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RIESZ POTENTIALS AND SOBOLEV-TYPE INEQUALITIES IN ORLICZ-MORREY SPACES OF AN INTEGRAL FORM

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Abstract. Our aim is to give Sobolev-type inequalities for Riesz potentials of functions in Orlicz-Morrey spaces of an integral form over non-doubling metric measure spaces as an extension of T. Ohno, T. Shimomura (2022). Our results are new even for the doubling metric measure spaces.

Keywords: Riesz potential; Sobolev's inequality; Orlicz-Morrey space; metric measure space; non-doubling measure

MSC 2020: 46E35, 46E30

1. Introduction

The space introduced by Morrey in 1938 (see [19]) has become a useful tool for the study of the existence and regularity of solutions of partial differential equations. For $0 < \alpha < N$ and a locally integrable function f on $\mathbb{R}^N$ the Riesz potential $U_\alpha f$ of order α is defined by

$$U_\alpha f(x) = \int_{\mathbb{R}^N} |x-y|^{\alpha-N} f(y)\,\mathrm{d}y.$$

The well-known Sobolev inequality for $U_\alpha f$ was studied on Morrey spaces in [1], on generalized Morrey spaces in [20], on Orlicz spaces in [16] and on Orlicz-Morrey spaces in [21], see also [14], [15], [25], [32].

Mizuta and the second author in [17] studied a Sobolev-type inequality for $U_\alpha f$ for locally integrable functions f on $\mathbb{R}^N$ satisfying

$$\sup_{x\in G}\biggl(\int_0^{d_G} r^{\nu-N}\varphi_1(r)\biggl(\int_{B(x,r)} |f(y)|^p \varphi_2(|f(y)|)\,\mathrm{d}y\biggr)\frac{\mathrm{d}r}{r}\biggr)^{1/p} < \infty, \tag{1.1}$$

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where G is a bounded open set in $\mathbb{R}^N$, $1 < p < \infty$, $0 < \nu \leqslant N$, $d_G = \sup\{d(x,y)\colon x, y \in G\}$ and φ_i $(i = 1, 2)$ are positive monotone functions on the interval $(0, \infty)$ satisfying

(φ) there exists a constant $c > 0$ such that

$$c^{-1}\varphi_i(r) \leqslant \varphi_i(r^2) \leqslant c\varphi_i(r)$$

whenever $r > 0$ for $i = 1, 2$;

(i) $r^{-\nu}\varphi_1(r)$ is nonincreasing on $(0, \infty)$;

(ii) $\varphi_2(r)$ is nondecreasing on $(0, \infty)$.

As in [24], we denote by (X, d, μ) a metric measure space, where X is a bounded set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. We often write X instead of (X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball in X centered at x with radius r and $d_X = \sup\{d(x, y)\colon x, y \in X\}$. We assume that $d_X < \infty$, $\mu(\{x\}) = 0$ for $x \in X$ and $0 < \mu(B(x, r)) < \infty$ for $x \in X$ and $r > 0$ for simplicity. We do not assume that μ has the so-called *doubling condition*. Recall that a Radon measure μ is said to be doubling if there exists a constant $c_0 > 0$ such that $\mu(B(x, 2r)) \leqslant c_0\mu(B(x, r))$ for all $x \in \operatorname{supp}(\mu)(= X)$ and $r > 0$, see [2]. Otherwise μ is said to be non-doubling, see [24], [29]. Note from [22], [23] that the doubling condition is not necessary for μ by using the modified maximal operator.

For $\tau \geqslant 1$ and $\alpha > 0$, we define the (modified) Riesz potential of order α for a locally integrable function f on X by

$$I_{\alpha,\tau}f(x) = \int_X \frac{d(x, y)^\alpha f(y)}{\mu(B(x, \tau d(x, y)))}\,\mathrm{d}\mu(y),$$

see, e.g., [5], [18], [26], [29]. Based on the idea of [34] by Stempak, we cannot remove the number τ in the non-doubling metric measure setting, see [28]. When $X = \mathbb{R}^N$ and $\mu = dx$, $I_{\alpha,\tau}f$ is equal to $U_\alpha f$. In the doubling metric measure setting we refer to e.g. [4], [10], [11], [24], [29] for other types of Riesz potentials.

In the previous paper [24], we established Sobolev-type inequalities for $I_{\alpha,\tau}f$ of functions in Morrey spaces $\mathcal{L}^{p,\omega,\theta}(X)$ of an integral form over non-doubling metric measure spaces as an extension of Theorem 5.4 of [17] from the Euclidean case.

In the present paper we will extend Theorem 3.3 of [24] from Morrey spaces of an integral form to Orlicz-Morrey spaces of an integral form. In fact, we establish a Sobolev-type inequality for $I_{\alpha,\tau}f$ of functions in Orlicz-Morrey spaces $\mathcal{L}^{\Phi,\omega,\theta}(X)$ of an integral form defined by general functions Φ and ω over non-doubling metric measure spaces X (see Theorem 4.4) as an extension of Theorem 3.3 of [24]. The result is new even in the case when X is a doubling metric measure space, see

Theorem 4.8. Our strategy is to apply the boundedness of the (modified) Hardy-Littlewood maximal operator M_λ (see Theorem 3.5) and Hedberg's trick, see [7]. See Sections 2 and 3 for the definitions of $\mathcal{L}^{\Phi,\omega,\theta}(X)$ and M_λ.

