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COPRIMALITY OF INTEGERS IN PIATETSKI-SHAPIRO SEQUENCES

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Abstract. We use the estimation of the number of integers n such that $\lfloor n^c \rfloor$ belongs to an arithmetic progression to study the coprimality of integers in $\mathbb{N}^c = \{\lfloor n^c \rfloor\}_{n\in\mathbb{N}}$, $c > 1$, $c \notin \mathbb{N}$.

Keywords: greatest common divisor; natural density; Piatetski-Shapiro sequence

MSC 2020: 11A05, 11K06

1. Introduction and results

Piatetski-Shapiro sequences are defined by

$$\mathbb{N}^c = \{\lfloor n^c \rfloor\}_{n\in\mathbb{N}} \quad (c > 1,\, c \notin \mathbb{N}),$$

where $\lfloor z \rfloor$ is the integer part of a real number z. They are named in honor of Piatetski-Shapiro (see [12]), who studied prime numbers in a sequence of the form $\lfloor f(n) \rfloor$, where $f(n)$ is a polynomial. The density for coprime pairs of integers is a classical result in number theory. In 1849, Dirichlet in [8] asserted that the proportion of coprime pairs of integers in $\{1, \ldots, n\}$,

$$\frac{1}{n^2}\#\{(n_1, n_2) \in \{1, \ldots, n\}^2 \colon \gcd(n_1, n_2) = 1\},$$

tends to $1/\zeta(2) = 6/\pi^2 \sim 0.608$. The proof is not trivial and can be found in the book of Hardy and Wright, see [10], Theorem 332. For further details, we refer to [2], [3], [4], [7], [9]. It is natural to study the coprimality of any pairs in

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other sequences. The coprimality of integers in Piatetski-Shapiro sequences first appeared in [11]. Lembek and Moser found that the number of positive integers not exceeding x, with $\gcd(n, \lfloor n^{1/2} \rfloor) = 1$, is

$$\frac{6}{\pi^2}x + O(x^{1/2}\log x).$$

In 2002, Delmer and Deshouillers in [5] proved that, for any positive real number c which is not an integer, the density of the integers n which are coprime to $\lfloor n^c \rfloor$ is $6/\pi^2$. Recently, Bergelson and Richter in [1] have extended this problem to functions in the Hardy field H. They have proved, under some natural conditions on the k-tuple $f_1, \ldots, f_k \in H$, that the density of the set

$$\{n \in \mathbb{N}\colon \gcd(n, \lfloor f_1(n) \rfloor, \ldots, \lfloor f_k(n) \rfloor) = 1\}$$

exists and equals $1/\zeta(k+1)$. Moreover, for $c_i > 0$, $c_i \notin \mathbb{N}$, $i = 1, \ldots, k$, they posed an interesting open question whether, with some conditions, is it true that natural density of the set

$$\{n \in \mathbb{N}\colon \gcd(\lfloor n^{c_1} \rfloor, \ldots, \lfloor n^{c_k} \rfloor) = 1\}$$

exists and equals $1/\zeta(k)$.

In this paper, we shall study the asymptotic formula for the number of Piatetski-Shapiro pairs $(\lfloor a^c \rfloor, \lfloor b^c \rfloor)$ that are coprime and $a, b \leqslant x$. We obtain the following results.

Theorem 1.1. *As $x \to \infty$, we have*

$$\sum_{\substack{a,b \leqslant x \\ \gcd(\lfloor a^c \rfloor, \lfloor b^c \rfloor) = 1}} 1 = \frac{1}{\zeta(2)}x^2 + \begin{cases} O(x^{(c+4)/3}) & \text{for } 1 < c \leqslant \frac{5}{4}, \\ O(x^{c+1/2}) & \text{for } \frac{5}{4} \leqslant c < \frac{3}{2}. \end{cases}$$

Theorem 1.2. *Let $k \geqslant 3$, and $1 < c < 2$. As $x \to \infty$, we have*

$$\sum_{\substack{a_1, \ldots, a_k \leqslant x \\ \gcd(\lfloor a_1^c \rfloor, \lfloor a_2^c \rfloor, \ldots, \lfloor a_k^c \rfloor) = 1}} 1 = \frac{1}{\zeta(k)}x^k + O(x^{k-(2-c)/3}).$$

Moreover, the following theorems give the coprimality on the different Piatetski-Shapiro sequences.

Theorem 1.3. *As $x \to \infty$, for $1 < c_1 < c_2 < \frac{3}{2}$, we have*

$$\sum_{\substack{a,b \leqslant x \\ \gcd(\lfloor a^{c_1} \rfloor, \lfloor b^{c_2} \rfloor) = 1}} 1 = \frac{1}{\zeta(2)}x^2 + \begin{cases} O(x^{(c_2+4)/3}) & \text{for } 1 < c_1 \leqslant \frac{5}{4}, \\ O(x^{1/2+(2c_1+c_2)/3}) & \text{for } \frac{5}{4} \leqslant c_1 < \frac{3}{2}. \end{cases}$$

Theorem 1.4. *Let $k \geqslant 3$. As $x \to \infty$, for $1 < c_1 \leqslant c_2 \leqslant \ldots \leqslant c_k < 2$, we have*

$$\sum_{\substack{a_1,\ldots,a_k \leqslant x \\ \gcd(\lfloor a_1^{c_1}\rfloor, \lfloor a_2^{c_2}\rfloor, \ldots, \lfloor a_k^{c_k}\rfloor)=1}} 1 = \frac{1}{\zeta(k)} x^k + O(x^{k-(2-c_k)/3}).$$

Here, the O-terms depend on k and c_k.

2. Lemmas

The main ingredient in the following proof is a good estimation for the number of integers n up to x such that $\lfloor n^c \rfloor$ belongs to an arithmetic progression. Deshouillers in [6] proved the following lemma.

Lemma 2.1. *For $1 < c < 2$. Let $x \in \mathbb{R}$ and $a, q \in \mathbb{Z}$ such that $0 \leqslant a < q \leqslant x^c$,*

$$\sum_{\substack{n \leqslant x \\ \lfloor n^c \rfloor \equiv a \pmod q}} 1 = \frac{x}{q} + O\Big(\min\Big(\frac{x^c}{q}, \frac{x^{(c+1)/3}}{q^{1/3}}\Big)\Big).$$

The following facts will be used in the proofs.

