3 - 2a_2^2| \leqslant \frac{2(1-\beta)(n+3)^{\alpha}[\lambda+1,q]_2}{(1+2\gamma+6\eta)[3,q]!\,(n+1)^{\alpha}}.$$

Putting $b_m = \varrho^{m-1}\mathrm{e}^{-\varrho}/(m-1)!$, $\varrho > 0$, in Theorem 2.1, we obtain the special case:

Corollary 2.6. *Let the function f given by equation* (1.1) *belong to the class* $\mathcal{M}_{\Sigma}^{q,\lambda}(\gamma,\eta,\beta,\varrho)$*, then*

$$|a_2| \leqslant \begin{cases} \dfrac{2(1-\beta)[\lambda+1,q]}{\varrho(1+\gamma+2\eta)[2,q]!}, & 0 \leqslant \beta < 1 - \dfrac{(1+\gamma+2\eta)^2([2,q]!)^2[\lambda+2,q]}{(1+2\gamma+6\eta)[3,q]!\,[\lambda+1,q]}, \\ \sqrt{\dfrac{4(1-\beta)[\lambda+1,q]_2}{\varrho^2(1+2\gamma+6\eta)[3,q]!}}, & 1 - \dfrac{(1+\gamma+2\eta)^2([2,q]!)^2[\lambda+2,q]}{(1+2\gamma+6\eta)[3,q]![\lambda+1,q]} \leqslant \beta < 1, \end{cases}$$

$$|a_3| \leqslant \frac{4(1-\beta)[\lambda+1,q]_2}{\varrho^2(1+2\gamma+6\eta)[3,q]!}, \quad \text{and} \quad |a_3 - 2a_2^2| \leqslant \frac{4(1-\beta)[\lambda+1,q]_2}{\varrho^2(1+2\gamma+6\eta)[3,q]!}.$$

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