For Trudinger-type inequalities in Orlicz-Morrey spaces of an integral form over non-doubling metric measure spaces, we refer to [9].

Throughout the paper, we let C denote various constants independent of the variables in question and $C(a,b,\ldots)$ be a constant that depends on $a,b,\ldots$ only. The symbol $g \sim h$ means that $C^{-1}h \leqslant g \leqslant Ch$ for some constant $C > 0$.

2. Orlicz-Morrey spaces of an integral form

In this section we define Orlicz-Morrey spaces $\mathcal{L}^{\Phi,\omega,\theta}(X)$ of an integral form. Before we define $\mathcal{L}^{\Phi,\omega,\theta}(X)$ we state the assumptions on the functions Φ and ω.

We consider a function

$$\Phi(t)\colon\ [0,\infty) \to [0,\infty)$$

satisfying the following conditions (Φ1)–(Φ3):

(Φ1) $\Phi(\cdot)$ is continuous on $[0,\infty)$;

(Φ2) $A_1 = \Phi(1) > 0$;

(Φ3) $t \mapsto \Phi(t)/t$ is almost increasing on $(0,\infty)$, i.e., there exists a constant $A_2 \geqslant 1$ such that

$$\frac{\Phi(t_1)}{t_1} \leqslant \frac{A_2\Phi(t_2)}{t_2} \quad \text{whenever } 0 < t_1 < t_2.$$

We write $\bar{\phi}(t) = \sup\limits_{0<s\leqslant t} (\Phi(s)/s)$ and $\bar{\Phi}(t) = \int_0^t \bar{\phi}(r)\,\mathrm{d}r$ for $t \geqslant 0$. Then $\bar{\Phi}(\cdot)$ is convex and

$$\Phi\Big(\frac{t}{2}\Big) \leqslant \bar{\Phi}(t) \leqslant A_2\Phi(t) \quad \text{for all } t \geqslant 0. \tag{2.1}$$

We also consider a weight function $\omega(r)\colon\ (0,\infty) \to (0,\infty)$ satisfying the following conditions:

(ω0) $\omega(\cdot)$ is continuous on $(0,\infty)$;

(ω1) $r \mapsto \omega(r)$ is almost increasing on $(0,\infty)$, i.e., there exists a constant $\tilde{c}_1 \geqslant 1$ such that

$$\omega(r_1) \leqslant \tilde{c}_1\omega(r_2) \quad \text{whenever } 0 < r_1 < r_2 < \infty;$$

(ω2) there exists a constant $\tilde{c}_2 > 1$ such that

$$\tilde{c}_2^{-1}\omega(r) \leqslant \omega(2r) \leqslant \tilde{c}_2\omega(r) \quad \text{whenever } r > 0;$$

(ω3) there exist constants $\omega_0 > 0$ and $\tilde{c}_3 \geqslant 1$ such that

$$\tilde{c}_3^{-1}r^{\omega_0} \leqslant \omega(r) \leqslant \tilde{c}_3 \quad \text{for all } 0 < r \leqslant 2d_X.$$

Let us write that $L_c(t) = \log(c+t)$ for $c > 1$ and $t \geqslant 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$.

Example 2.1. Let $0 < \sigma < \omega_0$ and $\beta \in \mathbb{R}$. Then

$$\omega(r) = r^{\sigma} L_e\Big(\frac{1}{r}\Big)^{\beta}$$

satisfies (ω0), (ω1), (ω2) and (ω3).

Recall that f is a locally integrable function on X if f is an integrable function on all balls B in X. Let $\theta \geqslant 1$. In connection with (1.1), given $\Phi(t)$ and $\omega(r)$ as above, we define the $\mathcal{L}^{\Phi,\omega,\theta}$ norm by

$$\|f\|_{\mathcal{L}^{\Phi,\omega,\theta}(X)} = \inf\Big\{\lambda > 0; \sup_{x\in X}\Big(\int_0^{2d_X} \frac{\omega(r)}{\mu(B(x,\theta r))}\Big(\int_{B(x,r)} \bar{\Phi}\Big(\frac{|f(y)|}{\lambda}\Big)\,\mathrm{d}\mu(y)\Big)\frac{\mathrm{d}r}{r}\Big) \leqslant 1\Big\}.$$

The space of all measurable functions f on X with $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta}(X)} < \infty$ is denoted by $\mathcal{L}^{\Phi,\omega,\theta}(X)$. The space $\mathcal{L}^{\Phi,\omega,\theta}(X)$ is called an *Orlicz-Morrey space of integral form*. Here note that $2d_X$ can be replaced by κd_X with $\kappa > 1$. In case $\Phi(t) = t^p$, $\mathcal{L}^{\Phi,\omega,\theta}(X)$ is denoted by $\mathcal{L}^{p,\omega,\theta}(X)$ for simplicity.

