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AUTOMORPHISM GROUP OF GREEN ALGEBRA OF WEAK HOPF ALGEBRA CORRESPONDING TO SWEEDLER HOPF ALGEBRA

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Abstract. Let $r(\mathfrak{w}_2^0)$ be the Green ring of the weak Hopf algebra $\mathfrak{w}_2^0$ corresponding to Sweedler's 4-dimensional Hopf algebra H_2, and let $\operatorname{Aut}(R(\mathfrak{w}_2^0))$ be the automorphism group of the Green algebra $R(\mathfrak{w}_2^0) = r(\mathfrak{w}_2^0) \otimes_{\mathbb{Z}} \mathbb{C}$. We show that the quotient group $\operatorname{Aut}(R(\mathfrak{w}_2^0))/C_2 \cong S_3$, where C_2 contains the identity map and is isomorphic to the infinite group $(\mathbb{C}^*, \times)$ and S_3 is the symmetric group of order 6.

Keywords: Green algebra; automorphism group; weak Hopf algebra

MSC 2020: 16W20, 19A22

1. Introduction

Let H be a finite-dimensional Hopf algebra over a field K. Let $r(H)$ be a free abelian group generated by the isomorphism classes $[V]$ of finite dimensional H-modules V modulo the relations $[M \oplus V] = [M] + [V]$. For any H-modules V and M, the multiplication of $r(H)$ is given by the tensor product, that is, $[M][V] = [M \otimes V]$. Then $r(H)$ is an associative ring with identity $[K]$ and it is called the *Green ring* of H (see [8]), where K is the trivial H-module. Notice that $r(H)$ is $\mathbb{Z}$-free with a $\mathbb{Z}$-basis $\{[V]\colon V \in \operatorname{ind}(H)\}$, where $\operatorname{ind}(H)$ denotes the category of finite dimensional indecomposable H-modules. Chen et al. in [4] gave the generators and relations of the Green rings of Taft algebras. Then Li and Zhang studied the Green rings of generalized Taft algebras and determined all nilpotent elements of the Green ring, see [11]. In [16], Su and Yang computed the Green rings of the weak Hopf algebras based on the generalized Taft Hopf algebras.

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The study of the automorphism group of Green rings and Green algebras has been recently popularized. In [9], Jia et al. proved that the automorphism group of the Green ring $r(H_2)$ of the Sweedler's 4-dimensional Hopf algebra H_2 is isomorphic to the Klein group K_4, and the automorphism group of the Green algebra $\mathbb{F}(H_2) = r(H_2) \otimes_{\mathbb{Z}} \mathbb{F}$ ($\mathbb{F}$ is a field with characteristic not equal to 2) is the semidirect product of cyclic group $\mathbb{Z}_2$ of order 2 and a specific group G, where $G = \mathbb{F} \setminus \{\frac{1}{2}\}$ with multiplication given by $a \cdot b = 1 - a - b + 2ab$. Then in [18], Zhao et al. considered the automorphism group of Green algebra of 9-dimensional Taft Hopf algebra H_3. Let $r(H_3)$ be the Green ring of H_3, $\mathbb{R}(H_3) = r(H_3) \otimes_{\mathbb{Z}} \mathbb{R}$ be the Green algebra of $r(H_3)$ over the real number field $\mathbb{R}$, and $\mathrm{Aut}(\mathbb{R}(H_3))$ be the automorphism group of $\mathbb{R}(H_3)$. They showed that the quotient group $\mathrm{Aut}(\mathbb{R}(H_3))/T_1$ is isomorphic to the direct product of the dihedral group D_6 of order 12 and $\mathbb{Z}_2$, where T_1 contains the identity map and is isomorphic to a group $H = \{(a,b) \in \mathbb{R}^2 \colon (a,b) \neq (-\frac{1}{3}, -\frac{1}{6})\}$ with multiplication given by $(a_1,b_1) \cdot (a_2,b_2) = (a_1 + a_2 + 2a_1a_2 - 4b_1b_2 + 2a_1b_2 + 2b_1a_2, b_1 + b_2 - 2a_1a_2 - 2b_1b_2 + 4a_1b_2 + 4b_1a_2)$. Su and Yang in [15] constructed a weak Hopf algebra based on the unique noncommutative and noncocommutative 8-dimensional semisimple Hopf algebra, and investigated the automorphism group of the Green ring of this weak Hopf algebra. It turned out that the automorphism group is isomorphic to D_6. Moreover, the Green algebras mentioned above are isomorphic to the quotient algebras of the polynomial algebras over different basic field in several variables.