Lemma 2.2. *Let $x \in \mathbb{R}$ and $d \in \mathbb{N}$. Let $k, l \in \mathbb{N}$ such that $k \geqslant 4$ and $1 \leqslant l < k$. For $1 < c_1 \leqslant c_2 \leqslant \ldots \leqslant c_k < 2$, we have*

$$\prod_{i=1}^{k}\Big(\frac{x}{d} + O\Big(\frac{x^{(c_i+1)/3}}{d^{1/3}}\Big)\Big) = \frac{x^k}{d^k} + \sum_{i=1}^{k} O\bigg(\frac{x^{(1/3)\sum_{j=0}^{i-1} c_{k-j} + k - 2i/3}}{d^{k-2i/3}}\bigg), \tag{2.1}$$

$$\prod_{i=l+1}^{k}\Big(\frac{x}{d} + O\Big(\frac{x^{(c_i+1)/3}}{d^{1/3}}\Big)\Big) = \frac{x^{k-l}}{d^{k-l}} + \sum_{i=1}^{k-l} O\bigg(\frac{x^{(1/3)\sum_{j=0}^{i-1} c_{k-j} + k - l - 2i/3}}{d^{k-l-2i/3}}\bigg), \tag{2.2}$$

$$\prod_{i=1}^{k}\Big(\frac{x}{d} + O\Big(\frac{x^{c_i}}{d}\Big)\Big) = \frac{x^k}{d^k} + O\bigg(\frac{x^{\sum_{j=1}^{k} c_j}}{d^k}\bigg), \tag{2.3}$$

and

$$\prod_{i=1}^{l}\Big(\frac{x}{d} + O\Big(\frac{x^{c_i}}{d}\Big)\Big) \prod_{i=l+1}^{k}\Big(\frac{x}{d} + O\Big(\frac{x^{(c_i+1)/3}}{d^{1/3}}\Big)\Big) \tag{2.4}$$

$$= \frac{x^k}{d^k} + O\bigg(\frac{x^{\sum_{j=1}^{l} c_j + k - l}}{d^k}\bigg) + \sum_{i=1}^{k-l} O\bigg(\frac{x^{(1/3)\sum_{j=0}^{i-1} c_{k-j} + k - l - 2i/3 + \sum_{j=1}^{l} c_j}}{d^{k-2i/3}}\bigg).$$

Proof. For $k \geqslant 4$, we write

$$\prod_{i=1}^{k}\Big(\frac{x}{d}+O\Big(\frac{x^{(c_i+1)/3}}{d^{1/3}}\Big)\Big)$$
$$=\frac{x^k}{d^k}+O\Big(\frac{x^{k-1}}{d^{k-1}}\sum_{i_1=1}^{k}\frac{x^{(c_{i_1}+1)/3}}{d^{1/3}}\Big)+O\Big(\frac{x^{k-2}}{d^{k-2}}\sum_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{k}\frac{x^{(c_{i_1}+c_{i_2}+2)/3}}{d^{2/3}}\Big)$$
$$+O\Big(\frac{x^{k-3}}{d^{k-3}}\sum_{\substack{i_1,i_2,i_3=1\\ i_1\neq i_2\neq i_3}}^{k}\frac{x^{(c_{i_1}+c_{i_2}+c_{i_3}+3)/3}}{d}\Big)+\ldots+O\Big(\frac{x^{(c_1+c_2+\ldots+c_k+k)/3}}{d^{k/3}}\Big).$$

From $1 < c_1 \leqslant c_2 \leqslant \ldots \leqslant c_k < 2$, we have

$$\prod_{i=1}^{k}\Big(\frac{x}{d}+O\Big(\frac{x^{(c_i+1)/3}}{d^{1/3}}\Big)\Big)$$
$$=\frac{x^k}{d^k}+O\Big(\frac{x^{k-1}}{d^{k-1}}\frac{x^{(c_k+1)/3}}{d^{1/3}}\Big)+O\Big(\frac{x^{k-2}}{d^{k-2}}\frac{x^{(c_k+c_{k-1}+2)/3}}{d^{2/3}}\Big)$$
$$+O\Big(\frac{x^{k-3}}{d^{k-3}}\frac{x^{(c_k+c_{k-1}+c_{k-2}+3)/3}}{d}\Big)+\ldots+O\Big(\frac{x^{(c_1+c_2+\ldots+c_k+k)/3}}{d^{k/3}}\Big),$$

and (2.1) follows. The proofs of (2.2)–(2.3) are similar to the proof of (2.1). The equation (2.4) is the product of (2.2) and (2.3). □

Lemma 2.3. *For $k \geqslant 2$ and $1 < c < 2$, we have*

$$\sum_{d\leqslant x^c}\mu(d)\frac{x^k}{d^k}=\frac{1}{\zeta(k)}x^k+O(x^{k+c-ck}),$$

where $\mu(n)$ denotes the Möbius function.

Proof. It follows from the well known identity $\sum_{n=1}^{\infty}\mu(n)/n^k = 1/\zeta(k)$ and the partial summation. □

3. Proof of theorems

Proof of Theorem 1.1. Let $1 < c < \frac{3}{2}$. In view of the identity

$$\sum_{d\mid n}\mu(d)=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{if } n>1,\end{cases}$$

we have

$$\sum_{\substack{a,b\leqslant x\\ \gcd(\lfloor a^c\rfloor,\lfloor b^c\rfloor)=1}} 1=\sum_{d\leqslant x^c}\mu(d)\sum_{\substack{a\leqslant x\\ d\mid\lfloor a^c\rfloor}}1\sum_{\substack{b\leqslant x\\ d\mid\lfloor b^c\rfloor}}1=\sum_{d\leqslant x^c}\mu(d)\Big(\sum_{\substack{n\leqslant x\\ \lfloor n^c\rfloor\equiv 0 \pmod d}}1\Big)^2.$$

In view of Lemma 2.1, we have

$$\begin{aligned}\sum_{\substack{a,b\leqslant x\\ \gcd(\lfloor a^c\rfloor,\lfloor b^c\rfloor)=1}} 1 &= \sum_{d\leqslant x^c}\mu(d)\Big(\frac{x}{d}+O\Big(\min\Big(\frac{x^c}{d},\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)\Big)^2\\
&=\sum_{d\leqslant x^{c-1/2}}\mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)^2\\
&\quad+\sum_{x^{c-1/2}\leqslant d\leqslant x^c}\mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^c}{d}\Big)\Big)^2\\
&=\sum_{d\leqslant x^c}\mu(d)\frac{x^2}{d^2}+O\Big(\sum_{d\leqslant x^{c-1/2}}\Big(\frac{x^{(c+4)/3}}{d^{4/3}}+\frac{x^{(2c+2)/3}}{d^{2/3}}\Big)\Big)\\
&\quad+O\Big(\sum_{x^{c-1/2}\leqslant d\leqslant x^c}\frac{x^{2c}}{d^2}\Big)\\
&=\sum_{d\leqslant x^c}\mu(d)\frac{x^2}{d^2}+O\Big(x^{(c+4)/3}+x^{c+1/2}\Big).\end{aligned}$$

