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## ON THE LEAST ALMOST-PRIME IN ARITHMETIC PROGRESSION

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*Abstract.* Let  $\mathcal{P}_r$  denote an almost-prime with at most  $r$  prime factors, counted according to multiplicity. Suppose that  $a$  and  $q$  are positive integers satisfying  $(a, q) = 1$ . Denote by  $\mathcal{P}_2(a, q)$  the least almost-prime  $\mathcal{P}_2$  which satisfies  $\mathcal{P}_2 \equiv a \pmod{q}$ . It is proved that for sufficiently large  $q$ , there holds

$$\mathcal{P}_2(a, q) \ll q^{1.8345}.$$

This result constitutes an improvement upon that of Iwaniec (1982), who obtained the same conclusion, but for the range 1.845 in place of 1.8345.

*Keywords:* almost-prime; arithmetic progression; linear sieve; Selberg's  $\Lambda^2$ -sieve

*MSC 2020:* 11N13, 11N35, 11N36

## 1. INTRODUCTION AND MAIN RESULT

Let  $\mathcal{P}_r$  denote an almost-prime with at most  $r$  prime factors, counted according to multiplicity. In this paper, we shall investigate the occurrence of almost-primes in arithmetic progressions. This problem corresponds to a well-known conjecture concerning prime numbers. The conjecture states that if  $(a, q) = 1$ , there exists a prime  $p$  satisfying

$$(1.1) \quad p \equiv a \pmod{q}, \quad p \leq q^2 \quad (q \geq 2).$$

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Indeed, the bound for  $p$  may presumably be reduced to  $p \ll q(\log q)^2$ . Unfortunately, we cannot prove (1.1) even on the assumption of the generalized Riemann hypothesis. The nearest approach seems to be the conditional estimate  $p \ll (\varphi(q))^2(\log q)^4$ , which follows from Theorem 6 of [9]. However, as an approach to approximate this conjecture, we can consider almost-primes in arithmetic progression. Many authors investigated this approximation in the past time. Denote by  $\mathcal{P}_2(a, q)$  the least almost-prime  $\mathcal{P}_2$  which satisfies  $\mathcal{P}_2 \equiv a \pmod{q}$ . In 1965, Levin [5] showed that  $\mathcal{P}_2(a, q) \ll q^{2.3696}$ . Later, Richert pointed out that by using the method in [4], the exponent can be replaced by  $\frac{25}{11} + \varepsilon$ . Afterwards, Halberstam and Richert gave the result that the exponent can be replaced by  $\frac{11}{5}$  in their monograph, see [1], Chapter 9. Motohashi in [7] in 1976 gave the exponent  $2 + \varepsilon$  subject to a certain unproved hypothesis. In 1978, Heath-Brown first gave an unconditional bound, stronger than (1.1), for almost-primes  $\mathcal{P}_2$ . He showed that  $\mathcal{P}_2(a, q) \ll q^{1.965}$ . After that, in 1982, Iwaniec in [3] improved Heath-Brown's result and derived that  $\mathcal{P}_2(a, q) \ll q^{1.845}$ .

In this paper, we shall continue to improve the result of Iwaniec [3] and establish the following theorem.

**Theorem 1.1.** *Suppose that  $a$  and  $q$  are positive integers satisfying  $(a, q) = 1$ . Let  $\mathcal{P}_2(a, q)$  be the least almost-prime  $\mathcal{P}_2$  which satisfies  $\mathcal{P}_2 \equiv a \pmod{q}$ . Then for sufficiently large  $q$ , there holds*

$$\mathcal{P}_2(a, q) \ll q^{1.8345}.$$

**Remark 1.1.** Our improvement comes from using distinct methods to deal with the different parts of the sifting sum with more delicate techniques, combined with linear sieve results of Iwaniec (see [2]) with bilinear forms for the remainder term and the two-dimensional sieve of Selberg.