Let K be an algebraically closed field with characteristic 0. There have been many results about the automorphisms of the polynomial algebras. In [6] and [12], the authors studied the automorphisms of the polynomial ring in two variables, respectively. More specifically, Perepechko gave all the automorphisms of the algebra $K[x,y]/(x^2, y^3, xy^2)$, see [13]. It is presented that the Nagata automorphism of the polynomial algebra $K[x,y,z]$ in three variables is wild, see [14]. Furthermore, Drensky and Yu in [7] investigated all z-automorphisms of $K[x,y,z]$. For the study of the automorphisms of the polynomial algebra $K[x_1, \dots, x_n]$ in n variables, one can see [2].

In this paper, we investigate the automorphism group $\mathrm{Aut}(R(\mathfrak{w}_2^0))$ of the Green algebra $R(\mathfrak{w}_2^0) = r(\mathfrak{w}_2^0) \otimes_{\mathbb{Z}} \mathbb{C}$, where $r(\mathfrak{w}_2^0)$ is the Green ring of the weak Hopf algebra $\mathfrak{w}_2^0$ corresponding to the Sweedler's 4-dimensional Hopf algebra H_2. It can be seen as a study of the automorphism group of the polynomial algebra in three variables modulo five generating relations over $\mathbb{C}$.

This paper is organized as follows. In Section 2, we first introduce the concept of Sweedler's 4-dimensional Hopf algebra H_2 and the corresponding weak Hopf algebra $\mathfrak{w}_2^0$. Then we recall the representation category of $\mathfrak{w}_2^0$, and the structure of the Green ring $r(\mathfrak{w}_2^0)$. In Section 3, we give the most important conclusion in this paper: the quotient group $\mathrm{Aut}(R(\mathfrak{w}_2^0))/C_2$ is isomorphic to the symmetric group S_3 of order 6, where C_2 contains the identity map and is isomorphic to the infinite group $(\mathbb{C}^*, \times)$. In Section 4, we list all solutions of $f(x)$, $f(y)$, $f(xy)$ and $f(z)$ meeting the generating relations of the Green ring $r(\mathfrak{w}_2^0)$ to prove Theorem 3.1.

2. Preliminaries

Throughout this section, we work over the field of complex numbers $\mathbb{C}$. Unless otherwise stated, all Hopf algebras and modules are defined over $\mathbb{C}$, all modules are left modules and finite dimensional. The letters $\mathbb{C}$ and $\mathbb{C}^*$ stand for the field of complex numbers and $\mathbb{C} \setminus \{0\}$, respectively.

Sweedler's 4-dimensional Hopf algebra H_2 is a special case of Taft algebras, see [17]. It is generated by two elements G and U subject to the relations

$$G^2 = 1, \quad U^2 = 0, \quad GU = -UG.$$

The comultiplication Δ, the counit ε and the antipode S are determined by

$$\begin{aligned} &\Delta(G) = G \otimes G, && \varepsilon(G) = 1, \quad S(G) = G^{-1} = G, \\ &\Delta(U) = U \otimes G + 1 \otimes U, && \varepsilon(U) = 0, \quad S(U) = GU. \end{aligned}$$

It is easy to see that the set $\{1, G, U, GU\}$ forms a PBW basis of H_2.

The concept of weak Hopf algebra studied in this paper is defined by Li in [10]. Aizawa and Isaac in [1] studied the weak Hopf algebras corresponding to $U_q(sl_n)$. By the works in [16], the weak Hopf algebra $\mathfrak{w}_2^0$ of 0-type corresponding to the Sweedler's 4-dimensional Hopf algebra H_2 is defined as follows, see [16]. As a bialgebra, $\mathfrak{w}_2^0 = B/(u - g^2u)$, where B is a bialgebra generated by g and x subject to the relations, see [3] and [5]

$$g = g^3, \quad ug = -gu, \quad u^2 = 0,$$

and the bialgebra structure is given by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad \Delta(u) = u \otimes g + g^2 \otimes u, \quad \varepsilon(u) = 0.$$

It follows from [16] that the bialgebra $\mathfrak{w}_2^0$ is a weak Hopf algebra equipped with the weak antipode $T\colon \mathfrak{w}_2^0 \to \mathfrak{w}_2^0$ given by

$$T(1) = 1, \quad T(g) = g, \quad T(u) = gu.$$

Note that $\dim_{\mathbb{C}}(\mathfrak{w}_2^0) = 5$, and the set $\{1, g, u, g^2, gu\}$ forms a PBW basis of $\mathfrak{w}_2^0$.