In view of Lemma 2.3, we have

$$\sum_{\substack{a,b\leqslant x\\ \gcd(\lfloor a^c\rfloor,\lfloor b^c\rfloor)=1}} 1=\frac{1}{\zeta(2)}x^2+O(x^{(c+4)/3}+x^{c+1/2}).$$

Since $\frac{1}{3}(c+4)\geqslant c+\frac{1}{2}$, when $c\leqslant\frac{5}{4}$, Theorem 1.1 follows. $\square$

P r o o f of Theorem 1.2. For $1<c<2$ and $k\geqslant 3$, we have

$$\begin{aligned}\sum_{\substack{a_1,a_2,a_3,\dots,a_k\leqslant x\\ \gcd(\lfloor a_1^c\rfloor,\lfloor a_2^c\rfloor,\lfloor a_3^c\rfloor,\dots,\lfloor a_k^c\rfloor)=1}} 1 &= \sum_{d\leqslant x^c}\mu(d)\sum_{\substack{a_1\leqslant x\\ d\mid\lfloor a_1^c\rfloor}}1\sum_{\substack{a_2\leqslant x\\ d\mid\lfloor a_2^c\rfloor}}1\sum_{\substack{a_3\leqslant x\\ d\mid\lfloor a_3^c\rfloor}}1\dots\sum_{\substack{a_k\leqslant x\\ d\mid\lfloor a_k^c\rfloor}}1\\
&=\sum_{d\leqslant x^c}\mu(d)\Big(\sum_{\substack{n\leqslant x\\ \lfloor n^c\rfloor\equiv 0 \pmod d}}1\Big)^k.\end{aligned}$$

In view of Lemma 2.1, we have

(3.1)

$$\sum_{\substack{a_1,a_2,a_3,\ldots,a_k\leqslant x\\ \gcd(\lfloor a_1^c\rfloor,\lfloor a_2^c\rfloor,\lfloor a_3^c\rfloor,\ldots,\lfloor a_k^c\rfloor)=1}} 1 = \sum_{d\leqslant x^c}\mu(d)\Big(\frac{x}{d}+O\Big(\min\Big(\frac{x^c}{d},\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)\Big)^k$$

$$= \sum_{d\leqslant x^c}\mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)^k$$

$$= \sum_{d\leqslant x^c}\mu(d)\Big(\frac{x^k}{d^k}+\sum_{i=1}^{k}\Big(O\Big(\frac{x^{k+i(c-2)/3}}{d^{k-2i/3}}\Big)\Big)\Big)$$

$$= x^k\sum_{d\leqslant x^c}\frac{\mu(d)}{d^k}+\sum_{i=1}^{k}\Big(O\Big(x^{k+i(c-2)/3}\sum_{d\leqslant x^c}\frac{1}{d^{k-2i/3}}\Big)\Big).$$

For $k>3$, we have $\sum_{d\leqslant x^c}(d^{k-2i/3})^{-1}=O(1)$, $1\leqslant i\leqslant k$. Thus, the second sum in the right hand side of (3.1) is bounded by $O(x^{k+(c-2)/3})$. Therefore, we have

$$\sum_{\substack{a_1,a_2,a_3,\ldots,a_k\leqslant x\\ \gcd(\lfloor a_1^c\rfloor,\lfloor a_2^c\rfloor,\lfloor a_3^c\rfloor,\ldots,\lfloor a_k^c\rfloor)=1}} 1 = x^k\sum_{d\leqslant x^c}\frac{\mu(d)}{d^k}+O(x^{k+(c-2)/3}). \tag{3.2}$$

In view of (3.2) and Lemma 2.3, we have

$$\sum_{\substack{a_1,a_2,a_3,\ldots,a_k\leqslant x\\ \gcd(\lfloor a_1^c\rfloor,\lfloor a_2^c\rfloor,\lfloor a_3^c\rfloor,\ldots,\lfloor a_k^c\rfloor)=1}} 1 = \frac{1}{\zeta(k)}x^k+O(x^{k+c-ck})+O(x^{k+(c-2)/3}).$$

The assertion of the case $k>3$ follows by comparing the value of c and $k>3$ in O-terms. For $k=3$, we have

$$\sum_{\substack{a_1,a_2,a_3\leqslant x\\ \gcd(\lfloor a_1^c\rfloor,\lfloor a_2^c\rfloor,\lfloor a_3^c\rfloor)=1}} 1 = x^3\sum_{d\leqslant x^c}\frac{\mu(d)}{d^3}+\sum_{i=1}^{3}\Big(O\Big(x^{3+i(c-2)/3}\sum_{d\leqslant x^{c-1/2}}\frac{1}{d^{3-2i/3}}\Big)\Big) \tag{3.3}$$

$$+\sum_{i=1}^{3}\Big(O\Big(x^{3-i+ci}\sum_{x^{c-1/2}\leqslant d\leqslant x^c}\frac{1}{d^3}\Big)\Big)$$

$$= x^3\sum_{d\leqslant x^c}\frac{\mu(d)}{d^3}+O(x^{(c+7)/3}+x^{(2c+5)/3}+x^{c+1}\log x)$$

$$+O(x^{c+1}).$$

In view of Lemma 2.3 and the comparison of the in O-terms in (3.3), the proof is completed. □

Proof of Theorem 1.3. For $1 < c_1 < c_2 < \frac{3}{2}$, we have

$$\sum_{\substack{s,b\leqslant x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor b^{c_2}\rfloor)=1}} 1 = \sum_{d\leqslant x^{c_1}} \mu(d) \sum_{\substack{a\leqslant x\\ d\mid\lfloor a^{c_1}\rfloor}} 1 \sum_{\substack{b\leqslant x\\ d\mid\lfloor b^{c_2}\rfloor}} 1$$
$$= \sum_{d\leqslant x^{c_1}} \mu(d) \sum_{\substack{a\leqslant x\\ \lfloor a^{c_1}\rfloor\equiv 0 \pmod d}} 1 \sum_{\substack{b\leqslant x\\ \lfloor b^{c_2}\rfloor\equiv 0 \pmod d}} 1.$$