## 2. NOTATION AND PRELIMINARIES

Throughout this paper, we always denote primes by  $p$ . The symbol  $\varepsilon$  always denotes an arbitrarily small positive constant, which may not be the same at different occurrences. As usual, we use  $\varphi(n)$ ,  $\mu(n)$ ,  $\tau(n)$  to denote Euler's function, Möbius' function, and Dirichlet divisor function, respectively. Moreover,  $\Omega(n)$  denotes the number of prime factors of  $n$ , counted according to multiplicity. Let  $(m_1, m_2, \dots, m_k)$  and  $[m_1, m_2, \dots, m_k]$  be the greatest common divisor and the least common multiple of  $m_1, m_2, \dots, m_k$ , respectively. Also,  $f(x) \ll g(x)$  means that  $f(x) = O(g(x))$ . The symbol  $\mathcal{P}_r$  always denotes an almost-prime with at most  $r$  prime factors, counted according to multiplicity.

Let  $\mathcal{A}$  be a finite sequence of integers, and  $\mathcal{P}$  a set of primes. For given  $z \geq 2$  we denote

$$P(z) = \prod_{\substack{p < z \\ p \in \mathcal{P}}} p.$$

Define the sifting function as

$$S(\mathcal{A}, \mathcal{P}, z) = |\{a \in \mathcal{A} : (a, P(z)) = 1\}|.$$

For  $d \mid P(z)$ , define  $\mathcal{A}_d = \{a \in \mathcal{A} : a \equiv 0 \pmod{d}\}$ . Moreover, we assume that  $|\mathcal{A}_d|$  may be written in the form

$$|\mathcal{A}_d| = \frac{\omega(d)}{d} X + r(\mathcal{A}, d),$$

where  $\omega(d)$  is a multiplicative function satisfying  $0 < \omega(p) < p$  for  $p \in \mathcal{P}$ ,  $X$  is an approximation to  $|\mathcal{A}|$  independent of  $d$ . In addition,  $\omega(d)d^{-1}X$  is regarded as a main term of  $|\mathcal{A}_d|$ ,  $r(\mathcal{A}, d)$  is regarded as an error term of  $|\mathcal{A}_d|$ , which is expected to be small on average over  $d$ . Also, we assume that the function  $\omega(p)$  is constant on average over  $p$  in  $\mathcal{P}$ , which means that

$$\prod_{\substack{z_1 \leq p < z_2 \\ p \in \mathcal{P}}} \left(1 - \frac{\omega(p)}{p}\right)^{-1} \leq \frac{\log z_2}{\log z_1} \left(1 + \frac{K}{\log z_1}\right)$$

for all  $z_2 > z_1 \geq 2$ , where  $K$  is a constant satisfying  $K \geq 1$ .

**Lemma 2.1.** *Let  $F(u)$  and  $f(u)$  be continuous functions which satisfy the following differential-difference equations:*

$$\begin{cases} F(u) = \frac{2e^\gamma}{u}, & f(u) = 0 & \text{for } 1 \leq u \leq 2, \\ (uF(u))' = f(u-1), & (uf(u))' = F(u-1) & \text{for } u \geq 2. \end{cases}$$

Then we have

$$\begin{aligned} F(u) &= \frac{2e^\gamma}{u} & \text{for } 0 < u \leq 3, \\ F(u) &= \frac{2e^\gamma}{u} \left(1 + \int_2^{u-1} \frac{\log(t-1)}{t} dt\right) & \text{for } 3 \leq u \leq 5, \\ f(u) &= \frac{2e^\gamma}{u} \left(\log(u-1) + \int_3^{u-1} \frac{dt_1}{t_1} \int_2^{t_1-1} \frac{\log(t_2-1)}{t_2} dt_2\right) & \text{for } 4 \leq u \leq 6. \end{aligned}$$

*Proof.* See [1], Chapter 8, (2.8) and [8], pages 126–127. □

### 3. PROOF OF THEOREM 1.1

Let  $\mathcal{A} = \{n: n \leq x, n \equiv a \pmod{q}\}$ , where  $(a, q) = 1$ ,  $x^{1/2} < q \leq x^{3/5}$ . Set

$$\mathcal{P} = \{p: p \nmid q\}, \quad M = x^{1-3\varepsilon}q^{-1}, \quad N = x^{1/2-4\varepsilon}q^{-3/4}, \quad D = MN.$$

Moreover, we put

$$\delta = 0.86, \quad \theta = 1.8345, \quad x = q^\theta, \quad y = q^\delta.$$