Theorem 5.5 of [16] implies that the Green ring $r(\mathfrak{w}_2^0)$ is isomorphic to the ring $\mathbb{Z}[x,y,z]$ modulo the generating relations

$$x^2-1,\quad y^2-xy-y,\quad xz-z,\quad yz-2z,\quad z^2-z. \tag{2.1}$$

Notice that the ring $r(\mathfrak{w}_2^0)\cong\mathbb{Z}[x,y,z]/(x^2-1,y^2-xy-y,xz-z,yz-2z,z^2-z)$ is $\mathbb{Z}$-spanned by $\{1,x,y,xy,z\}$.

3. Automorphism group of Green algebra $R(\mathfrak{w}_2^0)$

In this section, we will consider the automorphism group $\mathrm{Aut}(R(\mathfrak{w}_2^0))$ of the Green algebra $R(\mathfrak{w}_2^0):=r(\mathfrak{w}_2^0)\otimes_{\mathbb{Z}}\mathbb{C}$.

In order to study the structure of $\mathrm{Aut}(R(\mathfrak{w}_2^0))$, we shall define 6 classes of $\mathbb{C}$-linear maps $C_i=\{f(i,k)\colon\ k\in\mathbb{C}^*\}$ $(i=1,\dots,6)$ of $R(\mathfrak{w}_2^0)$ as follows:

▷ $C_1=\{f(1,k)\colon\ k\in\mathbb{C}^*\}$, where $f(1,k)$ is given by

$$x\mapsto x,\quad y\mapsto\frac{1+k}{2}y+\frac{1-k}{2}xy,\quad xy\mapsto\frac{1-k}{2}y+\frac{1+k}{2}xy,\quad z\mapsto\frac{1}{4}y+\frac{1}{4}xy-z.$$

▷ $C_2=\{f(2,k)\colon\ k\in\mathbb{C}^*\}$, where $f(2,k)$ is given by

$$x\mapsto x,\quad y\mapsto\frac{1+k}{2}y+\frac{1-k}{2}xy,\quad xy\mapsto\frac{1-k}{2}y+\frac{1+k}{2}xy,\ z\mapsto z.$$

▷ $C_3=\{f(3,k)\colon\ k\in\mathbb{C}^*\}$, where $f(3,k)$ is given by

$$x\mapsto x,\quad y\mapsto 1+x+\frac{k}{2}y-\frac{k}{2}xy-2z,\quad xy\mapsto 1+x-\frac{k}{2}y+\frac{k}{2}xy-2z,\quad z\mapsto\frac{1}{4}y+\frac{1}{4}xy-z.$$

▷ $C_4=\{f(4,k)\colon\ k\in\mathbb{C}^*\}$, where $f(4,k)$ is given by

$$x\mapsto x,\ y\mapsto 1+x+\frac{k}{2}y-\frac{k}{2}xy-2z,\ xy\mapsto 1+x-\frac{k}{2}y+\frac{k}{2}xy-2z,\ z\mapsto\frac{1}{2}+\frac{1}{2}x-\frac{1}{4}y-\frac{1}{4}xy.$$

▷ $C_5=\{f(5,k)\colon\ k\in\mathbb{C}^*\}$, where $f(5,k)$ is given by

$$x\mapsto x,\ y\mapsto 1+x-\frac{1-k}{2}y-\frac{1+k}{2}xy+2z,\ xy\mapsto 1+x-\frac{1+k}{2}y-\frac{1-k}{2}xy+2z,\ z\mapsto z.$$

▷ $C_6=\{f(6,k)\colon\ k\in\mathbb{C}^*\}$, where $f(6,k)$ is given by

$$x\mapsto x,\ y\mapsto 1+x-\frac{1-k}{2}y-\frac{1+k}{2}xy+2z,\ xy\mapsto 1+x-\frac{1+k}{2}y-\frac{1-k}{2}xy+2z,$$
$$z\mapsto\frac{1}{2}+\frac{1}{2}x-\frac{1}{4}y-\frac{1}{4}xy.$$

