Suixin He; Shuangping Tao

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THE FACTORIZATION OF THE WEIGHTED HARDY SPACE IN TERMS OF MULTILINEAR CALDERÓN-ZYGMUND OPERATORS

Suixin He, Shuangping Tao, Lanzhou

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Abstract. We give a constructive proof of the factorization theorem for the weighted Hardy space in terms of multilinear Calderón-Zygmund operators. The result is also new even in the linear setting. As an application, we obtain the characterization of weighted BMO space via the weighted boundedness of commutators of the multilinear Calderón-Zygmund operators.

Keywords: weighted Hardy space; weighted BMO space; multilinear Calderón-Zygmund operator; weak factorization

MSC 2020: 42B20, 42B35

1. Introduction and statement of main results

The theory of Hardy space has been developed systematically during the past half-century. One of the milestones in this area is due to Coifman, Rochberg and Weiss (see [1]), who gives a constructive proof of the weak factorizations of the classical Hardy space H^1 in terms of Riesz transforms. Later on, Li and Wick in [5] provided a deeper study of the Hardy and BMO spaces associated to the Neumann Laplacian, and they also obtained the classical results in the multilinear setting in [6]. Also, Duong, Li, Wick and Yang in 2016 in [2] obtained the results in the Bessel setting. Recently, Wang and Zhu in [9] considered the weak factorizations of the classical Hardy space for multilinear fractional integral operator and the classical results in the weighted setting in [10].

Inspired by the above articles, we provide a proof of the weak factorization theorem for weighted Hardy space. The results are certainly a contribution to the recent new

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progress by Li and Wick (see [6]) in weighted setting. The arguments in the paper closely follow the arguments in [6] and [10]. However, some of the techniques do not apply to the weighted setting. Therefore, it needs some tedious calculations in applications.

Throughout this paper, by $A \leqslant B$ we mean that $A \leqslant CB$ with a positive constant C independent of the appropriate quantities. If $A \leqslant B$ and $B \leqslant A$, we write $A \approx B$ and say that A and B are equivalent.

1.1. Multilinear Calderón-Zygmund operators. Let us recall that m-linear Calderón-Zygmund operator T is a bounded operator which satisfies

$$\|T(f_1,\dots,f_m)\|_{L^p} \leqslant C\|f_1\|_{L^{p_1}} \times \dots \times \|f_m\|_{L^{p_m}}$$

for some $1 < p_1,\dots,p_m < \infty$ with $1/p = 1/p_1 + \dots + 1/p_m$ and the function K, defined off the diagonal $y_0 = y_1 = \dots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfies the conditions as follows:

(1) The function K satisfies the size condition

$$|K(y_0,y_1,\dots,y_m)| \leqslant \frac{C}{\left(\sum_{k=1}^m |y_k - y_0|\right)^{mn}}.$$

(2) The function K satisfies the regularity condition: Whenever $\varepsilon > 0$ and all $1 \leqslant i \leqslant m$ if $|y_i - y_i'| \leqslant \frac{1}{2} \max\limits_{0\leqslant k\leqslant m} |y_0 - y_k|$,

$$|K(y_0,\dots,y_i,\dots,y_m) - K(y_0,\dots,y_i',\dots,y_m)| \leqslant \frac{C|y_i - y_i'|^{\varepsilon}}{\left(\sum_{k=1}^m |y_k - y_0|\right)^{mn+\varepsilon}}.$$

Then we say K is an m-linear Calderón-Zygmund kernel. If $x \notin \bigcap\limits_{i=1}^m \mathrm{supp} f_i$, then

$$T(f_1,\dots,f_m)(x) = \int_{\mathbb{R}^{mn}} K(x,y_1,\dots,y_m) \prod_{j=1}^m f_j(y_j)\, \mathrm{d}y_1 \dots \mathrm{d}y_m,$$

where $f_1,\dots,f_m$ are m functions on $\mathbb{R}^n$ with $\bigcap\limits_{j=1}^m \mathrm{supp}(f_j) \neq \emptyset$.

We also define that T is mn-homogeneous if T satisfies

$$|T(\chi_{B_0},\dots,\chi_{B_m})(x)| \geqslant \frac{C}{M^{mn}}$$

for balls $B_0 = B(x_0,r),\dots,B_m = B(x_m,r)$ satisfying $|x_l - x_0| = Mr$ for $l = 1,\dots,m$ and $x_0 \in \mathbb{R}^n$, where $r > 0$ and $M \geqslant 10$ is a positive constant.

1.2. Muckenhoupt weights. We need the notion of weighted L^p space: $L^p(w) = L^p(\mathbb{R}^n, \omega\,\mathrm{d}x)$ denotes the collection of measurable functions f on ω such that

$$\|f\|_{L^p(\omega)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\,\mathrm{d}x\right)^{1/p} < \infty.$$

We recall the definition of A_p weight introduced by Muckenhoupt in [7], which gives the characterization of all weights $\omega(x)$ such that the Hardy-Littlewood maximal operator

$$M(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y$$

is bounded on $L^p(\omega)$. For $1 < p < \infty$ and a nonnegative locally integrable function ω on $\mathbb{R}^n$, the weight ω is in the Muckenhoupt A_p class if it satisfies the condition

$$[\omega]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}\,\mathrm{d}x\right)^{p-1} < \infty.$$

And a weight function ω belongs to the class A_1 if

$$[\omega]_{A_1} := \frac{1}{|Q|}\int_Q \omega(x)\,\mathrm{d}x\left(\operatorname*{ess\,sup}_{x\in Q} \omega(x)^{-1}\right) < \infty.$$