See [3] for another space corresponding to Orlicz-Morrey spaces of an integral form.

Remark 2.2. In [17], Mizuta and the second author treated the case when $X = \mathbb{R}^N$ and $\Phi(t)$ and $\omega(r)$ are of the form $\Phi(t) = t^p\varphi_2(t)$, and $\omega(r) = r^{\nu}\varphi_1(r)$ as in (1.1).

We shall also consider the following conditions for $\Phi(t)$: Let $p \geqslant 1$ and $q \geqslant 1$ be given.

($\Phi3; 0; p$) $t \mapsto t^{-p}\Phi(t)$ is almost increasing on $(0, 1]$, i.e., there exists a constant $A_{2,0,p} \geqslant 1$ such that

$$t_1^{-p}\Phi(t_1) \leqslant A_{2,0,p}\, t_2^{-p}\,\Phi(t_2) \quad \text{whenever } 0 < t_1 < t_2 \leqslant 1;$$

($\Phi3; \infty; q$) $t \mapsto t^{-q}\Phi(t)$ is almost increasing on $[1, \infty)$, i.e., there exists a constant $A_{2,\infty,q} \geqslant 1$ such that

$$t_1^{-q}\Phi(t_1) \leqslant A_{2,\infty,q}\, t_2^{-q}\Phi(t_2) \quad \text{whenever } 1 \leqslant t_1 < t_2.$$

Remark 2.3. If $\Phi(t)$ satisfies ($\Phi3; 0; p$), then it satisfies ($\Phi3; 0; p'$) for $1 \leqslant p' \leqslant p$. If $\Phi(t)$ satisfies ($\Phi3; \infty; q$), then it satisfies ($\Phi3; \infty; q'$) for $1 \leqslant q' \leqslant q$.

Example 2.4. Let $1 < p_1 < \infty$ and $q_j \in \mathbb{R}$ for $j = 1, \dots, k$. Then,

$$\Phi_{p_1,\{q_j\}}(t) = t^{p_1} \prod_{j=1}^{k} (L_e^{(j)}(t))^{q_j}$$

satisfies (Φ1), (Φ2) and (Φ3). This function satisfies ($\Phi3; 0; p$) and ($\Phi3; \infty; q$) for $1 \leqslant p < p_1$ and $1 \leqslant q < p_1$ in general, and for $1 \leqslant p \leqslant p_1$ and $1 \leqslant q \leqslant p_1$ in case $q_j \geqslant 0$ for all $j = 1, \dots, k$, see [13].

3. Boundedness of the maximal operator

For a locally integrable function f on X and $\lambda \geqslant 1$, the Hardy-Littlewood maximal function $M_\lambda f$ is defined by

$$M_\lambda f(x) = \sup_{r>0} \frac{1}{\mu(B(x, \lambda r))} \int_{B(x,r)} |f(y)| \, \mathrm{d}\mu(y).$$

For $\lambda \geqslant 1$, we say that X satisfies $(M\lambda)$ if there exists a constant $C > 0$ such that

$$\mu(\{x \in X \colon \; M_\lambda f(x) > k\}) \leqslant \frac{C}{k} \int_X |f(y)| \, \mathrm{d}\mu(y) \tag{3.1}$$

for all measurable functions $f \in L^1(X)$ and $k > 0$. In (3.1), we cannot reduce the number λ any more, see [34].

Note from [8] that X satisfies $(M\lambda)$ for any $\lambda > 0$ if μ satisfies the doubling condition. For $(M\lambda)$, see e.g. [6], [22], [23], [27], [35], [36] and Remark 2.2 of [24].

We know the following result.

Lemma 3.1 ([30], Lemma 2.1)**.** *Let $1 < p < \infty$ and let $\lambda \geqslant 1$. Suppose X satisfies $(M\lambda)$. Then there exists a constant $C > 0$ such that*

$$\int_X \{M_\lambda f(x)\}^p \, \mathrm{d}\mu(x) \leqslant C$$

for all measurable functions f on X with $\|f\|_{L^p(X)} \leqslant 1$.

By Lemma 3.1, we can prove the next lemma.

Lemma 3.2 ([24], Theorem 2.4)**.** *Let $1 \leqslant \theta_1 < \theta_2$ and $\lambda > \theta_1(\theta_2 + 1)/(\theta_2 - \theta_1)$. Assume that X satisfies $(M\lambda)$. Further suppose*

$(\omega 1')$ $r \mapsto r^{-\varepsilon_1}\omega(r)$ is almost increasing in $(0, d_X]$ for some $\varepsilon_1 > 0$.

If $p > 1$, then there is a constant $C > 0$ such that

$$\|M_\lambda f\|_{\mathcal{L}^{p,\omega,\theta_2}(X)} \leqslant C \|f\|_{\mathcal{L}^{p,\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{p,\omega,\theta_1}(X)$.