Let A_i be the matrix of $f(i,k)$ with respect to the given basis $\{1,x,y,xy,z\}$ for $i=1,\dots,6$. Then we have

$$A_1=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&\frac{1+k}{2}&\frac{1-k}{2}&\frac{1}{4}\\0&0&\frac{1-k}{2}&\frac{1+k}{2}&\frac{1}{4}\\0&0&0&0&-1\end{pmatrix},\qquad A_2=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&\frac{1+k}{2}&\frac{1-k}{2}&0\\0&0&\frac{1-k}{2}&\frac{1+k}{2}&0\\0&0&0&0&1\end{pmatrix},$$

$$A_3=\begin{pmatrix}1&0&1&1&0\\0&1&1&1&0\\0&0&\frac{k}{2}&-\frac{k}{2}&\frac{1}{4}\\0&0&-\frac{k}{2}&\frac{k}{2}&\frac{1}{4}\\0&0&-2&-2&-1\end{pmatrix},\qquad A_4=\begin{pmatrix}1&0&1&1&\frac{1}{2}\\0&1&1&1&\frac{1}{2}\\0&0&\frac{k}{2}&-\frac{k}{2}&-\frac{1}{4}\\0&0&-\frac{k}{2}&\frac{k}{2}&-\frac{1}{4}\\0&0&-2&-2&0\end{pmatrix},$$

$$A_5=\begin{pmatrix}1&0&1&1&0\\0&1&1&1&0\\0&0&-\frac{1-k}{2}&-\frac{1+k}{2}&0\\0&0&-\frac{1+k}{2}&-\frac{1-k}{2}&0\\0&0&2&2&1\end{pmatrix},\qquad A_6=\begin{pmatrix}1&0&1&1&\frac{1}{2}\\0&1&1&1&\frac{1}{2}\\0&0&-\frac{1-k}{2}&-\frac{1+k}{2}&-\frac{1}{4}\\0&0&-\frac{1+k}{2}&-\frac{1-k}{2}&-\frac{1}{4}\\0&0&2&2&0\end{pmatrix}.$$

It is easy to check that $|A_i|=-k\neq 0$ $(i=1,4,5)$ and $|A_i|=k\neq 0$ $(i=2,3,6)$. In Section 4 we prove that $C_i=\{f(i,k)\colon k\in\mathbb{C}^*\}$ $(i=1,\dots,6)$ are all distinct 6 classes of algebra automorphisms of $R(\mathfrak{w}_2^0)$. Hence, we have the following theorem.

Theorem 3.1. $\operatorname{Aut}(R(\mathfrak{w}_2^0))=\bigcup_{i=1}^{6} C_i$.

Next, we will show that C_2 is a normal abelian subgroup of $\operatorname{Aut}(R(\mathfrak{w}_2^0))$ and $\operatorname{Aut}(R(\mathfrak{w}_2^0))/C_2\cong S_3$, where S_3 is the symmetric group of order 6.

Lemma 3.2. *Let $f(i,k)\in C_i$, $i=1,\dots,6$. Then the following holds:*

$$f^{-1}(i,k)=f\Big(i,\frac{1}{k}\Big)\in C_i,\quad i=1,2,4,5,$$
$$f^{-1}(3,k)=f\Big(6,\frac{1}{k}\Big)\in C_6,\quad f^{-1}(6,k)=f\Big(3,\frac{1}{k}\Big)\in C_3.$$

P r o o f. Here, we show that $f^{-1}(3,k) = f(6,1/k)$, and other cases can be obtained similarly. By straightforward computation, we have

$$\begin{aligned}
f(3,k_1)f(6,k_2)(y) &= f(3,k_1)\Big(1+x-\frac{1-k_2}{2}y-\frac{1+k_2}{2}xy+2z\Big)\\
&= 1+x-\frac{1-k_2}{2}\Big(1+x+\frac{k_1}{2}y-\frac{k_1}{2}xy-2z\Big)\\
&\quad -\frac{1+k_2}{2}\Big(1+x-\frac{k_1}{2}y+\frac{k_1}{2}xy-2z\Big)+2\Big(\frac{1}{4}y+\frac{1}{4}xy-z\Big)\\
&= \frac{1+k_1k_2}{2}y+\frac{1-k_1k_2}{2}xy = f(2,k_1k_2)(y)
\end{aligned}$$