In view of Lemma 2.1, we have

$$\sum_{\substack{a,b\leqslant x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor b^{c_2}\rfloor)=1}} 1 = \sum_{d\leqslant x^{c_1}} \mu(d)\Big(\frac{x}{d}+O\Big(\min\Big(\frac{x^{c_1}}{d},\frac{x^{(c_1+1)/3}}{d^{1/3}}\Big)\Big)\Big) \times\Big(\frac{x}{d}+O\Big(\min\Big(\frac{x^{c_2}}{d},\frac{x^{(c_2+1)/3}}{d^{1/3}}\Big)\Big)\Big). \tag{3.4}$$

Since $1 < c_1 < c_2 < \frac{3}{2}$, the case $c_2 - c_1 > \frac{1}{2}$ does not hold. Thus, we split the sum in (3.4) into three parts. Then,

(3.5)
$$\begin{aligned}\sum_{\substack{a,b\leqslant x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor b^{c_2}\rfloor)=1}} 1 &= \sum_{d\leqslant x^{c_1-1/2}} \mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_1+1)/3}}{d^{1/3}}\Big)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_2+1)/3}}{d^{1/3}}\Big)\Big)\\
&+\sum_{x^{c_1-1/2}<d\leqslant x^{c_2-1/2}} \mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^{c_1}}{d}\Big)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_2+1)/3}}{d^{1/3}}\Big)\Big)\\
&+\sum_{x^{c_2-1/2}<d\leqslant x^{c_1}} \mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^{c_1}}{d}\Big)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{c_2}}{d}\Big)\Big)\\
&= \sum_{d\leqslant x^{c_1}} \mu(d)\frac{x^2}{d^2} + O\Big(\sum_{d\leqslant x^{c_1-1/2}}\Big(\frac{x^{(c_2+4)/3}}{d^{4/3}}+\frac{x^{(c_1+c_2+2)/3}}{d^{2/3}}\Big)\Big)\\
&+O\Big(\sum_{x^{c_1-1/2}<d\leqslant x^{c_2-1/2}}\Big(\frac{x^{1+c_1}}{d^2}+\frac{x^{(3c_1+c_2+1)/3}}{d^{4/3}}\Big)\Big)\\
&+O\Big(\sum_{x^{c_2-1/2}\leqslant d\leqslant x^{c_1}}\frac{x^{c_1+c_2}}{d^2}\Big)\\
&= \sum_{d\leqslant x^{c_1}} \mu(d)\frac{x^2}{d^2} + O(x^{(c_2+4)/3}+x^{(2c_1+c_2+3)/6}).\end{aligned}$$

In view of (3.5) and Lemma 2.3, we have

$$\sum_{\substack{a,b\leqslant x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor b^{c_2}\rfloor)=1}} 1 = \frac{1}{\zeta(2)}x^2 + O(x^{(c_2+4)/3}+x^{(2c_1+c_2+3)/6}).$$

By comparing the value of c_1 and c_2 in the O-terms, Theorem 1.3 follows. □

P r o o f of Theorem 1.4. First, we prove Theorem 1.4 in the case $k = 3$. For $1 < c_1 \leqslant c_2 \leqslant c_3 < 2$, we have,

$$\sum_{\substack{a_1,a_2,a_3\leqslant x\\ \gcd(\lfloor a_1^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor)=1}} 1 = \sum_{d\leqslant x^{c_1}} \mu(d) \sum_{\substack{a_1\leqslant x\\ d\mid\lfloor a_1^{c_1}\rfloor}} 1 \sum_{\substack{a_2\leqslant x\\ d\mid\lfloor a_2^{c_2}\rfloor}} 1 \sum_{\substack{a_3\leqslant x\\ d\mid\lfloor a_3^{c_3}\rfloor}} 1$$
$$= \sum_{d\leqslant x^{c_1}} \mu(d) \sum_{\substack{a_1\leqslant x\\ \lfloor a_1^{c_1}\rfloor\equiv 0 \pmod d}} 1 \sum_{\substack{a_2\leqslant x\\ \lfloor a_2^{c_2}\rfloor\equiv 0 \pmod d}} 1 \sum_{\substack{a_3\leqslant x\\ \lfloor a_3^{c_3}\rfloor\equiv 0 \pmod d}} 1.$$

In view of Lemma 2.1, we have

$$\sum_{\substack{a_1,a_2,a_3\leqslant x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor)=1}} 1 = \sum_{d\leqslant x^{c_1}} \mu(d)\Big(\frac{x}{d} + O\Big(\min\Big(\frac{x^{c_1}}{d}, \frac{x^{(c_1+1)/3}}{d^{1/3}}\Big)\Big)\Big)$$
$$\times \Big(\frac{x}{d} + O\Big(\min\Big(\frac{x^{c_2}}{d}, \frac{x^{(c_2+1)/3}}{d^{1/3}}\Big)\Big)\Big)$$
$$\times \Big(\frac{x}{d} + O\Big(\min\Big(\frac{x^{c_3}}{d}, \frac{x^{(c_3+1)/3}}{d^{1/3}}\Big)\Big)\Big). \tag{3.6}$$

Next, we separate the consideration into 3 cases.

Case 1: $c_1 - \frac{1}{2} \leqslant c_2 - \frac{1}{2} \leqslant c_3 - \frac{1}{2} < c_1$. We split the sum in (3.6) into four parts. Then we have

$$\sum_{\substack{a_1,a_2,a_3\leqslant x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor)=1}} 1$$
$$= \sum_{d\leqslant x^{c_1-1/2}} \mu(d)\Big(\frac{x}{d} + O\Big(\frac{x^{(c_1+1)/3}}{d^{1/3}}\Big)\Big)\Big(\frac{x}{d} + O\Big(\frac{x^{(c_2+1)/3}}{d^{1/3}}\Big)\Big)\Big(\frac{x}{d} + O\Big(\frac{x^{(c_3+1)/3}}{d^{1/3}}\Big)\Big)$$
$$+ \sum_{x^{c_1-1/2}<d\leqslant x^{c_2-1/2}} \mu(d)\Big(\frac{x}{d} + O\Big(\frac{x^{c_1}}{d}\Big)\Big)\Big(\frac{x}{d} + O\Big(\frac{x^{(c_2+1)/3}}{d^{1/3}}\Big)\Big)\Big(\frac{x}{d} + O\Big(\frac{x^{(c_3+1)/3}}{d^{1/3}}\Big)\Big)$$
$$+ \sum_{x^{c_2-1/2}<d\leqslant x^{c_3-1/2}} \mu(d)\Big(\frac{x}{d} + O\Big(\frac{x^{c_1}}{d}\Big)\Big)\Big(\frac{x}{d} + O\Big(\frac{x^{c_2}}{d}\Big)\Big)\Big(\frac{x}{d} + O\Big(\frac{x^{(c_3+1)/3}}{d^{1/3}}\Big)\Big)$$
$$+ \sum_{x^{c_3-1/2}<d\leqslant x^{c_1}} \mu(d)\Big(\frac{x}{d} + O\Big(\frac{x^{c_1}}{d}\Big)\Big)\Big(\frac{x}{d} + O\Big(\frac{x^{c_2}}{d}\Big)\Big)\Big(\frac{x}{d} + O\Big(\frac{x^{c_3}}{d}\Big)\Big).$$