We write $A_\infty = \bigcup_{1\leqslant p<\infty} A_p$. For $\omega \in A_\infty$ there exist $0 < \varepsilon$, $L < \infty$ such that for all measurable subsets S of cube Q,

$$\frac{\omega(S)}{\omega(Q)} \leqslant C\left(\frac{|S|}{|Q|}\right)^{\varepsilon} \tag{1.1}$$

and

$$\left(\frac{|S|}{|Q|}\right)^{L} \leqslant C\frac{\omega(S)}{\omega(Q)}. \tag{1.2}$$

In the celebrated work [4] Lerner et al. established a theory of weights adapted to the multilinear setting and resolved the problems proposed in [3]. For $1 < p_1, \ldots, p_m < \infty$, $\vec{p} = (p_1, p_2, \ldots, p_m)$, and p such that $1/p_1 + \ldots + 1/p_m = 1/p$, a vector weight $\vec{\omega} = (\omega_1, \omega_2, \ldots, \omega_m)$ belongs to $A_{\vec{p}}$ if

$$[\vec{\omega}]_{A_{\vec{p}}} = \sup_Q \left(\frac{1}{|Q|}\int_Q \prod_{i=1}^m \omega_i(x)^{p/p_i}\,\mathrm{d}x\right)\prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q \omega_i(x)^{1-p_i'}\,\mathrm{d}x\right)^{p/p_i'} < \infty.$$

For brevity, we will often use the notation $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$ in the first integral.

1.3. Atomic decomposition of the weighted Hardy spaces. Let $0 < p \leqslant 1 < q \leqslant \infty$, $s \in \mathbb{Z}$ satisfying $s \geqslant [n(1/p-1)]$. A function a is called a (p, q, s) atom if there exists a cube Q such that

(i) a is supported in Q;

(ii) $\|a\|_{L^q(\omega)} \leqslant \omega(Q)^{1/q-1/p}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\gamma \,\mathrm{d}x = 0$ with $|\gamma| \leqslant s$.

Let us recall the definition of $H^p(\omega)$ using the above atoms.

$$H^p(\omega) = \Big\{ f \in \mathcal{S}' \colon\ f(x) \overset{\mathcal{S}'}{=} \sum_k \lambda_k a_k(x),\ \text{each } a_k \text{ is } a(p,q,s) \text{ atom, and } \sum_k |\lambda_k|^p < \infty \Big\},$$

setting $H^p(\omega)$ norm of f by

$$\|f\|_{H^p(\omega)} = \inf \Big(\sum_k |\lambda_k|^p \Big)^{1/p},$$

where the infimum is taken over all decompositions of $f = \sum\limits_k \lambda_k a_k$ above.

1.4. Main results. Our main result is then the following factorization result for $H^1(\omega)$ in terms of the multilinear operator Π_l. This result is new even in the linear case. The multilinear operator Π_l is defined as

$$\Pi_l(g, h_1, \ldots, h_m)(x) := h_l (T^*)_l (h_l, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x) - gT(h_1, \ldots, h_m)(x),$$

where $(T^*)_l$ is the lth partial adjoint of T. It is easy to see that $(T^*)_l$ is also an m-linear Calderón-Zygmund operator.

Theorem 1.1. *Let $1 \leqslant l \leqslant m$, $1 < p_1, \ldots, p_m, p < \infty$, $1/p_1 + \ldots + 1/p_m = 1/p$ and $\omega \in A_1$. Then for any $f \in H^1(\omega)$ there exists sequences $\{\lambda_s^k\} \in \ell_1$ and functions $g_s^k \in L^{p'}(\omega)$, $h_{s,1}^k \in L^{p_1}(\omega), \ldots, h_{s,m}^k \in L^{p_m}(\omega)$ such that*

$$f = \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_s^k \ \Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k) \quad \text{in } \mathcal{S}'. \tag{1.3}$$

Moreover,

$$\|f\|_{H^1(\omega)} \approx \inf \Big\{ \sum_{k=1}^\infty \sum_{s=1}^\infty |\lambda_s^k| \|g_s^k\|_{L^{p'}(\omega)} \|h_{s,1}^k\|_{L^{p_1}(\omega)} \cdots \|h_{s,m}^k\|_{L^{p_m}(\omega)} \Big\},$$

where the infimum above is taken over all possible representations as in (1.3).

As a direct application, we will give the characterization of the weighted the BMO space via the boundedness of commutators of the multilinear singular integral operator. In analogy with the linear case, we define the lth possible multilinear commutators of the multilinear Calderón-Zygmund operator T as

$$[b,T]_l(f_1,\ldots,f_m)(x)=T(f_1,\ldots,bf_l,\ldots,f_m)(x)-bT(f_1,\ldots,f_m)(x).$$

It is proved in [8] that if $\omega\in A_1$, $1<p_1,\ldots,p_m<\infty$ and $1/p=1/p_1+\ldots+1/p_m$, then the commutator $[b,T]$ is bounded from $L^{p_1}(\omega)\times\ldots\times L^{p_m}(\omega)$ to $L^p(\omega^{1-p})$ if and only if $b\in\mathrm{BMO}(\omega)$, that is,

$$\|b\|_{\mathrm{BMO}(\omega)}:=\sup_Q\frac{1}{\omega(Q)}\int_Q|b(x)-b_Q|\,\mathrm{d}x<\infty.$$

The methods used in [8] lie in expanding the kernel locally by Fourier series, which leads to a very strong assumption on the corresponding kernel. In this paper, we give a characterization of the weighted BMO space for the mn-homogeneous Calderón-Zygmund operators, using the duality theorem between $H^1(\omega)$ and $\mathrm{BMO}(\omega)$.