and

$$\begin{aligned}
f(6,k_2)f(3,k_1)(y) &= f(6,k_2)\Big(1+x+\frac{k_1}{2}y-\frac{k_1}{2}xy-2z\Big)\\
&= 1+x+\frac{k_1}{2}\Big(1+x-\frac{1-k_2}{2}y-\frac{1+k_2}{2}xy+2z\Big)\\
&\quad -\frac{k_1}{2}\Big(1+x-\frac{1+k_2}{2}y-\frac{1-k_2}{2}xy+2z\Big)\\
&\quad -2\Big(\frac{1}{2}+\frac{1}{2}x-\frac{1}{4}y-\frac{1}{4}xy\Big)\\
&= \frac{1+k_1k_2}{2}y+\frac{1-k_1k_2}{2}xy = f(2,k_1k_2)(y).
\end{aligned}$$

Similarly, one can check that $f(3,k_1)f(6,k_2)(xy) = f(6,k_2)f(3,k_1)(xy) = f(2,k_1k_2)(xy)$ and $f(3,k_1)f(6,k_2)(z) = f(6,k_2)f(3,k_1)(z) = f(2,k_1k_2)(z)$. Obviously, $f(3,k_1)f(6,k_2)(x) = f(6,k_2)f(3,k_1)(x) = f(2,k_1k_2)(x)$. It follows that $f(3,k_1)f(6,k_2) = f(6,k_2)f(3,k_1) = f(2,k_1k_2)$. Thus, it is easy to see $f^{-1}(3,k) = f(6,1/k)$. □

Lemma 3.3. *C_2 is a normal abelian subgroup of* $\mathrm{Aut}(R(\mathfrak{w}_2^0))$ *and* $C_2 \cong (\mathbb{C}^*,\times)$.

P r o o f. We first show that C_2 is an abelian subgroup of $\mathrm{Aut}(R(\mathfrak{w}_2^0))$. It is obvious that any $f(2,k) \in C_2$ ($k \in \mathbb{C}^*$) is an algebra automorphism and $f(2,1) \in C_2$ is the identity automorphism of $R(\mathfrak{w}_2^0)$. For any $f(2,k_1)$ and $f(2,k_2)$ in C_2 we have $f(2,k_1)f(2,k_2)(x) = x = f(2,k_1k_2)(x)$ and $f(2,k_1)f(2,k_2)(z) = f(2,k_1k_2)(z)$.

$$\begin{aligned}
f(2,k_1)f(2,k_2)(y) &= f(2,k_1)\Big(\frac{1+k_2}{2}y+\frac{1-k_2}{2}xy\Big)\\
&= \frac{1+k_2}{2}\Big(\frac{1+k_1}{2}y+\frac{1-k_1}{2}xy\Big)+\frac{1-k_2}{2}\Big(\frac{1-k_1}{2}y+\frac{1+k_1}{2}xy\Big)\\
&= \frac{1+k_1k_2}{2}y+\frac{1-k_1k_2}{2}xy = f(2,k_1k_2)(y).
\end{aligned}$$

Similarly, one can check that $f(2,k_1)f(2,k_2)(xy) = f(2,k_1k_2)(xy)$. The above statement shows that C_2 is closed under the composition of linear maps, and the commutativity is clear. Moreover, for any $f(2,k) \in C_2$, by direct computation, we have

$$f(2,k)f\Big(2,\frac{1}{k}\Big) = f(2,1),$$

which implies that $f^{-1}(2,k) = f(2,1/k)$. It is easy to check that $C_2 \cong (\mathbb{C}^*,\times)$ under the map $\psi\colon C_2 \to (\mathbb{C}^*,\times)$, $\psi(f(2,k)) = k$.

Now, it remains to prove that C_2 is normal. In order to show this statement, we only need to prove that

$$f(i,k_i)f(2,k_2)f^{-1}(i,k_i) \in C_2,$$

where $f(i,k_i) \in C_i$, $i = 1,\ldots,6$. We will consider $i = 3$, and the other cases can be handled in the same way. By direct computation, we have

$$f(3,k_3)f(2,k_2) = f(3,k_2k_3),$$

which implies $f(3,k_3)f(2,k_2) \in C_3$. By Lemma 3.2, we obtain $f^{-1}(3,k_3) \in C_3$. Using Lemma 3.2 again, we get

$$f(3,k_3)f(2,k_2)f^{-1}(3,k_3) \in C_2.$$

□

Recall that a group G is called virtually abelian if it has a normal subgroup H such that G/H is finite. The following theorem tells us that $\operatorname{Aut}(R(\mathfrak{w}_2^0))$ is virtually abelian.