Remark 3.3. Note that ($\omega 1'$) implies ($\omega 1$). Let $\omega(r) = r^{\sigma} L_e(1/r)^{\beta}$ be as in Example 2.1. Then note that ($\omega 1'$) holds for $0 < \varepsilon_1 < \sigma$.

Remark 3.4. In [18], [29], [33], the parameters θ_1 and θ_2 were needed to prove the boundedness of M_λ on non-doubling Morrey spaces. We refer to [31] for modified Morrey spaces over metric measure spaces.

We can show the next theorem using Lemma 3.2.

Theorem 3.5. *Let $1 \leqslant \theta_1 < \theta_2$ and $\lambda > \theta_1(\theta_2 + 1)/(\theta_2 - \theta_1)$. Suppose $\Phi(t)$ satisfies ($\Phi 3; 0; p$) and ($\Phi 3; \infty; q$) for $p > 1$ and $q > 1$. Assume that X satisfies ($M\lambda$) and ($\omega 1'$) holds. Then there is a constant $C > 0$ such that*

$$\|M_\lambda f\|_{\mathcal{L}^{\Phi,\omega,\theta_2}(X)} \leqslant C \|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{\Phi,\omega,\theta_1}(X)$.

P r o o f. Set $p_0 = \min(p, q)$. Then $p_0 > 1$. Consider the function

$$\Phi_0(t) = \Phi(t)^{1/p_0}.$$

Then note from Remark 2.3 that $\Phi_0(t)$ satisfies ($\Phi 1$), ($\Phi 2$) and ($\Phi 3$). Therefore, as in (2.1), there exists a convex function $\widetilde{\Phi}_0(t)$ such that

$$\Phi_0\Big(\frac{t}{2}\Big) \leqslant \widetilde{\Phi}_0(t) \leqslant C_1 \Phi_0(t)$$

for all $t \geqslant 0$ and some constant $C_1 > 0$. Let f be a nonnegative measurable function on X with $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)} \leqslant \frac{1}{2}$. Jensen's inequality implies

$$\Phi_0\Big(\frac{M_\lambda f(x)}{2}\Big) \leqslant \widetilde{\Phi}_0(M_\lambda f(x)) \leqslant M_\lambda[\widetilde{\Phi}_0(f(\cdot))](x) \leqslant C_1 M_\lambda[\Phi_0(f(\cdot))](x),$$

so that

$$\Phi\Big(\frac{M_\lambda f(x)}{2}\Big) \leqslant C_1^{p_0} [M_\lambda[\Phi_0(f(\cdot))](x)]^{p_0}$$

for all $x \in X$. Since $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)} \leqslant \frac{1}{2}$, by (2.1), we obtain

$$\int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_1 r))} \Big(\int_{B(z,r)} \Phi_0(f(y))^{p_0} \,\mathrm{d}\mu(y)\Big) \frac{\mathrm{d}r}{r}$$
$$= \int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_1 r))} \Big(\int_{B(z,r)} \Phi(f(y)) \,\mathrm{d}\mu(y)\Big) \frac{\mathrm{d}r}{r} \leqslant 1$$

for all $z \in X$. Hence, applying Lemma 3.2 to $\Phi_0(f(y))$, we obtain by (2.1)

$$\begin{aligned}
&\int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_2 r))}\Big(\int_{B(z,r)} \bar{\Phi}\Big(\frac{M_\lambda f(x)}{2}\Big)\,\mathrm{d}\mu(x)\Big)\frac{\mathrm{d}r}{r} \\
&\qquad \leqslant A_2 \int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_2 r))}\Big(\int_{B(z,r)} \Phi\Big(\frac{M_\lambda f(x)}{2}\Big)\,\mathrm{d}\mu(x)\Big)\frac{\mathrm{d}r}{r} \\
&\qquad \leqslant A_2 C_1^{p_0} \int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_2 r))}\Big(\int_{B(z,r)} [M_\lambda[\Phi_0(f(\cdot))](x)]^{p_0}\,\mathrm{d}\mu(x)\Big)\frac{\mathrm{d}r}{r} \leqslant C
\end{aligned}$$

for all $z \in X$. This completes the proof of the theorem. □

Example 3.6. Theorem 3.5 applies e.g. to the following non-doubling functions

$$\Phi_1(t) = \mathrm{e}^{p_1 t} - p_1 t - 1 \quad (0 < p_1 < \infty), \quad \Phi_2(t) = \mathrm{e}^t t^{p_1} \quad (1 \leqslant p_1 < \infty),$$
$$\Phi_3(t) = \mathrm{e}^{t^{p_1}} - 1 \quad (1 \leqslant p_1 < \infty)$$

which satisfy (Φ1), (Φ2) and (Φ3). For the conditions on p and q such that $(\Phi 3; 0; p)$ and $(\Phi 3; \infty; q)$ hold, see Examples 3–5 of [13] for details.

4. Sobolev-type inequality

We recall a lemma for an auxiliary function with certain properties which we are going to need.