Theorem 1.2. *Let $b\in L^1_{\mathrm{loc}}$, $1\leqslant l\leqslant m$, $0<\alpha<mn$, $1<p_1,\ldots,p_m,q<\infty$, $1/p_1+\ldots+1/p_m-1/p=\alpha/n$ and $\vec{\omega}\in A_{\vec{p},q}$. The commutator $[b,T]_l$ is bounded from $L^{p_1}(\omega)\times\ldots\times L^{p_m}(\omega)$ to $L^p(\omega^{1-p})$ if and only if $b\in\mathrm{BMO}(\omega)$.*

2. Auxiliary lemmas

In 2018, Li and Wick in [6] showed a technical lemma about certain H^1 as follows.

Lemma 2.1. *Suppose f is a function defined on $\mathbb{R}^n$ satisfying $\int_{\mathbb{R}^n}f(x)\,\mathrm{d}x=0$ and $|f(x)|\leqslant\chi_{B(x_0,1)}(x)+\chi_{B(y_0,1)}(x)$, where $|x_0-y_0|:=M>10$. Then we have*

$$\|f\|_{H^1}\leqslant C_n\log M.$$

In Lemma 2.1, we see that the function f satisfies

$$|f(x)|\leqslant h_1(x)\chi_{B(x_0,1)}(x)+h_2(x)\chi_{B(y_0,1)}(x)$$

with $h_1(x)\equiv h_2(x)\equiv 1$. However, we need to consider the case when h_1, h_2 do not belong to L^∞ arising from the weighted setting. Therefore, it needs some tedious calculations in applications.

Lemma 2.2. *Let $\omega \in A_1$, $1 < q \leqslant \infty$. If the function f satisfies the estimates:*

(i) $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = 0$,

(ii) *there exist $x_1, x_2 \in \mathbb{R}^n$ and $r > 0$ such that*

$$|f(x)| \leqslant h_1(x)\chi_{B_1}(x) + h_2(x)\chi_{B_2}(x) \quad \text{for } |x_1 - x_2| \geqslant 4r,$$

where $\|h_i\|_{L^q(\omega)} \leqslant C\omega(B_i)^{-1/q'}$ and $B_i = B(x_i, r)$, $i = 1, 2$ then there exists a positive constant C independent of x_1, x_2, r such that

$$\|f\|_{H^1(\omega)} \leqslant C \log \frac{|x_1 - x_2|}{r}.$$

P r o o f. Assume that $f := f_1 + f_2$, where $|f_i| \leqslant h_i$ and supp $f_i \subset B_i$ for $i = 1, 2$. We will show that f has the following atomic decomposition:

$$f = \sum_{i=1}^{2} \sum_{j=1}^{J_0+1} \lambda_i^j a_i^j, \tag{2.1}$$

where J_0 is the smallest integer larger than $\log|x_1 - x_2|/r$ and for each j, a_i^j is an atom and λ_i^j a real number satisfying

$$|\lambda_i^j| \leqslant 1. \tag{2.2}$$

To this end, for $i = 1, 2$ we write

$$f_i(x) = [f_i(x) - \tilde{\lambda}_i^1 \chi_{B_i}] + \tilde{\lambda}_i^1 \chi_{B_i} =: f_i^1(x) + \tilde{\lambda}_i^1 \chi_{B_i},$$

where

$$\tilde{\lambda}_i^1 := \frac{1}{|B_i|} \int_{B_i} f_i(x)\,\mathrm{d}x.$$

By (ii) and the direct calculation, we get $\lambda_i^1 := \|f_i^1\|_{L^q(\omega)} \omega(B_i)^{1/q'} \leqslant 1$. We write $a_i^1 := f_i^1/\lambda_i^1$. From the fact that

$$\|a_i^1\|_{L^q(\omega)} = \frac{\|f_i^1\|_{L^q(\omega)}}{\lambda_i^1} \leqslant \omega(B_i)^{1/q-1},$$

we know that a_i^1 is a $(1, q, 0)$-atom supported on B_i and λ_i^1 satisfies (2.2). We further write

$$\tilde{\lambda}_i^1 \chi_{B_i} = \tilde{\lambda}_i^1 \chi_{B_i} - \tilde{\lambda}_i^2 \chi_{2B_i} + \tilde{\lambda}_i^2 \chi_{2B_i} =: f_i^2 + \tilde{\lambda}_i^2 \chi_{2B_i},$$

where

$$\tilde{\lambda}_i^2 := \frac{1}{|2B_i|} \int_{2B_i} f_i(x)\,\mathrm{d}x.$$

Let $\lambda_i^2 := \|f_i^2\|_{L^q(\omega)} \omega(2B_i)^{1/q'}$ and $a_i^2 := f_i^2/\lambda_i^2$. Then we see that a_i^2 is an atom supported on $2B_i$ and

$$|\lambda_i^2| \leqslant (|\tilde{\lambda}_i^1| + |\tilde{\lambda}_i^2|)\omega(2B_i) \leqslant 1.$$