Theorem 3.4. $\operatorname{Aut}(R(\mathfrak{w}_2^0))/C_2 = \{f(1,1)C_2, C_2, f(3,1)C_2,\ldots,f(6,1)C_2\} \cong S_3$, *where* S_3 *is the symmetric group of order* 6, *and* $C_i = f(i,1)C_2$, $i = 1,\ldots,6$.

Proof. We first prove that

$$\operatorname{Aut}(R(\mathfrak{w}_2^0))/C_2 = \{f(1,1)C_2, C_2, f(3,1)C_2,\ldots,f(6,1)C_2\},$$

and $C_i = f(i,1)C_2$ $(i = 1,\ldots,6)$. Note that the definition of C_i tells us that $C_i \cap C_j = \emptyset$ for any $i \neq j \in \{1,\ldots,6\}$. So it is enough to show that for any $f(i,k_i) \in C_i$ there exists an element $f(2,k_2) \in C_2$ such that $f(i,k_i) = f(i,1)f(2,k_2) \in f(i,1)C_2$. We consider $i = 1$, and the other cases can be handled in the same way. Take $k_2 = k_1$. It is obvious that $f(1,1)f(2,k_1)(x) = x = f(1,k_1)(x)$ and $f(1,1)f(2,k_1)(z) = \frac{1}{4}y + \frac{1}{4}xy - z = f(1,k_1)(z)$.

$$\begin{aligned} f(1,1)f(2,k_1)(y) &= f(1,1)\Big(\frac{1+k_1}{2}y + \frac{1-k_1}{2}xy\Big) \\ &= \frac{1+k_1}{2}y + \frac{1-k_1}{2}xy = f(1,k_1)(y). \end{aligned}$$

Similarly, one can get $f(1,1)f(2,k_1)(xy) = f(1,k_1)(xy)$. Now it remains to show $\mathrm{Aut}(R(\mathfrak{w}_2^0))/C_2 \cong S_3$. We already know that the order of $\mathrm{Aut}(R(\mathfrak{w}_2^0))/C_2$ is 6. We only need to prove that $\mathrm{Aut}(R(\mathfrak{w}_2^0))/C_2$ is not commutative. Notice that

$$f(1,k_1)f(3,k_3) = f(5,k_1k_3) \quad \text{and} \quad f(3,k_3)f(1,k_1) = f(4,k_1k_3),$$

which implies that $\mathrm{Aut}(R(\mathfrak{w}_2^0))/C_2$ is not an abelian group. Hence, we obtain that $\mathrm{Aut}(R(\mathfrak{w}_2^0))/C_2 \cong S_3$. □

4. The proof of Theorem 3.1

Throughout this section, let A_g denote the corresponding coefficient matrix of a $\mathbb{C}$-linear map $g\colon\ R(\mathfrak{w}_2^0) \to R(\mathfrak{w}_2^0)$, and let $|A_g|$ denote the determinant of A_g. Otherwise stated, we always let $f \in \mathrm{Aut}(R(\mathfrak{w}_2^0))$. One should know that $|A_f| \neq 0$. The aim of this section is to prove Theorem 3.1. For this reason, we need to consider all distinct solutions of $f(x)$, $f(y)$, $f(xy)$ and $f(z)$ meeting the generating relations (2.1). We first consider the condition $(f(x))^2 = 1$. Assume that

$$f(x) = k_1 + k_2x + k_3y + k_4xy + k_5z, \quad k_i \in \mathbb{C},\ i = 1,2,3,4,5.$$

Then we have $(f(x))^2 = 1$ since $x^2 = 1$, i.e.,

$$(k_1 + k_2x + k_3y + k_4xy + k_5z)^2 = 1.$$

By straightforward computation,

$$\begin{aligned}(f(x))^2 = (k_1^2 + k_2^2) + 2k_1k_2x + (2k_1k_3 + 2k_2k_4 + 2k_3k_4 + k_3^2 + k_4^2)y \\ + (2k_1k_4 + 2k_2k_3 + 2k_3k_4 + k_3^2 + k_4^2)xy \\ + (2k_1k_5 + 2k_2k_5 + 4k_3k_5 + 4k_4k_5 + k_5^2)z.\end{aligned}$$