By the same reason as in the proof of (3.5), we have

$$\sum_{\substack{a_1,a_2,a_3\leqslant x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor)=1}} 1$$
$$= \sum_{d\leqslant x^{c_1-1/2}} \mu(d)\Big(\frac{x^3}{d^3} + O\Big(\frac{x^{(c_3+7)/3}}{d^{7/3}}\Big) + O\Big(\frac{x^{(c_2+c_3+5)/3}}{d^{5/3}}\Big) + O\Big(\frac{x^{(c_1+c_2+c_3+3)/3}}{d}\Big)\Big)$$

$$+\sum_{x^{c_1-1/2}<d\leqslant x^{c_2-1/2}}\mu(d)\Big(\frac{x^3}{d^3}+O\Big(\frac{x^{2+c_1}}{d^3}\Big)+O\Big(\frac{x^{(3c_1+c_3+4)/3}}{d^{7/3}}\Big)$$

$$+O\Big(\frac{x^{(3c_1+c_2+c_3+2)/3}}{d^{5/3}}\Big)\Big)$$

$$+\sum_{x^{c_2-1/2}<d\leqslant x^{c_3-1/2}}\mu(d)\Big(\frac{x^3}{d^3}+O\Big(\frac{x^{c_1+c_2+1}}{d^3}\Big)+O\Big(\frac{x^{(3c_1+3c_2+c_3+1)/3}}{d^{7/3}}\Big)\Big)$$

$$+\sum_{x^{c_3-1/2}<d\leqslant x^{c_1}}\mu(d)\Big(\frac{x^3}{d^3}+O\Big(\frac{x^{c_1+c_2+c_3}}{d^3}\Big)\Big)$$

$$=x^3\sum_{d\leqslant x^{c_1}}\frac{\mu(d)}{d^3}+O\Big(x^{(c_3+7)/3}\Big)+O\Big(x^{(3c_1-c_2+c_3+3)/3}\Big)+O\Big(x^{c_1+c_2-c_3+1}\Big).$$

Since $2c_1-(c_2-c_1)<4$, we have $\frac{1}{3}(c_3+7)>\frac{1}{3}(3c_1-c_2+c_3+3)$ and $\frac{1}{3}(3c_1-c_2+c_3+3)\geqslant c_1+c_2-c_3+1$. Thus,

$$\sum_{\substack{a_1,a_2,a_3\leqslant x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor)=1}}1=x^3\sum_{d\leqslant x^{c_1}}\frac{\mu(d)}{d^3}+O\Big(x^{(c_3+7)/3}\Big).$$

In view of Lemma 2.3, for $c_1-\frac{1}{2}\leqslant c_2-\frac{1}{2}\leqslant c_3-\frac{1}{2}<c_1$,

$$\sum_{\substack{a_1,a_2,a_3\leqslant x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor)=1}}1=\frac{1}{\zeta(3)}x^3+O(x^{(c_3+7)/3}).\tag{3.7}$$

Case 2: $c_1-\frac{1}{2}\leqslant c_2-\frac{1}{2}<c_1\leqslant c_3-\frac{1}{2}$. We have

(3.8)

$$\sum_{\substack{a_1,a_2,a_3\leqslant x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor)=1}}1$$

$$=\sum_{d\leqslant x^{c_1-1/2}}\mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_1+1)/3}}{d^{1/3}}\Big)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_2+1)/3}}{d^{1/3}}\Big)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_3+1)/3}}{d^{1/3}}\Big)\Big)$$

$$+\sum_{x^{c_1-1/2}<d\leqslant x^{c_2-1/2}}\mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^{c_1}}{d}\Big)\Big)$$

$$\times\Big(\frac{x}{d}+O\Big(\frac{x^{(c_2+1)/3}}{d^{1/3}}\Big)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_3+1)/3}}{d^{1/3}}\Big)\Big)$$

$$+\sum_{x^{c_2-1/2}<d\leqslant x^{c_1}}\mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^{c_1}}{d}\Big)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{c_2}}{d}\Big)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_3+1)/3}}{d^{1/3}}\Big)\Big).$$

Case 3: $c_1 - \frac{1}{2} < c_1 \leqslant c_2 - \frac{1}{2}$. We have

(3.9)

$$\sum_{\substack{a_1,a_2,a_3\leqslant x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor)=1}} 1$$

$$= \sum_{d\leqslant x^{c_1-1/2}} \mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_1+1)/3}}{d^{1/3}}\Big)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_2+1)/3}}{d^{1/3}}\Big)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_3+1)/3}}{d^{1/3}}\Big)\Big)$$

$$+ \sum_{x^{c_1-1/2}<d\leqslant x^{c_1}} \mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^{c_1}}{d}\Big)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_2+1)/3}}{d^{1/3}}\Big)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{(c_3+1)/3}}{d^{1/3}}\Big)\Big).$$

By the same calculation as in Case 1, we obtain (3.8) and (3.9) as

$$\sum_{\substack{a_1,a_2,a_3\leqslant x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor)=1}} 1 = \frac{1}{\zeta(3)}x^3 + O\Big(x^{(c_3+7)/3}\Big). \tag{3.10}$$

From (3.7) and (3.10), Theorem 1.4 follows for $k=3$.