Continuing in this process with $j \in 2, 3, \ldots, J_0$,

$$\tilde{\lambda}_i^j := \frac{1}{|2^j B_i|} \int_{2^j B_i} f_i(x)\,\mathrm{d}x, \quad f_i^j := \tilde{\lambda}_i^{j-1} \chi_{2^{j-1} B_i} - \tilde{\lambda}_i^j \chi_{2^j B_i},$$
$$\lambda_i^j := \|f_i^j\|_{L^q(\omega)} \omega(2^j B_i)^{1/q'}, \quad a_i^j := \frac{f_i^j}{\lambda_i^j},$$

we obtain that

$$f = \sum_{i=1}^{2} \Big[\sum_{j=1}^{J_0} f_i^j \Big] + \sum_{i=1}^{2} \tilde{\lambda}_i^{J_0} \chi_{2^{J_0} B_i} = \sum_{i=1}^{2} \Big[\sum_{j=1}^{J_0} \lambda_i^j a_i^j \Big] + \sum_{i=1}^{2} \tilde{\lambda}_i^{J_0} \chi_{2^{J_0} B_i},$$

where each i and j, a_i^j is a $(1, q, 0)$-atom and $\lambda_i^j \leqslant 1$.

For $\sum_{i=1}^{2} \tilde{\lambda}_i^{J_0} \chi_{2^{J_0} B_i}$ we set

$$\tilde{\lambda}^{J_0} := \frac{1}{|B((x_1 + x_2)/2, 2^{J_0+1} r)|} \int_{B(x_1, r)} f_1(x)\,\mathrm{d}x.$$

By the cancellation condition (i) of f, we arrive at

$$\tilde{\lambda}^{J_0} = -\frac{1}{|B((x_1 + x_2)/2, 2^{J_0+1} r)|} \int_{B(x_2, r)} f_2(x)\,\mathrm{d}x.$$

It follows that

$$\begin{aligned}\sum_{i=1}^{2} \tilde{\lambda}_i^{J_0} \chi_{B(x_i, 2^{J_0} r)} &= [\tilde{\lambda}_1^{J_0} \chi_{B(x_1, 2^{J_0} r)} - \tilde{\lambda}^{J_0} \chi_{B((x_1+x_2)/2, 2^{J_0+1} r)}] \\ &\quad + [\tilde{\lambda}^{J_0} \chi_{B((x_1+x_2)/2, 2^{J_0+1} r)} + \tilde{\lambda}_2^{J_0} \chi_{B(x_2, 2^{J_0} r)}] \\ &=: \sum_{i=1}^{2} f_i^{J_0+1}.\end{aligned}$$

For $i = 1, 2$, let

$$\lambda_i^{J_0+1} := \|f_i^{J_0+1}\|_{L^q(\omega)} \omega\Big(B\Big(\frac{x_1 + x_2}{2}, 2^{J_0+1} r\Big)\Big) \quad \text{and} \quad a_i^{J_0+1} := \frac{f_i^{J_0+1}}{\lambda_i^{J_0+1}}.$$

Also, $a_i^{J_0+1}$ is a $(1, q, 0)$-atom and $\lambda_i^{J_0+1}$ satisfies (2.2). Thus, we have that (2.1) holds, which implies that $f \in H^1(\omega)$ with

$$\|f\|_{H^1(\omega)} \leqslant \sum_{i=1}^{2} \sum_{j=1}^{J_0+1} |\lambda_i^j| \leqslant \log \frac{|x_1 - x_2|}{r}.$$

This finishes the proof of Lemma 2.1. □

Lemma 2.3. *Suppose* $1 \leqslant l \leqslant m$, $1 < p_1, \dots, p_m < \infty$ *with*

$$\frac{1}{p_1} + \ldots + \frac{1}{p_m} = \frac{1}{p}.$$

There exists a positive constant C *such that for any* $g \in L^{p'}(\omega)$ *and* $h_i \in L^{p_i}(\omega)$, $i = 1, 2, \dots, m$,

$$\|\Pi_l(g, h_1, \dots, h_m)\|_{H^1(\omega)} \leqslant C\|g\|_{L^{p'}(\omega)}\|h_1\|_{L^{p_1}(\omega)} \dots \|h_m\|_{L^{p_m}(\omega)}.$$

Proof. Note that for any $g \in L^{p'}(\omega)$ and $h_i \in L^{p_i}(\omega)$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |g(x)T(h_1, \dots, h_m)(x)|\omega(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n} |g(x)|\omega(x)^{1/p'}|T(h_1, \dots, h_m)(x)|\omega(x)^{1/p}\,\mathrm{d}x \\ &\leqslant \|g\|_{L^{p'}(\omega)}\|T(h_1, \dots, h_m)\|_{L^p(\omega)} \\ &\leqslant C\|g\|_{L^{p'}(\omega)}\prod_{i=1}^{m}\|h_i\|_{L^{p_i}(\omega)}.\end{aligned}$$

A direct calculation gives us that

$$\frac{1}{p_1} + \ldots \frac{1}{p_{l-1}} + \frac{1}{p'} + \frac{1}{p_{l+1}} + \ldots + \frac{1}{p_m} = \frac{1}{p_l'},$$

then we obtain the weighted boundedness of the operator

$$(T^*)_l \colon L^{p_1}(\omega) \times \ldots \times L^{p_{l-1}}(\omega) \times L^{p'}(\omega) \times L^{p_{l+1}}(\omega) \times \ldots \times L^{p_m}(\omega) \to L^{p_l'}(\omega),$$

since $(T^*)_l$ is also an m-linear Calderón-Zygmund operator. This implies that $\Pi_l(g, h_1, \dots, h_m)(x) \in L^1(\omega)$ by Hölder's inequality. Moreover,