Hence, we get

$$\begin{cases} k_1^2 + k_2^2 = 1, \\ 2k_1k_2 = 0, \\ 2k_1k_3 + 2k_2k_4 + 2k_3k_4 + k_3^2 + k_4^2 = 0, \\ 2k_1k_4 + 2k_2k_3 + 2k_3k_4 + k_3^2 + k_4^2 = 0, \\ 2k_1k_5 + 2k_2k_5 + 4k_3k_5 + 4k_4k_5 + k_5^2 = 0. \end{cases} \tag{4.1}$$

By direct calculation, system of equations (4.1) has 16 distinct solutions in $\mathbb{C}$:

$$\pm(1,0,0,0,0), \quad \pm\Big(1,0,-\frac{1}{2},-\frac{1}{2},0\Big), \quad \pm(1,0,0,0,-2), \quad \pm\Big(1,0,-\frac{1}{2},-\frac{1}{2},2\Big),$$
$$\pm(0,1,0,0,0), \quad \pm\Big(0,1,-\frac{1}{2},-\frac{1}{2},0\Big), \quad \pm(0,1,0,0,-2), \quad \pm\Big(0,1,-\frac{1}{2},-\frac{1}{2},2\Big).$$

Note that $f(x) \neq \pm 1$, hence $f(x)$ has the following 14 possibilities:

$$\begin{aligned}
&f(1,x) = 1 - 2z, && f(2,x) = 1 - \frac{1}{2}y - \frac{1}{2}xy,\\
&f(3,x) = 1 - \frac{1}{2}y - \frac{1}{2}xy + 2z, && f(4,x) = -1 + 2z,\\
&f(5,x) = -1 + \frac{1}{2}y + \frac{1}{2}xy, && f(6,x) = -1 + \frac{1}{2}y + \frac{1}{2}xy - 2z,\\
&f(7,x) = x, && f(8,x) = x - 2z,\\
&f(9,x) = x - \frac{1}{2}y - \frac{1}{2}xy, && f(10,x) = x - \frac{1}{2}y - \frac{1}{2}xy + 2z,\\
&f(11,x) = -x, && f(12,x) = -x + 2z,\\
&f(13,x) = -x + \frac{1}{2}y + \frac{1}{2}xy, && f(14,x) = -x + \frac{1}{2}y + \frac{1}{2}xy - 2z.
\end{aligned}$$

We continue to compute the case $(f(z))^2 = f(z)$. By straightforward calculation, one can check that $f(z)$ has the following 14 possibilities:

$$\begin{aligned}
&f(1,z) = \frac{1}{4}y + \frac{1}{4}xy, && f(2,z) = \frac{1}{4}y + \frac{1}{4}xy - z,\\
&f(3,z) = z, && f(4,z) = 1 - z,\\
&f(5,z) = 1 - \frac{1}{4}y - \frac{1}{4}xy, && f(6,z) = 1 - \frac{1}{4}y - \frac{1}{4}xy + z,\\
&f(7,z) = \frac{1}{2} + \frac{1}{2}x, && f(8,z) = \frac{1}{2} + \frac{1}{2}x - z,\\
&f(9,z) = \frac{1}{2} + \frac{1}{2}x - \frac{1}{4}y - \frac{1}{4}xy, && f(10,z) = \frac{1}{2} + \frac{1}{2}x - \frac{1}{4}y - \frac{1}{4}xy + z,\\
&f(11,z) = \frac{1}{2} - \frac{1}{2}x, && f(12,z) = \frac{1}{2} - \frac{1}{2}x + z,\\
&f(13,z) = \frac{1}{2} - \frac{1}{2}x + \frac{1}{4}y + \frac{1}{4}xy, && f(14,z) = \frac{1}{2} - \frac{1}{2}x + \frac{1}{4}y + \frac{1}{4}xy - z.
\end{aligned}$$

Before approaching the possible cases of $f(y)$ and $f(xy)$, we need the following three lemmas.