Lemma 4.1 ([12], Lemma 5.1)**.** *Let $F(t)$ be a positive function on $(0,\infty)$ satisfying the following conditions:*

(F1) *$F(\cdot)$ is continuous on $(0,\infty)$;*

(F2) *$K_1 = F(1) > 0$;*

(F3) *$t \mapsto t^{-\varepsilon'} F(t)$ is almost increasing for some $\varepsilon' > 0$; i.e., there exists a constant $K_2 \geqslant 1$ such that*

$$t_1^{-\varepsilon'} F(t_1) \leqslant K_2 t_2^{-\varepsilon'} F(t_2) \quad \text{whenever } 0 < t_1 < t_2.$$

Set $F^{-1}(s) = \sup\{t > 0\colon F(t) < s\}$ for $s > 0$. Then:

(1) *$F^{-1}(\cdot)$ is nondecreasing.*

(2) $F^{-1}(\lambda t) \leqslant (K_2\lambda)^{1/\varepsilon'} F^{-1}(t)$ *for all $t > 0$ and $\lambda \geqslant 1$.*

(3) $F(F^{-1}(t)) = t$ *for all $t > 0$.*

(4) $K_2^{-1/\varepsilon'} t \leqslant F^{-1}(F(t)) \leqslant K_2^{2/\varepsilon'} t$ *for all $t > 0$.*

(5) $\min\Big\{1, \Big(\dfrac{s}{K_1K_2}\Big)^{1/\varepsilon'}\Big\} \leqslant F^{-1}(s) \leqslant \max\{1, (K_1K_2 s)^{1/\varepsilon'}\}$ *for all $s > 0$.*

Remark 4.2. Note that $F(t) = \Phi(t)$ is a function satisfying (F1), (F2) and (F3) with $K_1 = A_1, K_2 = A_2$ and $\varepsilon' = 1$.

We consider the following condition:

($\Phi\omega\alpha$) There exist constants $\varepsilon_2 > 0$ and $A_4 \geqslant 1$ such that

$$r_2^{\varepsilon_2+\alpha}\Phi^{-1}(\omega(r_2)^{-1}) \leqslant A_4 r_1^{\varepsilon_2+\alpha}\Phi^{-1}(\omega(r_1)^{-1})$$

for all $x \in X$ whenever $0 < r_1 < r_2 < d_X$.

Lemma 4.3. *Let $1 \leqslant \theta < \tau$. Assume that ($\Phi\omega\alpha$) holds. Then there exists a constant $C > 0$ such that*

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,\mathrm{d}\mu(y) \leqslant C\delta^\alpha \Phi^{-1}(\omega(\delta)^{-1})$$

for all $x \in X$, $0 < \delta < \frac{1}{2}d_X$ and nonnegative $f \in \mathcal{L}^{\Phi,\omega,\theta}(X)$ with $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta}(X)} \leqslant 1$.

P r o o f. Let f be a nonnegative measurable function with $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta}(X)} \leqslant \frac{1}{2}$. Let $x \in X$ and $0 < \delta < \frac{1}{2}d_X$. We find by ($\Phi$3) and Lemma 4.1 (3) that

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,\mathrm{d}\mu(y)\\
&\quad\leqslant \int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))}\Phi^{-1}(\omega(d(x,y))^{-1})\,\mathrm{d}\mu(y)\\
&\qquad + A_2\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\\
&\qquad \times \frac{f(y)^{-1}\Phi(f(y))}{\{\Phi^{-1}(\omega(d(x,y))^{-1})\}^{-1}\Phi(\Phi^{-1}(\omega(d(x,y))^{-1}))}\,\mathrm{d}\mu(y)\\
&\quad= \int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))}\Phi^{-1}(\omega(d(x,y))^{-1})\,\mathrm{d}\mu(y)\\
&\qquad + A_2\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha\omega(d(x,y))\Phi^{-1}(\omega(d(x,y))^{-1})}{\mu(B(x,\tau d(x,y)))}\Phi(f(y))\,\mathrm{d}\mu(y)\\
&\quad= I_1 + A_2 I_2.
\end{aligned}$$

Let j_0 be the smallest integer such that $\tau^{j_0}\delta \geqslant d_X$. By ($\omega$1), ($\omega$2), Lemma 4.1 (2) and ($\Phi\omega\alpha$), we have

$$I_1 = \sum_{j=1}^{j_0}\int_{B(x,\tau^j\delta)\setminus B(x,\tau^{j-1}\delta)} \frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))}\Phi^{-1}(\omega(d(x,y))^{-1})\,\mathrm{d}\mu(y)$$

$$\leqslant C\sum_{j=1}^{j_0}(\tau^j\delta)^\alpha\Phi^{-1}(\omega(\tau^j\delta)^{-1}) \leqslant C\int_\delta^{\tau d_X}\varrho^\alpha\Phi^{-1}(\omega(\varrho)^{-1})\frac{\mathrm{d}\varrho}{\varrho}$$
$$\leqslant C\int_\delta^{d_X}\varrho^\alpha\Phi^{-1}(\omega(\varrho)^{-1})\frac{\mathrm{d}\varrho}{\varrho} \leqslant C\delta^\alpha\Phi^{-1}(\omega(\delta)^{-1}).$$