$$\int_{\mathbb{R}^n} \Pi_l(g, h_1, \dots, h_m)(x)\,\mathrm{d}x = 0.$$

Hence, for $b \in \mathrm{BMO}(\omega)$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} b(x)\Pi_l(g, h_1, \dots, h_m)(x)\,\mathrm{d}x\right| &= \left|\int_{\mathbb{R}^n} g(x)[b, T]_l(h_1, \dots, h_m)(x)\,\mathrm{d}x\right| \\ &= \left|\int_{\mathbb{R}^n} g(x)\omega(x)^{1/p'}[b, T]_l(h_1, \dots, h_m)(x)\omega(x)^{-1/p'}\,\mathrm{d}x\right| \\ &\leqslant \|g\|_{L^{p'}(\omega)} \cdot \|[b, T]_l(h_1, \dots, h_m)\|_{L^p(\omega^{1-p})} \\ &\leqslant C\|h_1\|_{L^{p_1}(\omega)} \dots \|h_m\|_{L^{p_m}(\omega)}\|g\|_{L^{p'}(\omega)}\|b\|_{\mathrm{BMO}(\omega)}.\end{aligned}$$

Therefore, $\Pi_l(g, h_1, \dots, h_m)$ is in $H^1(\omega)$ with

$$\|\Pi_l(g, h_1, \dots, h_m)\|_{H^1(\omega)} \leqslant C\|g\|_{L^{p'}(\omega)}\|h_1\|_{L^{p_1}(\omega)} \dots \|h_m\|_{L^{p_m}(\omega)}.$$

The proof of Lemma 2.2 is completed. □

The arguments in the paper closely follow the arguments in [6] and the ideas from [6] are further combined with some modifications arising from the weighted setting.

Lemma 2.4. *Let $1 \leqslant l \leqslant m$, $1 < p_1, \ldots, p_m, q < \infty$, $1/p_1 + \ldots + 1/p_m = 1/p$ and $\omega \in A_1$. For every $H^1(\omega)$-atom $a(x)$ there exists $g \in L^{p'}(\omega)$ and $h_i \in L^{p_i}(\omega)$, $i = 1, 2, \ldots, m$ and a large positive number M (depending only on ε) such that*

$$\|a - \Pi_l(g, h_1, h_2, \ldots, h_m)\|_{H^1(\omega)} < \varepsilon$$

and that $\|g\|_{L^{p'}(\omega)} \|h_1\|_{L^{p_1}(\omega)} \ldots \|h_m\|_{L^{p_m}(\omega)} \leqslant CM^{mn(1+L)}$, where L is defined in (1.2).

P r o o f. Let $a(x)$ be an $H^1(\omega)$-atom supported in $B(x_0, r)$, satisfying that

$$\int_{\mathbb{R}^n} a(x)\,\mathrm{d}x = 0 \quad \text{and} \quad \|a\|_{L^q(\omega)} \leqslant \omega(B(x_0, r))^{1/q-1}.$$

Fix $1 \leqslant l \leqslant m$. Now select $y_l \in \mathbb{R}^n$ so that $y_{l,i} - x_{0,i} = Mr/\sqrt{n}$, where $x_{0,i}$ (or $y_{l,i}$) is the ith coordinate of x_0 (or y_l) for $i = 1, 2, \ldots, n$. Note that for this y_l we have $|x_0 - y_l| = Mr$. Similarly to the relation of x_0 and y_l, we choose y_1 such that x_0 and y_1 satisfies the same relationship as x_0 and y_l. Then by induction we choose $y_2, \ldots, y_{l-1}, y_{l+1}, \ldots, y_m$. We write $B_i = B(y_i, r)$ and set

$$g(x) := \chi_{B_l}(x), \quad h_j(x) := \chi_{B_j}(x),\ j \neq l,$$
$$h_l(x) = \frac{a(x)}{(T^*)_l(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0)} \chi_{B_l}(x).$$

It follows from the specific choice of the functions $h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m$ that

$$|(T^*)_l(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0)| \geqslant \int_{B_1 \times \ldots \times B_m} \frac{1}{(|x_0 - z_1| + \ldots + |x_0 - z_m|)^{mn}}\,\mathrm{d}z_1 \ldots \mathrm{d}z_m \geqslant CM^{-mn}.$$

The definitions of the functions $g(x)$ and $h_j(x)$ give us that $\operatorname{supp} g = B(y_l, r)$ and $\operatorname{supp} h_j = B(y_j, r)$. Moreover,

$$\|g\|_{L^{p'}(\omega)} = \omega(B_l)^{1/p'} \quad \text{and} \quad \|h_j\|_{L^{p_j}(\omega)} = \omega(B_j)^{1/p_j}$$

for $i = 1, \ldots, l-1,\ l+1, \ldots, m$. Also,

$$\|h_l\|_{L^{p_l}(\omega)} = \frac{1}{|(T^*)_l(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0)|} \|a\|_{L^{p_l}(\omega)} \leqslant CM^{mn} \omega(B_l)^{-1/p_l'}.$$

From inequality (1.2) we have

$$\omega(B_j) \leqslant \omega((M+1)B_l) \leqslant M^{nL}\omega(B_l).$$

It is easy to see that

$$\|g\|_{L^{p'}(\omega)}\|h_1\|_{L^{p_1}(\omega)} \dots \|h_m\|_{L^{p_m}(\omega)} \leqslant M^{mn(1+L)}.$$

Next, we have

$$\begin{aligned}
&a(x) - \Pi_l(g, h_1, h_2, \dots, h_m)(x) \\
&\quad = a(x) - [h_l(T^*)_l(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m) - gT(h_1, \dots, h_m)(x)] \\
&\quad = a(x)\frac{\Upsilon}{(T^*)_l(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0)} + g(x)T(h_1, \dots, h_m)(x) \\
&\quad =: W_1(x) + W_2(x),
\end{aligned}$$

where

$$\Upsilon = (T^*)_l(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0) - (T^*)_l(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x).$$

It is obvious that $W_1(x)$ is supported on $B(x_0, r)$ and $W_2(x)$ is supported on $B(y_0, r)$.