We write  $S(\mathcal{A}, z)$  as abbreviation of  $S(\mathcal{A}, \mathcal{P}, z)$  for convenience. For the notations defined as above, we have  $M < y < D$  and consider the weighted sum with Richert's weights of logarithmic type

$$W(\mathcal{A}; z, y) = \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1}} \left( 1 - \frac{1}{\lambda} \sum_{\substack{z \leq p < y \\ p|n}} \left( 1 - \frac{\log p}{\log y} \right) \right),$$

where  $z = D^{5/23}$ ,  $\lambda = 3 - \log x / \log y - \varepsilon$ . For convenience, we write

$$\mathcal{W}(n) = 1 - \frac{1}{\lambda} \sum_{\substack{z \leq p < y \\ p|n}} \left( 1 - \frac{\log p}{\log y} \right).$$

Then we have

$$(3.1) \quad W(\mathcal{A}; z, y) = \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1 \\ \Omega(n) \leq 2}} \mathcal{W}(n) + \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1 \\ \Omega(n) \geq 3 \\ \mu(n) \neq 0}} \mathcal{W}(n) + \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1 \\ \Omega(n) \geq 3 \\ \mu(n) = 0}} \mathcal{W}(n).$$

Obviously, we have

$$(3.2) \quad \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1 \\ \Omega(n) \geq 3 \\ \mu(n)=0}} \mathcal{W}(n) \ll \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1 \\ \mu(n)=0}} \tau(n) \ll x^\varepsilon \sum_{\substack{z \leq p \leq x^{1/2} \\ p \nmid q}} \sum_{\substack{k \leq (x-a_1)/q \\ k \equiv -a_1 \bar{q} \pmod{p^2}}} 1 \\ \ll x^\varepsilon \sum_{z \leq p \leq x^{1/2}} \left( \frac{x}{p^2 q} + 1 \right) \ll x^\varepsilon \left( \frac{x}{qz} + x^{1/2} \right) = o\left( \frac{x^{1-\varepsilon}}{\varphi(q)} \right),$$

where the integer  $a_1$  satisfies  $1 \leq a_1 \leq q$  and  $a_1 \equiv a \pmod{q}$ , while the integer  $\bar{q}$  satisfies  $q\bar{q} \equiv 1 \pmod{p^2}$  with respect to the prime  $p$  satisfying  $z \leq p \leq x^{1/2}$  and  $p \nmid q$ .

For a given integer  $n$  with  $n \leq x$ ,  $n \equiv a \pmod{q}$ ,  $(a, q) = 1$ ,  $(n, P(z)) = 1$  and  $\mu(n) \neq 0$ , the weight  $\mathcal{W}(n)$  in the sum  $W(\mathcal{A}; z, y)$  satisfies

$$(3.3) \quad 1 - \frac{1}{\lambda} \sum_{\substack{z \leq p < y \\ p|n}} \left(1 - \frac{\log p}{\log y}\right) \leq \frac{1}{\lambda} \left( \lambda - \sum_{p|n} \left(1 - \frac{\log p}{\log y}\right) \right) \\ = \frac{1}{\lambda} \left( 3 - \frac{\log x}{\log y} - \varepsilon - \Omega(n) + \frac{\log n}{\log y} \right) < \frac{1}{\lambda} (3 - \Omega(n)),$$

and thus  $\mathcal{W}(n) < 0$  for  $\Omega(n) \geq 3$ . From (3.1)–(3.3) we know that

$$(3.4) \quad \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1 \\ \Omega(n) \leq 2}} \mathcal{W}(n) = W(\mathcal{A}; z, y) - \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1 \\ \Omega(n) \geq 3 \\ \mu(n) \neq 0}} \mathcal{W}(n) + o\left(\frac{x^{1-\varepsilon}}{\varphi(q)}\right) \\ \geq W(\mathcal{A}; z, y) + o\left(\frac{x^{1-\varepsilon}}{\varphi(q)}\right).$$

For  $W(\mathcal{A}; z, y)$  we have

$$(3.5) \quad W(\mathcal{A}; z, y) = \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1}} 1 - \frac{1}{\lambda} \sum_{\substack{z \leq p < y \\ p \nmid q}} \left(1 - \frac{\log p}{\log y}\right) \sum_{\substack{n \in \mathcal{A} \\ n \equiv 0 \pmod{p} \\ (n, P(z))=1}} 1 \\ = S(\mathcal{A}, z) - \frac{1}{\lambda} \sum_{\substack{z \leq p < y \\ p \nmid q}} \left(1 - \frac{\log p}{\log y}\right) S(\mathcal{A}_p, z).$$