Next, for $\gamma = \tau\theta^{-1} > 1$, let j_1 be the smallest positive integer such that $\gamma^{j_1/2}\delta \geqslant d_X$. Then we have by $(\Phi\omega\alpha)$ and Lemma 4.1 (2)

$$I_2 \leqslant C\delta^\alpha\Phi^{-1}(\omega(\delta)^{-1})\int_{X\setminus B(x,\delta)}\frac{\omega(d(x,y))}{\mu(B(x,\tau d(x,y)))}\Phi(f(y))\,\mathrm{d}\mu(y)$$
$$= C\delta^\alpha\Phi^{-1}(\omega(\delta)^{-1})\sum_{j=1}^{j_1}\int_{B(x,\gamma^{j/2}\delta)\setminus B(x,\gamma^{(j-1)/2}\delta)}\frac{\omega(d(x,y))}{\mu(B(x,\tau d(x,y)))}\Phi(f(y))\,\mathrm{d}\mu(y).$$

Hence, by $(\omega 1)$ and $(\omega 2)$,

$$I_2 \leqslant C\delta^\alpha\Phi^{-1}(\omega(\delta)^{-1})\sum_{j=1}^{j_1}\frac{\omega(\gamma^{j/2}\delta)}{\mu(B(x,\gamma^{(j+1)/2}\theta\delta))}\int_{B(x,\gamma^{j/2}\delta)}\Phi(f(y))\,\mathrm{d}\mu(y)$$
$$\leqslant C\delta^\alpha\Phi^{-1}(\omega(\delta)^{-1})\sum_{j=1}^{j_1}\int_{\gamma^{j/2}\delta}^{\gamma^{(j+1)/2}\delta}\frac{\omega(t)}{\mu(B(x,\theta t))}\left(\int_{B(x,t)}\Phi(f(y))\,\mathrm{d}\mu(y)\right)\frac{\mathrm{d}t}{t}$$
$$\leqslant C\delta^\alpha\Phi^{-1}(\omega(\delta)^{-1})\int_{\gamma^{1/2}\delta}^{\gamma d_X}\frac{\omega(t)}{\mu(B(x,\theta t))}\left(\int_{B(x,t)}\Phi(f(y))\,\mathrm{d}\mu(y)\right)\frac{\mathrm{d}t}{t}$$
$$\leqslant C\delta^\alpha\Phi^{-1}(\omega(\delta)^{-1})\int_0^{2d_X}\frac{\omega(t)}{\mu(B(x,\theta t))}\left(\int_{B(x,t)}\Phi(f(y))\,\mathrm{d}\mu(y)\right)\frac{\mathrm{d}t}{t}$$
$$\leqslant C\delta^\alpha\Phi^{-1}(\omega(\delta)^{-1}).$$

Thus, we obtain the result required. □

Before we state our main theorem we present assumptions for the function in the Sobolev-type inequality. We consider a function

$$\Psi(t)\colon\ [0,\infty)\to[0,\infty)$$

that satisfies $(\Phi1)$–$(\Phi3)$ and

$(\Psi\Phi)$ there exists a constant $A'\geqslant 1$ such that

$$\Psi(t(\omega^{-1}(\Phi(t)^{-1}))^\alpha)\leqslant A'\Phi(t)\quad\text{for all } t\geqslant 1.$$

As an application of M_λ, we establish a Sobolev-type inequality for $I_{\alpha,\tau}f$ of functions in $\mathcal{L}^{\Phi,\omega,\theta}(X)$ over non-doubling metric measure spaces, which is an extension of Theorem 3.3 of [24].

Theorem 4.4. *Let X be a non-doubling metric measure space. Let $1 \leqslant \theta_1 < \theta_2$ and $\theta_1(\theta_2+1)/(\theta_2-\theta_1) < \lambda \leqslant \tau$. Suppose $\Phi(t)$ satisfies $(\Phi 3; 0; p)$ and $(\Phi 3; \infty; q)$ for $p > 1$ and $q > 1$. Assume that X satisfies $(M\lambda)$, and $(\omega 1')$ and $(\Phi\omega\alpha)$ hold. Then there exists a constant $C > 0$ such that*

$$\|I_{\alpha,\tau} f\|_{\mathcal{L}^{\Psi,\omega,\theta_2}(X)} \leqslant C \|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{\Phi,\omega,\theta_1}(X)$.