We first estimate $W_1(x)$. For $x \in B(x_0, r)$ we have

$$\begin{aligned}
|W_1(x)| &= |a(x)|\frac{|\Upsilon|}{|(T^*)_l(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0)|} \\
&\leqslant \frac{|a(x)|}{M^{-mn}} \int_{\prod_{j=1}^{m} B(y_j, r)} \frac{|x - x_0|^{\varepsilon}}{\left(\sum_{i=1, i\neq l}^{m} |z_l - z_i| + |z_l - x_0|\right)^{mn+\varepsilon}} \,\mathrm{d}z_1 \dots \mathrm{d}z_m \\
&\leqslant \frac{|a(x)|}{M^{\varepsilon}}
\end{aligned}$$

with

$$\Upsilon = (T^*)_l(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0) - (T^*)_l(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x).$$

Hence, we conclude that

$$|W_1(x)| \leqslant \frac{|a(x)|}{M^{\varepsilon}}\chi_{B(x_0, r)}(x).$$

Next, we estimate $W_2(x)$. From the definition of $g(x)$ and $h_i(x)$ we have

$$\begin{aligned}
&|T(h_1,\dots,h_m)(x)|\\
&\quad= \frac{1}{|(T^*)_l(h_1,\dots,h_{l-1},g,h_{l+1},\dots,h_m)(x_0)|}\\
&\qquad\times\biggl|\int_{\prod_{j\neq l} B(y_j,r)\times B(x_0,r)}\bigl(K(z_1,\dots,z_{l-1},x_0,z_{l+1},\dots,z_m)(x_0)\\
&\qquad - K(z_1,\dots,z_{l-1},x,z_{l+1},\dots,z_m)(x_0)\bigr)a(z_l)\prod_{j\neq l}h_j(z_j)\,\mathrm{d}z_1\dots\mathrm{d}z_m\biggr|\\
&\quad\leqslant M^{mn}\int_{\prod_{j\neq l} B(y_j,r)\times B(x_0,r)}\frac{|a(z_l)||x_0-x|^{\varepsilon}}{\bigl(\sum_{s=1}^m|x_0-z_s|\bigr)^{mn+\varepsilon}}\,\mathrm{d}z_1\,\mathrm{d}z_1\dots\mathrm{d}z_m\\
&\quad\leqslant M^{mn}\frac{r^{\varepsilon}r^{(m-1)n}}{(Mr)^{mn+\varepsilon}}\int_{B(y_l,r)}|a(z_l)|\,\mathrm{d}z_l\leqslant\frac{1}{M^{\varepsilon}r^n}\|a\|_{L^q(\omega)}\biggl(\int_{B_l}\omega(z_l)^{1-q}\,\mathrm{d}z_l\biggr)^{1/q'}\\
&\quad\leqslant\frac{1}{M^{\varepsilon}\omega(B_l)},
\end{aligned}$$

where in the second equality we use the cancellation property of the atom $a(z_l)$. It follows that

$$|W_2(x)|\leqslant\frac{1}{M^{\varepsilon}r^n}\chi_{B(y_l,r)}(x).$$

The estimates of $W_1(x)$ and $W_2(x)$ imply that

$$|a(x)-\Pi_l(g,h_1,\dots,h_m)(x)|\leqslant\frac{|a(x)|}{M^{\varepsilon}}\chi_{B(x_0,r)}(x)+\frac{1}{M^{\varepsilon}\omega(B_l)}\chi_{B(y_l,r)}(x). \tag{2.3}$$

Notice that

$$\int_{\mathbb{R}^n}[a(x)-\Pi_l(g,h_1,\dots,h_m)(x)]\,\mathrm{d}x=0, \tag{2.4}$$

because the atom $a(x)$ has cancellation property and the second integral equals 0 just by the definitions of Π_l. Then inequality (2.3) and cancellation (2.4), together with Lemma 2.1, show that

$$\|a(x)-\Pi_l(g,h_1,\dots,h_m)(x)\|_{H^1(\omega)}\leqslant C\frac{\log M}{M^{\varepsilon}}.$$

For M sufficiently large such that

$$\frac{C\log M}{M^{\varepsilon}}<\varepsilon.$$

Thus, the result follows from here. □

With this approximation result above, we can give the proof of the main Theorem 1.1.

P r o o f of Theorem 1.1. By Lemma 2.2, it is obvious that

$$\|\Pi_l(g,h_1,\ldots,h_m)(x)\|_{H^1(\omega)} \leqslant C\|g\|_{L^{p'}(\omega)}\|h_1\|_{L^{p_1}(\omega)}\ldots\|h_m\|_{L^{p_m}(\omega)}.$$

It is immediate that for any representation of f as in (1.3), i.e.,

$$f=\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}\lambda_s^k\Pi_l(g_s^k,h_{s,1}^k,\ldots,h_{s,m}^k)(x)$$

with

$$\|f\|_{H^1(\omega)}\leqslant C\inf\left\{\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|\lambda_s^k|\|g_s^k\|_{L^{p'}(\omega)}\|h_{s,1}^k\|_{L^{p_1}(\omega)}\ldots\|h_{s,m}^k\|_{L^{p_m}(\omega)}\right\},$$

where the infimum above is taken over all possible representations of f that satisfy (1.3).