We appeal to Theorem 1 of [2] for linear sieve results with bilinear forms for the remainder term, which simply gives

$$(3.6) \quad S(\mathcal{A}, z) \geq \frac{x}{\varphi(q)} V(z) \left( f\left(\frac{23}{5}\right) + O((\log D)^{-1/3}) \right) - R^-,$$

where  $f$  is the function given in Lemma 2.1,

$$R^- = \sum_{l < \exp(8\varepsilon^{-3})} \sum_{\substack{m < M \\ (mn, q)=1}} \sum_{n < N} a_m^-(l) b_n^-(l) r(\mathcal{A}, mn),$$

and

$$(3.7) \quad V(z) = \prod_{p < z} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log z} \left(1 + O\left(\frac{1}{\log z}\right)\right)$$

by the Mertens' prime number theorem (see [6]). By Theorem 5 of Iwaniec [3], one has

$$(3.8) \quad R^- \ll \frac{x^{1-\varepsilon}}{\varphi(q)}.$$

From Lemma 2.1, (3.6), (3.7) and (3.8), we obtain

$$(3.9) \quad S(\mathcal{A}, z) \geq \frac{x}{\varphi(q) \log D} \left\{ 2 \left( \log \frac{18}{5} + \int_3^{18/5} \frac{dt_1}{t_1} \int_2^{t_1-1} \frac{\log(t_2-1)}{t_2} dt_2 \right) \right\} (1 + O(\varepsilon)).$$

For the second term in (3.5) we write it in three parts:

$$(3.10) \quad \sum_{\substack{z \leq p < y \\ p|q}} \left( 1 - \frac{\log p}{\log y} \right) S(\mathcal{A}_p, z) \\ = \sum_{\substack{z \leq p < D^{8/23} \\ p|q}} \left( 1 - \frac{\log p}{\log y} \right) S(\mathcal{A}_p, z) + \sum_{\substack{D^{8/23} \leq p < M \\ p|q}} \left( 1 - \frac{\log p}{\log y} \right) S(\mathcal{A}_p, z) \\ + \sum_{\substack{M \leq p < y \\ p|q}} \left( 1 - \frac{\log p}{\log y} \right) S(\mathcal{A}_p, z).$$

Henceforth, we shall use two distinct methods to deal with the sums in (3.10). For the first and the second sum in (3.10), we shall appeal to linear sieve results of Iwaniec with bilinear forms for the remainder term. On the other hand, we will treat the third sum by the two-dimensional sieve of Selberg, which is the key point that leads to more exponent saving than that obtained by Iwaniec.

Now, we deal with the first sum in (3.10). For each  $S(\mathcal{A}_p, z)$ , by Theorem 1 of [2], we derive that

$$S(\mathcal{A}_p, z) \leq \frac{x}{p\varphi(q)} V(z) \left( F \left( \frac{\log(D/p)}{\log z} \right) + O(\log^{-1/3} D) \right) \\ + \sum_{l < \exp(8\varepsilon^{-3})} \sum_{\substack{m < M/p \\ (mn, q) = 1}} \sum_{n < N} a_m^+(l) b_n^+(l) r(\mathcal{A}, pmn) \\ = \frac{x(2 + O(\varepsilon))}{p\varphi(q) \log(D/p)} \left( 1 + \int_2^{\log(D/p)/\log z - 1} \frac{\log(t-1)}{t} dt \right) \\ + \sum_{l < \exp(8\varepsilon^{-3})} \sum_{\substack{m < M/p \\ (mn, q) = 1}} \sum_{n < N} a_m^+(l) b_n^+(l) r(\mathcal{A}, pmn),$$

where  $|a_m^+(l)| \leq 1$ ,  $|b_n^+(l)| \leq 1$ . Summing over  $p \in [z, D^{8/23}]$ ,  $p \nmid q$  interpreting that  $pm$  as one variable of the summation while  $n$  as the other, then, according to Theorem 5 of [3], the final remainder term arising is  $\ll x^{1-\varepsilon}/\varphi(q)$ . Therefore, we deduce that