Next, we prove Theorem 1.4 in the case $k \geqslant 4$. For $1 < c_1 \leqslant c_2 \leqslant \ldots \leqslant c_k < 2$, we have

$$\sum_{\substack{a_1,a_2,a_3,\ldots,a_k\leqslant x\\ \gcd(\lfloor a_1^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor,\ldots,\lfloor a_k^{c_k}\rfloor)=1}} 1 = \sum_{d\leqslant x^{c_1}} \mu(d) \sum_{\substack{a_1\leqslant x\\ d\mid\lfloor a_1^{c_1}\rfloor}} 1 \sum_{\substack{a_2\leqslant x\\ d\mid\lfloor a_2^{c_2}\rfloor}} 1 \sum_{\substack{a_3\leqslant x\\ d\mid\lfloor a_3^{c_3}\rfloor}} 1 \ldots \sum_{\substack{a_k\leqslant x\\ d\mid\lfloor a_k^{c_k}\rfloor}} 1$$

$$= \sum_{d\leqslant x^{c_1}} \mu(d) \prod_{i=1}^{k} \sum_{\substack{n\leqslant x\\ \lfloor n^{c_i}\rfloor\equiv 0 \pmod d}} 1.$$

In view of Lemma 2.1, we have

(3.11)

$$\sum_{\substack{a_1,a_2,a_3,\ldots,a_k\leqslant x\\ \gcd(\lfloor a_1^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor,\ldots,\lfloor a_k^{c_k}\rfloor)=1}} 1 = \sum_{d\leqslant x^{c_1}} \mu(d) \prod_{i=1}^{k}\Big(\frac{x}{d}+O\Big(\min\Big(\frac{x^{c_i}}{d},\frac{x^{(c_i+1)/3}}{d^{1/3}}\Big)\Big)\Big).$$

Firstly, we consider the case of $c_1 > c_k - \frac{1}{2}$. Since the condition on the O-terms, we split the sum (3.11) into the following $k+1$ sums.

$$\sum_{\substack{a_1,a_2,a_3,\ldots,a_k\leqslant x\\ \gcd(\lfloor a_1^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor,\ldots,\lfloor a_k^{c_k}\rfloor)=1}} 1 = \sum_{d\leqslant x^{c_1-1/2}} \mu(d) \prod_{i=1}^{k}\Big(\frac{x}{d}+O\Big(\frac{x^{(c_i+1)/3}}{d^{1/3}}\Big)\Big)$$

$$+\sum_{l=1}^{k-1}\sum_{x^{c_l-1/2}<d\leqslant x^{c_{l+1}-1/2}}\mu(d)\prod_{i=1}^{l}\Big(\frac{x}{d}+O\Big(\frac{x^{c_i}}{d}\Big)\Big)\prod_{i=l+1}^{k}\Big(\frac{x}{d}+O\Big(\frac{x^{(c_i+1)/3}}{d^{1/3}}\Big)\Big)$$

$$+\sum_{x^{c_k-1/2}<d\leqslant x^{c_1}}\mu(d)\prod_{i=1}^{k}\Big(\frac{x}{d}+O\Big(\frac{x^{c_i}}{d}\Big)\Big).$$

In view of (2.1), (2.3) and (2.4) in Lemma 2.2, we have

$$(3.13)\qquad \sum_{\substack{a_1,a_2,a_3,\ldots,a_k\leqslant x\\ \gcd(\lfloor a_1^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor,\ldots,\lfloor a_k^{c_k}\rfloor)=1}} 1$$

$$=\sum_{d\leqslant x^{c_1-1/2}}\mu(d)\bigg(\frac{x^k}{d^k}+\sum_{i=1}^{k}O\bigg(\frac{x^{(1/3)\sum_{j=0}^{i-1}c_{k-j}+k-2i/3}}{d^{k-2i/3}}\bigg)\bigg)$$

$$+\sum_{l=1}^{k-1}\sum_{x^{c_l-1/2}<d\leqslant x^{c_{l+1}-1/2}}\mu(d)\bigg(\frac{x^k}{d^k}+O\bigg(\frac{x^{\sum_{j=1}^{l}c_j+k-l}}{d^k}\bigg)\bigg)$$

$$+\sum_{i=1}^{k-l}O\bigg(\frac{x^{(1/3)\sum_{j=0}^{i-1}c_{k-j}+k-l-2i/3+\sum_{j=1}^{l}c_j}}{d^{k-2i/3}}\bigg)$$

$$+\sum_{x^{c_k-1/2}<d\leqslant x^{c_1}}\mu(d)\bigg(\frac{x^k}{d^k}+O\bigg(\frac{x^{\sum_{j=1}^{k}c_j}}{d^k}\bigg)\bigg).$$

We note that, for $1\leqslant i\leqslant k$, $k-\frac{2i}{3}>1$ and

$$\sum_{d\leqslant x^{c_1-1/2}}\frac{1}{d^{k-2i/3}}=O(1).$$

Then
(3.13)

$$\sum_{d\leqslant x^{c_1-1/2}}\mu(d)\bigg(\sum_{i=1}^{k}O\bigg(\frac{x^{(1/3)\sum_{j=0}^{i-1}c_{k-j}+k-2i/3}}{d^{k-2i/3}}\bigg)\bigg)=\sum_{i=1}^{k}O\big(x^{(1/3)\sum_{j=0}^{i-1}c_{k-j}+k-2i/3}\big)$$

$$=O(x^{c_k/3+k-2/3}).$$

For the O-term of the second sum in (3.13), we have

$$\sum_{l=1}^{k-1}\sum_{x^{c_l-1/2}<d\leqslant x^{c_{l+1}-1/2}}\mu(d)O\bigg(\frac{x^{\sum_{j=1}^{l}c_j+k-l}}{d^k}\bigg)=O\bigg(\sum_{l=1}^{k-1}x^{\sum_{j=1}^{l}c_j+k-l+(1-k)(c_l-1/2)}\bigg).$$

Next, we show that, for $1\leqslant l<k$,

$$\sum_{j=1}^{l}c_j+k-l+(1-k)\Big(c_l-\frac{1}{2}\Big)\leqslant\frac{1}{3}c_k+k-\frac{2}{3}.$$

First, we assume that

$$c_1 + k - 1 + (1-k)\Big(c_1 - \frac{1}{2}\Big) > \frac{1}{3}c_k + k - \frac{2}{3}.$$

Then

$$c_1 > \frac{c_k+1}{3} + (k-1)\Big(c_1 - \frac{1}{2}\Big) > \frac{c_k+1}{3} + 4\Big(c_1 - \frac{1}{2}\Big) > \frac{2}{3} + \frac{3}{2} = \frac{13}{6} > 2.$$

This is impossible. Next, we assume that for $1 < l < k$

$$\sum_{j=1}^{l} c_j + k - l + (1-k)\Big(c_l - \frac{1}{2}\Big) > \frac{1}{3}c_k + k - \frac{2}{3}.$$

Then

$$\sum_{j=1}^{l} c_j + \frac{2}{3} > l + (k-1)\Big(c_l - \frac{1}{2}\Big) + \frac{1}{3}c_k.$$