Next, we will show that the other inequality holds and that it is possible to obtain such a decomposition for any $f\in H^1(\omega)$. Applying the atomic decomposition, for any $f\in H^1(\omega)$ we can find a sequence $\{\lambda_s^1\}\in\ell^1$ and a sequence of $H^1(\omega)$-atom $\{a_s^1\}$ so that $f=\sum\limits_{s=1}^{\infty}\lambda_s^1a_s^1$ and $\sum\limits_{s=1}^{\infty}|\lambda_s^1|\leqslant C\|f\|_{H^1(\omega)}$.

We adopt some arguments from [1] (or [6], [9]). Let $\varepsilon>0$ be small enough such that $C\varepsilon<1$. For each atom a_s^1 we apply Lemma 2.4 to find the functions $g_s^1\in L^{p'}(\omega)$, $h_{s,1}^1\in L^{p_1}(\omega),\ldots,h_{s,m}^1\in L^{p_m}(\omega)$ with

$$\left\|a_s^1-\prod_{j,l}(g_s^1,h_{s,1}^1,\ldots,h_{s,m}^1)\right\|_{H^1(\omega)}<\varepsilon\quad\forall\, s$$

and

$$\|g_s^1\|_{L^{p'}(\omega)}\|h_1\|_{L^{p_1}(\omega)}\ldots\|h_m\|_{L^{p_m}(\omega)}\leqslant C(\varepsilon,L),$$

where $C(\varepsilon,L)=CM^{mn(1+L)}$ is a constant depending on ε and L. We write

$$f=\sum_{s=1}^{\infty}\lambda_s^1a_s^1=:M_1+E_1,$$

where

$$M_1=\sum_{s=1}^{\infty}\lambda_s^1\Pi_l(g_s^1,h_{s,1}^1,\ldots,h_{s,m}^1)(x),$$

$$E_1=\sum_{s=1}^{\infty}\lambda_s^1(a_s^1-\Pi_l(g_s^1,h_{s,1}^1,\ldots,h_{s,m}^1)).$$

Notice that

$$\|E_1\|_{H^1(\omega)}\leqslant\sum_{s=1}^{\infty}|\lambda_s^1|\ \|a_s^1-\Pi_l(g_s^1,h_{s,1}^1,\ldots,h_{s,m}^1)\|_{H^1(\omega)}\leqslant\varepsilon\sum_{s=1}^{\infty}|\lambda_s^1|\leqslant\varepsilon C\|f\|_{H^1(\omega)}.$$

Meanwhile, we can also find a sequence $\{\lambda_s^2\} \in l^1$ and a sequence of $H^1(\omega)$-atom $\{a_s^2\}$ such that $E_1 = \sum\limits_{s=1}^{\infty} \lambda_s^2 a_s^2$ and

$$\sum_{s=1}^{\infty} |\lambda_s^2| \leqslant C\|E_1\|_{H^1(\omega)} \leqslant \varepsilon C^2 \|f\|_{H^1(\omega)}.$$

Again, by applying Lemma 2.4 to each atom a_s^2, there exists $g_s^2 \in L^{p'}(\omega)$, $h_{s,1}^2 \in L^{p_1}(\omega), \ldots, h_{s,m}^2 \in L^{p_m}(\omega)$ with

$$\Big\| a_s^2 - \prod_{j,l}(g_s^2, h_{s,1}^2, \ldots, h_{s,m}^2) \Big\|_{H^1(\omega)} < \varepsilon \quad \forall\, s.$$

We then have that $E_1 = M_2 + E_2$ with

$$M_2 = \sum_{s=1}^{\infty} \lambda_s^2 a_s^2 = \sum_{s=1}^{\infty} \lambda_s^2 \Pi_l(g_s^2, h_{s,1}^2, \ldots, h_{s,m}^2)(x),$$
$$E_2 = \sum_{s=1}^{\infty} \lambda_s^2 (a_s^2 - \Pi_l(g_s^2, h_{s,1}^2, \ldots, h_{s,m}^2)).$$

Observe that

$$\|E_2\|_{H^1(\omega)} \leqslant \sum_{s=1}^{\infty} |\lambda_s^2|\ \|a_s^2 - \Pi_l(g_s^2, h_{s,1}^2, \ldots, h_{s,m}^2)\|_{H^1(\omega)}$$
$$\leqslant \varepsilon \sum_{s=1}^{\infty} |\lambda_s^2| \leqslant (\varepsilon C)^2 \|f\|_{H^1(\omega)},$$

then

$$f = M_1 + E_1 = M_1 + M_2 + E_2 = \sum_{k=1}^{2} \sum_{s=1}^{\infty} \lambda_s^k \Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k) + E_2.$$

The same argument above shows that for each $1 \leqslant k \leqslant K$ produces functions $g_s^k \in L^{p'}(\omega)$, $h_{s,1}^k \in L^{p_1}(\omega), \ldots, h_{s,m}^k \in L^{p_m}(\omega)$ with

$$\|g_s^k\|_{L^{q'}(\mu_{\varpi})} \|h_{s,1}^k\|_{L^{p_1}(\omega_1^{p_1})} \cdots \|h_{s,m}^k\|_{L^{p_m}(\omega_m^{p_m})} \leqslant C(\varepsilon, L) \quad \forall\, s,$$

sequences $\{\lambda_s^k\} \in l^1$ with $\|\lambda_s^k\|_{l_1} \leqslant \varepsilon^{k-1} C^k \|f\|_{H^1(\omega)}$, and a function $E_K \in H^1(\omega)$ with

$$\|E_K\|_{H^1(\omega)} \leqslant (C\varepsilon)^K \|f\|_{H^1(\omega)} \quad \text{and} \quad f = \sum_{k=1}^{K} \sum_{s=1}^{\infty} \lambda_s^k \Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k) + E_k.$$