$$\begin{aligned}
 (3.11) \quad & \sum_{\substack{z \leq p < D^{8/23} \\ p \nmid q}} \left(1 - \frac{\log p}{\log y}\right) S(\mathcal{A}_p, z) \\
 & \leq \sum_{z \leq p < D^{8/23}} \frac{\log(y/p)}{\log y} \frac{x(2 + O(\varepsilon))}{p\varphi(q) \log(D/p)} \\
 & \quad \times \left(1 + \int_2^{\log(D/p)/\log z - 1} \frac{\log(t-1)}{t} dt\right) + O\left(\frac{x^{1-\varepsilon}}{\varphi(q)}\right) \\
 & = \frac{x(2 + O(\varepsilon)) \log D}{\varphi(q) \log D \log y} \sum_{z \leq p < D^{8/23}} \frac{\log(y/p)}{p \log(D/p)} \\
 & \quad \times \left(1 + \int_2^{\log(D/p)/\log z - 1} \frac{\log(t-1)}{t} dt\right) + O\left(\frac{x^{1-\varepsilon}}{\varphi(q)}\right).
 \end{aligned}$$

By partial summation and by prime number theorem, it is easy to derive that

$$\begin{aligned}
 (3.12) \quad & \frac{\log D}{\log y} \sum_{z \leq p < D^{8/23}} \frac{\log(y/p)}{p \log(D/p)} \left(1 + \int_2^{\log(D/p)/\log z - 1} \frac{\log(t-1)}{t} dt\right) \\
 & = \frac{6\theta - 7}{4\delta} \int_{(30\theta - 35)/92}^{(12\theta - 14)/23} \frac{\delta - \beta}{\beta(3\theta/2 - 7/4 - \beta)} \\
 & \quad \times \left(1 + \int_2^{(108\theta - 126 - 92\beta)/(30\theta - 35)} \frac{\log(t-1)}{t} dt\right) d\beta + O(\varepsilon).
 \end{aligned}$$

Next, we shall deal with the second sum in (3.10), which is similar to the first sum. For each  $S(\mathcal{A}_p, z)$ , by Theorem 1 of [2], we get

$$\begin{aligned}
 S(\mathcal{A}_p, z) & \leq \frac{x}{p\varphi(q)} V(z) \left(F\left(\frac{\log(D/p)}{\log z}\right) + O(\log^{-1/3} D)\right) \\
 & \quad + \sum_{l < \exp(8\varepsilon^{-3})} \sum_{\substack{m < M/p \\ (mn, q) = 1}} \sum_{n < N} a_m^+(l) b_n^+(l) r(\mathcal{A}, pmn) \\
 & = \frac{x(2 + O(\varepsilon))}{p\varphi(q) \log(D/p)} + \sum_{l < \exp(8\varepsilon^{-3})} \sum_{\substack{m < M/p \\ (mn, q) = 1}} \sum_{n < N} a_m^+(l) b_n^+(l) r(\mathcal{A}, pmn),
 \end{aligned}$$

where  $|a_m^+(l)| \leq 1$ ,  $|b_n^+(l)| \leq 1$ . Summing over  $p \in [D^{8/23}, M)$ ,  $p \nmid q$  an interpretation that  $pm$  as one variable of the summation while  $n$  as the other, then, according to

Theorem 5 of [3], the final remainder term arising is  $\ll x^{1-\varepsilon}/\varphi(q)$ . Hence, once gets

$$\begin{aligned}
 (3.13) \quad & \sum_{\substack{D^{8/23} \leq p < M \\ p \nmid q}} \left(1 - \frac{\log p}{\log y}\right) S(\mathcal{A}_p, z) \\
 & \leq \sum_{D^{8/23} \leq p < M} \frac{\log(y/p)}{\log y} \frac{x(2 + O(\varepsilon))}{p\varphi(q) \log(D/p)} + O\left(\frac{x^{1-\varepsilon}}{\varphi(q)}\right) \\
 & = \frac{x(2 + O(\varepsilon)) \log D}{\varphi(q) \log D} \frac{\log D}{\log y} \sum_{D^{8/23} \leq p < M} \frac{\log(y/p)}{p \log(D/p)} + O\left(\frac{x^{1-\varepsilon}}{\varphi(q)}\right).
 \end{aligned}$$