Proof. Let f be a nonnegative measurable function on X such that

$$\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)} \leqslant 1.$$

Let $x \in X$ and $0 < \delta < \frac{1}{2} d_X$. By Lemma 4.3, we find

$$\begin{aligned} I_{\alpha,\tau} f(x) &= \int_{B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))} \, \mathrm{d}\mu(y) + \int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))} \, \mathrm{d}\mu(y) \\ &\leqslant C\{\delta^\alpha M_\lambda f(x) + \delta^\alpha \Phi^{-1}(\omega(\delta)^{-1})\}. \end{aligned}$$

If $\omega^{-1}(\Phi(M_\lambda f(x))^{-1}) \geqslant \frac{1}{2} d_X$, then, taking $\delta = \frac{1}{2} d_X$, we have $I_{\alpha,\tau} f(x) \leqslant C$ by Lemma 4.1, $(\omega 1)$ and $(\omega 3)$. If $\omega^{-1}(\Phi(M_\lambda f(x))^{-1}) < \frac{1}{2} d_X$, then take $\delta = \omega^{-1}(\Phi(M_\lambda f(x))^{-1})$. Thus, we have

$$I_{\alpha,\tau} f(x) \leqslant C M_\lambda f(x) (\omega^{-1}(\Phi(M_\lambda f(x))^{-1}))^\alpha$$

by Lemma 4.1. Therefore, we obtain

$$I_{\alpha,\tau} f(x) \leqslant C_1 \max\{M_\lambda f(x)(\omega^{-1}(\Phi(M_\lambda f(x))^{-1}))^\alpha, 1\},$$

so that by $(\Psi\Phi)$, we have

$$\Psi\Big(\frac{I_{\alpha,\tau} f(x)}{C_1}\Big) \leqslant C\{\Psi(M_\lambda f(x)(\omega^{-1}(\Phi(M_\lambda f(x))^{-1}))^\alpha) + 1\} \leqslant C\{\Phi(M_\lambda f(x)) + 1\}.$$

Therefore, we obtain by Theorem 3.5

$$\begin{aligned} &\int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_2 r))} \Big(\int_{B(z,r)} \Psi\Big(\frac{I_{\alpha,\tau} f(x)}{C_1}\Big) \, \mathrm{d}\mu(x) \Big) \frac{\mathrm{d}r}{r} \\ &\quad \leqslant C \Big\{ \int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_2 r))} \Big(\int_{B(z,r)} \Phi(M_\lambda f(x)) \, \mathrm{d}\mu(x) \Big) \frac{\mathrm{d}r}{r} + \int_0^{2d_X} \omega(r) \frac{\mathrm{d}r}{r} \Big\} \\ &\quad \leqslant C \end{aligned}$$

for all $z \in X$ since there exists a constant $C_2 > 0$ such that

$$\int_0^{2d_X} \omega(r) \frac{\mathrm{d}r}{r} = \int_0^{2d_X} r^{-\varepsilon_1} \omega(r) r^{\varepsilon_1} \frac{\mathrm{d}r}{r} \leqslant C \int_0^{2d_X} r^{\varepsilon_1} \frac{\mathrm{d}r}{r} \leqslant C_2$$

by $(\omega 1')$ and $(\omega 3)$. This completes the proof of the theorem. □

Remark 4.5. Let $\Phi_{p_1,\{q_j\}}(t) = t^{p_1} \prod_{j=1}^{k} (L_e^{(j)}(t))^{q_j} = t^{p_1} Q(t)$ be as in Example 2.4 and $\omega(r) = r^{\sigma} L_e(1/r)^{\beta}$ be as in Example 2.1. Note that the condition $(\Phi\omega\alpha)$ holds if $\sigma/p_1 - \alpha > 0$. Set

$$\Psi(t) = [\Phi_{p_1,\{q_j\}}(t)]^{p_1^*/p_1} L_e(t)^{p_1^*\alpha\beta/\sigma},$$

where $1/p_1^* = 1/p_1 - \alpha/\sigma$. Since $\omega^{-1}(r) \sim r^{1/\sigma} L_e(1/r)^{-\beta/\sigma}$, we see that

$$t(\omega^{-1}(\Phi_{p_1,\{q_j\}}(t)^{-1}))^{\alpha} \sim t^{p_1/p_1^*} Q(t)^{-\alpha/\sigma} L_e(t)^{-\alpha\beta/\sigma}.$$

Therefore,

$$\Psi(t(\omega^{-1}(\Phi_{p_1,\{q_j\}}(t)^{-1}))^{\alpha}) \leqslant C t^{p_1} Q(t) = C \Phi_{p_1,\{q_j\}}(t).$$

Thus, $\Psi(t)$ satisfies the condition $(\Psi\Phi)$.

The next example shows that $\mathcal{L}^{\Phi_{p_1,\{q_j\}},\omega,\theta}(X) \neq \{0\}$.

Example 4.6. Let $X_1 = \{(x_1, 0) \in \mathbb{R}^2 \colon 0 \leqslant x_1 < 1\}$ and $X_2 = \{(x_1, x_2) \in \mathbb{R}^2 \colon |(x_1, x_2)| < 1,\ x_1 < 0\}$, and define $(X, d, \mu) = (X_1, d_2, m_1) \cup (X_2, d_2, m_2)$, where d_2 denotes the 2-dimensional Euclidean distance and m_i denotes the i-dimensional Lebesgue measure. It is easy to show that μ is non-doubling. Since X is a separable metric space, X satisfies $(M\lambda)$ for $\lambda \geqslant 2$, see [27].