Letting $K \to \infty$ gives the desired decomposition of

$$f = \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \lambda_s^k \Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k).$$

We conclude that

$$\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} |\lambda_s^k| \leqslant \sum_{k=1}^{\infty} \varepsilon^{-1}(C\varepsilon)^K \|f\|_{H^1(\omega)} = \frac{C}{1-\varepsilon C}\|f\|_{H^1(\omega)}.$$

Thus, we have completed the proof of Theorem 1.1. □

Finally, we dispense with the proof of Theorem 1.2.

Proof of Theorem 1.2. The upper bound in this theorem is contained in [8]. For the lower bound, suppose that $f \in H^1(\omega)$, using the weak factorization in Theorem 1.1 and the weighted boundedness of $[b, T]_l$, we obtain

$$\begin{aligned} \langle b, f\rangle_{L^2} &= \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \lambda_s^k \langle b, \Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k)\rangle_{L^2} \\ &= \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \lambda_s^k \langle g_s^k, [b, T]_l(h_{s,1}^k, \ldots, h_{s,m}^k)\rangle_{L^2}. \end{aligned}$$

Hence, we have that

$$\begin{aligned} |\langle b, f\rangle_{L^2}| &\leqslant \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} |\lambda_s^k| \ \|g_s^k\|_{L^{p'}(\omega)} \|[b, T]_l(h_{s,1}^k, \ldots, h_{s,m}^k)\|_{L^p(\omega^{1-p})} \\ &\leqslant \|[b, T]_l \colon L^{p_1}(\omega) \times \ldots \times L^{p_m}(\omega) \to L^p(\omega^{1-p})\| \\ &\quad \times \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} |\lambda_s^k| \|g_s^k\|_{L^{p'}(\omega)} \prod_{j=1}^{m} \|h_{s,j}^k\|_{L^{p_j}(\omega)} \\ &\leqslant C\|[b, T]_l \colon L^{p_1}(\omega) \times \ldots \times L^{p_m}(\omega) \to L^p(\omega^{1-p})\| \ \|f\|_{H^1(\omega)}. \end{aligned}$$

From the duality theorem between $H^1(\omega)$ and $\mathrm{BMO}(\omega)$ we get

$$\begin{aligned} \|b\|_{\mathrm{BMO}(\omega)} &\approx \sup_{\|f\|_{H^1(\omega)} \leqslant 1} |\langle b, f\rangle_{L^2}| \\ &\leqslant C\|[b, T]_l \colon L^{p_1}(\omega) \times \ldots \times L^{p_m}(\omega) \to L^p(\omega^{1-p})\|, \end{aligned}$$

it follows that $b \in \mathrm{BMO}(\omega)$. □

References

[1] *R. R. Coifman, R. Rochberg, G. Weiss*: Factorization theorems for Hardy spaces in several variables. Ann. Math. (2) *103* (1976), 611–635. zbl MR doi

[2] *X. T. Duong, J. Li, B. D. Wick, D. Yang*: Factorization for Hardy spaces and characterization for BMO spaces via commutators in the Bessel setting. Indiana Univ. Math. J. *66* (2017), 1081–1106. zbl MR doi

[3] *L. Grafakos, R. H. Torres*: Maximal operator and weighted norm inequalities for multilinear singular integrals. Indiana Univ. Math. J. *51* (2002), 1261–1276. zbl MR doi

[4] *A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres, R. Trujillo-González*: New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory. Adv. Math. *220* (2009), 1222–1264. zbl MR doi

[5] *J. Li, B. D. Wick*: Characterizations of $H^1_{\Delta_N}(\mathbb{R}^n)$ and $\mathrm{BMO}_{\Delta_N}(\mathbb{R}^n)$ via weak factorizations and commutators. J. Funct. Anal. *272* (2017), 5384–5416. zbl MR doi

[6] *J. Li, B. D. Wick*: Weak factorizations of the Hardy space $H^1(\mathbb{R}^n)$ in terms of multilinear Riesz transforms. Can. Math. Bull. *60* (2017), 571–585. zbl MR doi

[7] *B. Muckenhoupt*: Weighted norm inequalities for the Hardy maximal function. Trans. Am. Math. Soc. *165* (1972), 207–226. zbl MR doi

[8] *D. H. Wang, J. Zhou, Z. D. Teng*: Characterizations of weighted BMO space and its application. Acta. Math. Sin., Engl. Ser. *37* (2021), 1278–1292. zbl MR doi

[9] *D. Wang, R. Zhu*: Weak factorizations of the Hardy space in terms of multilinear fractional integral operator. Available at `https://arxiv.org/abs/2112.06249v1` (2021), 12 pages.

[10] *D. Wang, R. Zhu, L. Shu*: The factorizations of $H^{\varrho}(\mathbb{R}^n)$ via multilinear Calderón-Zygmund operators on weighted Lebesgue spaces. Available at `https://arxiv.org/abs/2112.06252v1` (2021), 22 pages.

Authors' address: Suixin He, Shuangping Tao (corresponding author), College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu, 730070, P. R. China, e-mail: `hesuixinmath@126.com`, `taosp@nwnu.edu.cn`.