By partial summation and by prime number theorem, one gets

$$\begin{aligned}
 (3.14) \quad & \frac{\log D}{\log y} \sum_{D^{8/23} \leq p < M} \frac{\log(y/p)}{p \log(D/p)} \\
 & = \frac{6\theta - 7}{4\delta} \int_{(12\theta-14)/23}^{\theta-1} \frac{\delta - \beta}{\beta(3\theta/2 - 7/4 - \beta)} d\beta + O(\varepsilon).
 \end{aligned}$$

Finally, we shall deal with the third sum, which appears in (3.10), in a different manner without appealing to Theorem 1 of Iwaniec [2]. We begin with ignoring the fact that  $p$  is a prime and obtaining

$$\begin{aligned}
 (3.15) \quad & \sum_{\substack{M \leq p < y \\ p \nmid q}} \left(1 - \frac{\log p}{\log y}\right) S(\mathcal{A}_p, z) = \sum_{\substack{M \leq p < y \\ (p,q)=1}} \left(1 - \frac{\log p}{\log y}\right) \sum_{\substack{m \in \mathcal{A} \\ m \equiv 0 \pmod{p} \\ (m, P(z))=1}} 1 \\
 & \leq \sum_{M \leq n < y} \left(1 - \frac{\log n}{\log y}\right) \sum_{\substack{m \in \mathcal{A} \\ m \equiv 0 \pmod{n} \\ (m, P(z))=1}} 1,
 \end{aligned}$$

where  $n$  runs over all integers in the interval  $[M, y)$ . Let  $\{\lambda^+(d)\}$  be an upper bound sieve of level  $D_1$ , i.e., a sequence of real numbers satisfying

$$|\lambda^+(d)| \leq 1, \quad \lambda^+(d) = 0 \quad \text{for } d \geq D_1 \quad \text{or} \quad \mu(d) = 0,$$

and

$$\sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda^+(d).$$

Then we get

$$\begin{aligned}
(3.16) \quad & \sum_{M \leq n < y} \left(1 - \frac{\log n}{\log y}\right) \sum_{\substack{m \in \mathcal{A} \\ m \equiv 0 \pmod{n} \\ (m, P(z))=1}} 1 \\
&= \sum_{M \leq n < y} \left(1 - \frac{\log n}{\log y}\right) \sum_{\substack{m \in \mathcal{A} \\ m \equiv 0 \pmod{n}}} \sum_{d|(m, P(z))} \mu(d) \\
&\leq \sum_{M \leq n < y} \left(1 - \frac{\log n}{\log y}\right) \sum_{\substack{m \in \mathcal{A} \\ m \equiv 0 \pmod{n}}} \sum_{d|(m, P(z))} \lambda^+(d) \\
&= \sum_{\substack{d < D_1 \\ d|P(z)}} \lambda^+(d) \sum_{M \leq n < y} \left(1 - \frac{\log n}{\log y}\right) \sum_{\substack{m \in \mathcal{A} \\ m \equiv 0 \pmod{[d, n]}}} 1 \\
&= \sum_{\substack{d < D_1 \\ d|P(z)}} \lambda^+(d) \sum_{M \leq n < y} \left(1 - \frac{\log n}{\log y}\right) \frac{x(1 + O(\varepsilon))}{[d, n]\varphi(q)} \\
&= \frac{x(1 + O(\varepsilon))}{\varphi(q)} \sum_{\substack{d < D_1 \\ d|P(z)}} \frac{\lambda^+(d)}{d} \sum_{M \leq n < y} \left(1 - \frac{\log n}{\log y}\right) \frac{(d, n)}{n}.
\end{aligned}$$