From $lc_l > \sum_{j=1}^{l} c_j$, we have $lc_l + \frac{2}{3} > l + (k-1)(c_l - \frac{1}{2}) + \frac{1}{3}c_k$. Then

$$l\Big(c_l - \frac{1}{2}\Big) + \frac{2}{3} > \frac{l}{2} + (k-1)\Big(c_l - \frac{1}{2}\Big) + \frac{1}{3}c_k \geqslant \frac{l}{2} + l\Big(c_l - \frac{1}{2}\Big) + \frac{1}{3}c_k > \frac{1}{2} + l\Big(c_l - \frac{1}{2}\Big) + \frac{1}{3}.$$

It is also impossible. Thus, we have

$$\sum_{l=1}^{k-1} \sum_{x^{c_l - 1/2} < d \leqslant x^{c_{l+1} - 1/2}} \mu(d) O\Big(\frac{x^{\sum_{j=1}^{l} c_j + k - l}}{d^k}\Big) = O(x^{c_k/3 + k - 2/3}). \tag{3.14}$$

For the O-term of the third sum in (3.13), we have

$$\begin{aligned}&\sum_{l=1}^{k-1} \sum_{x^{c_l - 1/2} < d \leqslant x^{c_{l+1} - 1/2}} \mu(d) \Big(\sum_{i=1}^{k-l} O\Big(\frac{x^{(1/3)\sum_{j=0}^{i-1} c_{k-j} + k - l - 2i/3 + \sum_{j=1}^{l} c_j}}{d^{k-2i/3}}\Big)\Big) \\ &\quad = \sum_{l=1}^{k-1} \sum_{i=1}^{k-l} O(x^{(1/3)\sum_{j=0}^{i-1} c_{k-j} + k - l - 2i/3 + \sum_{j=1}^{l} c_j + (1+2i/3-k)(c_l - 1/2)}).\end{aligned}$$

Now we show that for $1 \leqslant i \leqslant k - l$

$$\begin{aligned}&\frac{1}{3}\sum_{j=0}^{i-1} c_{k-j} + k - l - \frac{2i}{3} + \Big(1 + \frac{2i}{3} - k\Big)\Big(c_l - \frac{1}{2}\Big) \\ &\qquad\qquad \leqslant \frac{1}{3}\sum_{j=0}^{k-l-1} c_{k-j} + \frac{k-l}{3} + \Big(1 - \frac{k+2l}{3}\Big)\Big(c_l - \frac{1}{2}\Big).\end{aligned}$$

It suffices to show that

$$f(i)=\frac{1}{3}\sum_{j=0}^{i-1}c_{k-j}-\frac{2i}{3}+\Big(1+\frac{2i}{3}-k\Big)\Big(c_l-\frac{1}{2}\Big)$$

is strictly increasing. We assume $f(i)\geqslant f(i+1)$. Then

$$\begin{aligned}\frac{1}{3}\sum_{j=0}^{i-1}c_{k-j}-\frac{2i}{3}+\Big(1+\frac{2i}{3}-k\Big)\Big(c_l-\frac{1}{2}\Big)&\\ \geqslant \frac{1}{3}\sum_{j=0}^{i}c_{k-j}-\frac{2(i+1)}{3}+\Big(1+\frac{2(i+1)}{3}-k\Big)\Big(c_l-\frac{1}{2}\Big),&\\ \frac{2}{3}>\frac{2}{3}\Big(c_l-\frac{1}{2}\Big)+\frac{1}{3}c_{k-i}.&\end{aligned}$$

From $c_l-\frac{1}{2}>\frac{1}{2}$ then $\frac{2}{3}(c_l-\frac{1}{2})+\frac{1}{3}c_{k-i}>\frac{1}{3}(c_{k-i}+1)$. Thus, $\frac{2}{3}>\frac{1}{3}(c_{k-i}+1)$, which is impossible, so $f(i)$ is strictly increasing. Thus,

$$\begin{aligned}\sum_{l=1}^{k-1}\sum_{x^{c_l-1/2}<d\leqslant x^{c_{l+1}-1/2}}\mu(d)\Big(\sum_{i=1}^{k-l}O\Big(\frac{x^{(1/3)\sum_{j=0}^{i-1}c_{k-j}+k-l-2i/3+\sum_{j=1}^{l}c_j}}{d^{k-2i/3}}\Big)\Big)&\\ =\sum_{l=1}^{k-1}O\big(x^{(1/3)\sum_{j=0}^{k-l-1}c_{k-j}+(k-l)/3+\sum_{j=1}^{l}c_j+(1-(k+2l)/3)(c_l-1/2)}\big).&\end{aligned}$$

Next, we wish to show that

$$\frac{1}{3}\sum_{j=0}^{k-l-1}c_{k-j}+\frac{k-l}{3}+\sum_{j=1}^{l}c_j+\Big(1-\frac{k+2l}{3}\Big)\Big(c_l-\frac{1}{2}\Big)\leqslant\frac{c_k}{3}+k-\frac{2}{3}.$$

To show this, we assume that, for $1\leqslant l\leqslant k-1$,

$$\frac{1}{3}\sum_{j=0}^{k-l-1}c_{k-j}+\frac{k-l}{3}+\sum_{j=1}^{l}c_j+\Big(1-\frac{k+2l}{3}\Big)\Big(c_l-\frac{1}{2}\Big)>\frac{c_k}{3}+k-\frac{2}{3}.$$

Then

$$\frac{1}{3}\sum_{j=0}^{k-l-1}c_{k-j}+\sum_{j=1}^{l}c_j>\frac{2k}{3}+\frac{l-2}{3}+\Big(\frac{k+2l}{3}-1\Big)\Big(c_l-\frac{1}{2}\Big).$$

From

$$\frac{1}{3}\sum_{j=0}^{k-l-1}c_{k-j}<\frac{2}{3}(k-l-1)\quad\text{and}\quad\sum_{j=1}^{l}c_j<lc_l,$$

then

$$\frac{2}{3}(k-l-1)+lc_l > \frac{2k}{3}+\frac{l-2}{3}+\Big(\frac{k+2l}{3}-1\Big)\Big(c_l-\frac{1}{2}\Big),$$
$$lc_l > l+\Big(\frac{k+2l}{3}-1\Big)\Big(c_l-\frac{1}{2}\Big),$$
$$l\Big(c_l-\frac{1}{2}\Big) > \frac{l}{2}+\Big(\frac{k+2l}{3}-1\Big)\Big(c_l-\frac{1}{2}\Big),$$
$$0 > \frac{l}{2}+\Big(\frac{k-l}{3}-1\Big)\Big(c_l-\frac{1}{2}\Big).$$