Let $\theta \geqslant 1$. Consider the function

$$f(y) = d_2(0, y)^{-a} \chi_{X_2}(y)$$

for $y = (y_1, y_2) \in X_1 \cup X_2$ and $a < \min\{2/p_1, \sigma/p_1\}$. Then note that

$$\begin{aligned}
&\int_0^4 \frac{\omega(r)}{\mu(B(x,\theta r))} \biggl(\int_{B(x,r)} \Phi_{p_1,\{q_j\}}(|f(y)|)\,\mathrm{d}\mu(y)\biggr) \frac{\mathrm{d}r}{r} \\
&\quad\leqslant C \int_0^4 \frac{\omega(r)}{\mu(B(x,\theta r))} \biggl(\int_{B(0,r)\cap X_2} \Phi_{p_1,\{q_j\}}(|f(y)|)\,\mathrm{d}\mu(y)\biggr) \frac{\mathrm{d}r}{r} \\
&\quad\leqslant C \int_0^4 \frac{\omega(r)}{\mu(B(x,\theta r))} r^{2-ap_1} Q\Bigl(\frac{1}{r}\Bigr) \frac{\mathrm{d}r}{r} \leqslant C \int_0^4 r^{\sigma - ap_1} Q\Bigl(\frac{1}{r}\Bigr) L_e\Bigl(\frac{1}{r}\Bigr)^{\beta} \frac{\mathrm{d}r}{r} < \infty
\end{aligned}$$

for all $x = (x_1, x_2) \in X_1 \cup X_2$ since $\mu(B(x, \theta r)) \geqslant Cr^2$ for all $x \in X_1 \cup X_2$ and $0 < r < 4$. Therefore, $f \in \mathcal{L}^{\Phi_{p_1,\{q_j\}},\omega,\theta}(X)$, so that $\mathcal{L}^{\Phi_{p_1,\{q_j\}},\omega,\theta}(X) \neq \{0\}$.

Applying Theorem 4.4 to special $\Phi_{p_1,\{q_j\}}$ and ω given above, we obtain the following corollary.

Corollary 4.7. *Let X be a non-doubling metric measure space. Let $\Phi_{p_1,\{q_j\}}(t)$ and $\omega(r)$ be as in Examples 2.4 and 2.1. Let $1 \leqslant \theta_1 < \theta_2$ and $\theta_1(\theta_2+1)/(\theta_2-\theta_1) < \lambda \leqslant \tau$. Assume that X satisfies $(M\lambda)$. Then there exists a constant $C > 0$ such that*

$$\|I_{\alpha,\tau} f\|_{\mathcal{L}^{\Psi,\omega,\theta_2}(X)} \leqslant C\|f\|_{\mathcal{L}^{\Phi_{p_1,\{q_j\}},\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{\Phi_{p_1,\{q_j\}},\omega,\theta_1}(X)$, where

$$\Psi(t) = [\Phi_{p_1,\{q_j\}}(t)]^{p_1^*/p_1} L_e(t)^{p_1^*\alpha\beta/\sigma}$$

with $1/p_1^* = 1/p_1 - \alpha/\sigma > 0$.

Noting that for $c > 0$ there exists a constant $C > 0$ such that

$$C^{-1}\mu(B(x,r)) \leqslant \mu(B(x,cr)) \leqslant C\mu(B(x,r))$$

for all $x \in X$ and $0 < r < d_X$ by the doubling condition of μ, we can prove the following theorem for the doubling metric measure case as in the proof of Theorem 4.4.

Theorem 4.8. *Let X be a doubling metric measure space. Suppose $\Phi(t)$ satisfies $(\Phi 3; 0; p)$ and $(\Phi 3; \infty; q)$ for $p > 1$ and $q > 1$. Assume that $(\omega 1')$ and $(\Phi\omega\alpha)$ hold. Then there exists a constant $C > 0$ such that*

$$\|I_{\alpha,1} f\|_{\mathcal{L}^{\Psi,\omega,1}(X)} \leqslant C\|f\|_{\mathcal{L}^{\Phi,\omega,1}(X)}$$

for all $f \in \mathcal{L}^{\Phi,\omega,1}(X)$.

Corollary 4.9. *Let X be a doubling metric measure space. Let $\Phi_{p_1,\{q_j\}}(t)$ and $\omega(r)$ be as in Examples 2.4 and 2.1. Then there exists a constant $C > 0$ such that*

$$\|I_{\alpha,1} f\|_{\mathcal{L}^{\Psi,\omega,1}(X)} \leqslant C\|f\|_{\mathcal{L}^{\Phi_{p_1,\{q_j\}},\omega,1}(X)}$$

for all $f \in \mathcal{L}^{\Phi_{p_1,\{q_j\}},\omega,1}(X)$, where

$$\Psi(t) = [\Phi_{p_1,\{q_j\}}(t)]^{p_1^*/p_1} L_e(t)^{p_1^*\alpha\beta/\sigma}$$

with $1/p_1^* = 1/p_1 - \alpha/\sigma > 0$.

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