For the inner sum in (3.16), by partial summation, we have

$$\begin{aligned}
(3.17) \quad & \sum_{M \leq n < y} \left(1 - \frac{\log n}{\log y}\right) \frac{(d, n)}{n} \\
&= \sum_{v|d} \sum_{\substack{M/v \leq n_1 < y/v \\ (n_1, d/v)=1}} \left(1 - \frac{\log(n_1 v)}{\log y}\right) \frac{1}{n_1} \\
&= \sum_{v|d} \sum_{M/v \leq n_1 < y/v} \left(1 - \frac{\log(n_1 v)}{\log y}\right) \frac{1}{n_1} \sum_{\alpha|(n_1, d/v)} \mu(\alpha) \\
&= \sum_{v|d} \sum_{\alpha|d/v} \frac{\mu(\alpha)}{\alpha} \sum_{M/(v\alpha) \leq n_2 < y/(v\alpha)} \left(1 - \frac{\log(n_2 v\alpha)}{\log y}\right) \frac{1}{n_2} \\
&= \sum_{v|d} \sum_{\alpha|d/v} \frac{\mu(\alpha)}{\alpha} \int_{M/(v\alpha)}^{y/(v\alpha)} \left(1 - \frac{\log(tv\alpha)}{\log y}\right) \frac{dt}{t} + O\left(\frac{1}{M} \sum_{v|d} v \sum_{\alpha|d/v} 1\right) \\
&= \sum_{v|d} \sum_{\alpha|d/v} \frac{\mu(\alpha)}{\alpha} \int_M^y \left(1 - \frac{\log t}{\log y}\right) \frac{dt}{t} + O\left(\frac{1}{M} \sum_{v|d} v\tau\left(\frac{d}{v}\right)\right).
\end{aligned}$$

For the integral in (3.17), it is easy to see that

$$(3.18) \quad \int_M^y \left(1 - \frac{\log t}{\log y}\right) \frac{dt}{t} = \frac{1}{2 \log y} \left(\log \frac{y}{M}\right)^2.$$

In addition, we have

$$\begin{aligned}
 (3.19) \quad \omega_1(d) &:= \sum_{v|d} \sum_{\alpha|d/v} \frac{\mu(\alpha)}{\alpha} = \sum_{\alpha|d} \frac{\mu(\alpha)}{\alpha} \sum_{v|d/\alpha} 1 \\
 &= \sum_{\alpha|d} \frac{\mu(\alpha)}{\alpha} \tau\left(\frac{d}{\alpha}\right) = \prod_{p|d} \left(2 - \frac{1}{p}\right).
 \end{aligned}$$

Combining (3.15)–(3.19), we derive that

$$\begin{aligned}
 (3.20) \quad &\sum_{\substack{M \leq p < y \\ p \nmid q}} \left(1 - \frac{\log p}{\log y}\right) S(\mathcal{A}_p, z) \\
 &\leq \frac{x(1+O(\varepsilon))}{\varphi(q)} \sum_{\substack{d < D_1 \\ d|P(z)}} \frac{\lambda^+(d)}{d} \left( \frac{\omega_1(d)}{2 \log y} \left(\log \frac{y}{M}\right)^2 + O\left(\frac{1}{M} \sum_{v|d} v \tau\left(\frac{d}{v}\right)\right) \right) \\
 &= \frac{x(1+O(\varepsilon))}{2\varphi(q) \log y} \left(\log \frac{y}{M}\right)^2 \sum_{\substack{d < D_1 \\ d|P(z)}} \frac{\omega_1(d)}{d} \lambda^+(d) \\
 &\quad + O\left(\frac{x}{\varphi(q)M} \sum_{\substack{d < D_1 \\ d|P(z)}} |\lambda^+(d)| \sum_{v|d} \frac{\tau(d/v)}{d/v}\right) \\
 &= \frac{x(1+O(\varepsilon))}{2\varphi(q) \log y} \left(\log \frac{y}{M}\right)^2 \sum_{\substack{d < D_1 \\ d|P(z)}} \frac{\omega_1(d)}{d} \lambda^+(d) + O\left(\frac{x}{\varphi(q)M} \sum_{\substack{d < D_1 \\ d|P(z)}} \sum_{v|d} \frac{\tau(v)}{v}\right).
 \end{aligned}$$