This is impossible for $l \leqslant k-3$. For $l = k-1$, we see that $0 > \frac{1}{2}(k-1)-\frac{2}{3}(c_{k-1}-\frac{1}{2}) > \frac{1}{2}$. This is impossible. For $l = k-2$, we see that $0 > \frac{1}{2}(k-2)-\frac{1}{3}(c_{k-1}-\frac{1}{2}) > \frac{1}{2}$. This is impossible. Thus,

$$\text{(3.15)}\quad \sum_{l=1}^{k-1}\sum_{x^{c_l-1/2}<d\leqslant x^{c_{l+1}-1/2}}\mu(d)\Big(\sum_{i=1}^{k-l}O\Big(\frac{x^{(1/3)\sum_{j=0}^{i-1}c_{k-j}+k-l-2i/3+\sum_{j=1}^{l}c_j}}{d^{k-2i/3}}\Big)\Big) = O(x^{(c_k/3)+k-2/3}).$$

For the O-term of the last sum in (3.13), we have

$$\sum_{x^{c_k-1/2}<d\leqslant x^{c_1}}\mu(d)O\Big(\frac{x^{\sum_{j=1}^{k}c_j}}{d^k}\Big) = O\big(x^{\sum_{j=1}^{k}c_j+(1-k)(c_k-1/2)}\big).$$

Next, we show that,

$$\sum_{j=1}^{k}c_j+(1-k)\Big(c_k-\frac{1}{2}\Big) \leqslant \frac{c_k}{3}+k-\frac{2}{3}.$$

We first assume that,

$$\sum_{j=1}^{k}c_j+(1-k)\Big(c_k-\frac{1}{2}\Big) > \frac{c_k}{3}+k-\frac{2}{3}.$$

Then, we have

$$\sum_{j=1}^{k}c_j+\frac{2}{3} > \frac{c_k}{3}+k+(k-1)\Big(c_k-\frac{1}{2}\Big),$$
$$kc_k+\frac{2}{3} > \sum_{j=1}^{k}c_j+\frac{2}{3} > \frac{c_k}{3}+k+(k-1)\Big(c_k-\frac{1}{2}\Big).$$

Then

$$k\Big(c_k-\frac{1}{2}\Big)+\frac{2}{3} > \frac{c_k}{3}+\frac{k}{2}+(k-1)\Big(c_k-\frac{1}{2}\Big),\quad c_k-\frac{1}{2}+\frac{2}{3} > \frac{c_k}{3}+\frac{k}{2}.$$

Therefore,

$$\frac{4}{3}+\frac{1}{6}>\frac{2c_k}{3}+\frac{1}{6}>\frac{k}{2}>2.$$

This is impossible. Thus,

$$\sum_{x^{c_k-1/2}<d\leqslant x^{c_1}} \mu(d) O\Bigg(\frac{x^{\sum_{j=1}^{k} c_j}}{d^k}\Bigg) = O(x^{(c_k/3)+k-2/3}). \tag{3.16}$$

Inserting (3.14)–(3.17) to (3.13), we have

$$\sum_{\substack{a_1,a_2,a_3,\ldots,a_k\leqslant x\\ \gcd(\lfloor a_1^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor,\ldots,\lfloor a_k^{c_k}\rfloor)=1}} 1 = \sum_{d\leqslant x^{c_1}} \mu(d)\frac{x^k}{d^k} + O(x^{(c_k/3)+k-2/3}). \tag{3.17}$$

In view of Lemma 2.3 and (3.18) we have for $c_k-\frac{1}{2}<c_1<c_2<\ldots<c_k$,

$$\sum_{\substack{a_1,a_2,a_3,\ldots,a_k\leqslant x\\ \gcd(\lfloor a_1^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor,\ldots,\lfloor a_k^{c_k}\rfloor)=1}} 1 = \frac{1}{\zeta(k)}x^k + O(x^{(c_k/3)+k-2/3}). \tag{3.18}$$

Now, we consider the other cases. If $c_1>c_j-\frac{1}{2}$, $1\leqslant j\leqslant k-1$, we write the right hand side of (3.11) as

(3.19)

$$\begin{aligned}
&\sum_{d\leqslant x^{c_1}} \mu(d)\prod_{i=1}^{k}\Big(\frac{x}{d}+O\Big(\min\Big(\frac{x^{c_i}}{d},\frac{x^{(c_i+1)/3}}{d^{1/3}}\Big)\Big)\Big)\\
&= \sum_{d\leqslant x^{c_1-1/2}} \mu(d)\prod_{i=1}^{k}\Big(\frac{x}{d}+O\Big(\frac{x^{(c_i+1)/3}}{d^{1/3}}\Big)\Big)\\
&\quad+\sum_{l=1}^{j-1}\sum_{x^{c_l-1/2}<d\leqslant x^{c_{l+1}-1/2}} \mu(d)\prod_{i=1}^{l}\Big(\frac{x}{d}+O\Big(\frac{x^{c_i}}{d}\Big)\Big)\prod_{i=l+1}^{k}\Big(\frac{x}{d}+O\Big(\frac{x^{(c_i+1)/3}}{d^{1/3}}\Big)\Big)\\
&\quad+\sum_{x^{c_j-1/2}<d\leqslant x^{c_1}} \mu(d)\prod_{i=1}^{j}\Big(\frac{x}{d}+O\Big(\frac{x^{c_i}}{d}\Big)\Big)\prod_{i=j+1}^{k}\Big(\frac{x}{d}+O\Big(\frac{x^{(c_i+1)/3}}{d^{1/3}}\Big)\Big).
\end{aligned}$$

Like in the proof of (3.19), $O(x^{(c_k/3)+k-2/3})$ from the O-term of the first sum in the right hand side of (3.20) dominates all of O-terms. In view of Lemma 2.3, we get, for $c_1>c_j-\frac{1}{2}$ and $1\leqslant j\leqslant k-1$,

$$\sum_{\substack{a_1,a_2,a_3,\ldots,a_k\leqslant x\\ \gcd(\lfloor a_1^{c_1}\rfloor,\lfloor a_2^{c_2}\rfloor,\lfloor a_3^{c_3}\rfloor,\ldots,\lfloor a_k^{c_k}\rfloor)=1}} 1 = \frac{1}{\zeta(k)}x^k + O(x^{(c_k/3)+k-2/3}). \tag{3.20}$$

From (3.19) and (3.21), Theorem 1.4 follows. □

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