By noting that the function  $\omega_1(d)$  is multiplicative and satisfies the 2-dimensional sieve assumptions, we specify  $\lambda^+(d)$ 's to be that from Selberg's  $\Lambda^2$ -sieve and deduce that (for instance, one can see [1], page 197)

$$(3.21) \quad \sum_{\substack{d < D_1 \\ d|P(z)}} \frac{\omega_1(d)}{d} \lambda^+(d) = \frac{1}{G(D_1, z)} = \frac{\mathcal{V}(z)}{\sigma(s)} \left(1 + O\left(\frac{1}{\log z}\right)\right)$$

holds for  $z \leq D_1$ , where

$$\begin{aligned}
 (3.22) \quad s &= \frac{\log D_1}{\log z}, \quad \mathcal{V}(z) = \prod_{p < z} \left(1 - \frac{\omega_1(p)}{p}\right), \\
 \sigma(s) &= \frac{s^2}{8e^{2\gamma}} \quad \text{for } 0 < s \leq 2.
 \end{aligned}$$

By (3.19) and Mertens' prime number theorem (see [6]), we obtain

$$(3.23) \quad \mathcal{V}(z) = \prod_{p < z} \left(1 - \frac{1}{p}\right)^2 = \frac{e^{-2\gamma}}{\log^2 z} \left(1 + O\left(\frac{1}{\log z}\right)\right).$$

Taking  $D_1 = N^2$ , then  $z \leq D_1$ , and thus (3.21) holds. Moreover, the remainder term in (3.20) is

$$(3.24) \quad \begin{aligned} &\ll \frac{x}{\varphi(q)M} \sum_{d < D_1} \prod_{p|d} \left(1 + \frac{2}{p}\right) \ll \frac{x}{\varphi(q)M} \sum_{d < D_1} (\log \log d)^2 \\ &\ll \frac{x}{\varphi(q)M} D_1 (\log \log D_1)^2 = o\left(\frac{x}{\varphi(q) \log y}\right). \end{aligned}$$

From (3.20)–(3.24) we deduce that

$$(3.25) \quad \begin{aligned} \sum_{\substack{M \leq p < y \\ p|q}} \left(1 - \frac{\log p}{\log y}\right) S(\mathcal{A}_p, z) &\leq \frac{x(1 + O(\varepsilon))}{\varphi(q) \log y} \left(\frac{\log(y/M)}{\log N}\right)^2 \\ &= \frac{x(1 + O(\varepsilon)) \log D}{\varphi(q) \log D \log y} \left(\frac{\log(y/M)}{\log N}\right)^2 \\ &= \frac{x(1 + O(\varepsilon))}{\varphi(q) \log D} \frac{6\theta - 7}{\delta} \left(\frac{2(\delta - \theta + 1)}{2\theta - 3}\right)^2. \end{aligned}$$

Finally, combining (3.4), (3.5), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14) and (3.25), we conclude that

$$\begin{aligned} \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1 \\ \Omega(n) \leq 2}} \mathcal{W}(n) &\geq W(\mathcal{A}; z, y) + o\left(\frac{x^{1-\varepsilon}}{\varphi(q)}\right) \\ &\geq \frac{x(1 + O(\varepsilon))}{\varphi(q) \log D} \left\{ 2 \left( \log \frac{18}{5} + \int_3^{18/5} \frac{dt_1}{t_1} \int_2^{t_1-1} \frac{\log(t_2-1)}{t_2} dt_2 \right) \right. \\ &\quad - \frac{6\theta - 7}{2(3\delta - \theta)} \int_{(30\theta-35)/92}^{(12\theta-14)/23} \frac{\delta - \beta}{\beta(3\theta/2 - 7/4 - \beta)} \\ &\quad \times \left( 1 + \int_2^{(108\theta-126-92\beta)/(30\theta-35)} \frac{\log(t-1)}{t} dt \right) d\beta \\ &\quad - \frac{6\theta - 7}{2(3\delta - \theta)} \int_{(12\theta-14)/23}^{\theta-1} \frac{\delta - \beta}{\beta(3\theta/2 - 7/4 - \beta)} d\beta \\ &\quad \left. - \frac{6\theta - 7}{3\delta - \theta} \left(\frac{2(\delta - \theta + 1)}{2\theta - 3}\right)^2 \right\}. \end{aligned}$$

By recalling the parameter  $\delta = 0.86$  and  $\theta = 1.8345$ , then by a simple numerical calculation, we know that the number in the above brackets  $\{ \}$  is  $> 0.0004282583$ . This completes the proof of Theorem 1.1.  $\square$

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