Lemma 4.1. *Let $0 \neq v \in R(\mathfrak{w}_0^2)$. Then $v^2 = 0$ if and only if $v = k(y - xy)$, where $k \in \mathbb{C}^*$.*

Proof. Assume $0 \neq v = t_1 + t_2x + t_3y + t_4xy + t_5z$, where $t_i \in \mathbb{C}$ and $i = 1, 2, 3, 4, 5$. It is easy to obtain

$$\begin{aligned}
v^2 &= (t_1^2 + t_2^2) + 2t_1t_2x + (2t_1t_3 + 2t_2t_4 + 2t_3t_4 + t_3^2 + t_4^2)y\\
&\quad + (2t_1t_4 + 2t_2t_3 + 2t_3t_4 + t_3^2 + t_4^2)xy\\
&\quad + (2t_1t_5 + 2t_2t_5 + 4t_3t_5 + 4t_4t_5 + t_5^2)z\\
&= 0.
\end{aligned}$$

By

$$\begin{cases} t_1^2 + t_2^2 = 0, \\ 2t_1t_2 = 0, \\ 2t_1t_3 + 2t_2t_4 + 2t_3t_4 + t_3^2 + t_4^2 = 0, \\ 2t_1t_4 + 2t_2t_3 + 2t_3t_4 + t_3^2 + t_4^2 = 0, \\ 2t_1t_5 + 2t_2t_5 + 4t_3t_5 + 4t_4t_5 + t_5^2 = 0, \end{cases}$$

one can get $t_1 = t_2 = t_5 = 0$ and $t_3 = -t_4$ easily. □

Lemma 4.2. *$f(y) = f(xy) + k(y - xy)$, where $k \in \mathbb{C}$.*

Proof. Notice that $y^2 = xy + y$ implies that $y^2 = (xy)^2 = y(xy)$, i.e., $(f(y))^2 = (f(xy))^2 = f(y)f(xy)$. Hence, $(f(y) - f(xy))^2 = 0$. By Lemma 4.1, we have $f(y) = f(xy) + k(y - xy)$, $k \in \mathbb{C}$. □

Lemma 4.3. *$k(y - xy)f(xy) = 0$, $k \in \mathbb{C}$.*

Proof. If $k = 0$, it is obvious. Now we consider $k \neq 0$. By Lemma 4.2, $f(y) = f(xy) + k(y - xy)$. Then

$$(f(y))^2 = (f(xy) + k(y - xy))^2 = (f(xy))^2 + 2k(y - xy)f(xy).$$

Since $y(xy) = y + xy = y^2$, we have $f(y)f(xy) = (f(y))^2$. Note that

$$f(y)f(xy) = (f(xy) + k(y - xy))f(xy) = (f(xy))^2 + k(y - xy)f(xy).$$

Comparing $f(y)f(xy)$ and $(f(y))^2$, we get $k(y-xy)f(xy) = 0$, i.e., $(y-xy)f(xy) = 0$. □

Assume that

$$f(y) = k_6 + k_7x + k_8y + k_9xy + k_{10}z, \quad k_i \in \mathbb{C},\ i = 6, 7, 8, 9, 10.$$

By Lemma 4.2, we have

$$f(xy) = k_6 + k_7x + (k_8 - k)y + (k_9 + k)xy + k_{10}z, \quad k \in \mathbb{C}.$$

Now we consider the case $(f(y))^2 = f(y) + f(xy)$. By straightforward computation,

$$\begin{aligned}(f(y))^2 = (k_6^2 + k_7^2) + 2k_6k_7x + (2k_6k_8 + 2k_7k_9 + 2k_8k_9 + k_8^2 + k_9^2)y \\ + (2k_6k_9 + 2k_7k_8 + 2k_8k_9 + k_8^2 + k_9^2)xy \\ + (2k_6k_{10} + 2k_7k_{10} + 4k_8k_{10} + 4k_9k_{10} + k_{10}^2)z.\end{aligned}$$

Since $(f(y))^2 = f(y) + f(xy)$, one has

$$(4.2)\qquad \begin{cases} k_6^2 + k_7^2 = 2k_6, \\ 2k_6k_7 = 2k_7, \\ 2k_6k_8 + 2k_7k_9 + 2k_8k_9 + k_8^2 + k_9^2 = 2k_8 - k, \\ 2k_6k_9 + 2k_7k_8 + 2k_8k_9 + k_8^2 + k_9^2 = 2k_9 + k, \\ 2k_6k_{10} + 2k_7k_{10} + 4k_8k_{10} + 4k_9k_{10} + k_{10}^2 = 2k_{10}. \end{cases}$$

One can check that the system of equations (4.2) has 16 distinct solutions:

$$\Big(0,0,\frac{k}{2},-\frac{k}{2},0\Big),\quad \Big(0,0,\frac{k}{2},-\frac{k}{2},2\Big),\quad \Big(0,0,\frac{1+k}{2},\frac{1-k}{2},0\Big),\quad \Big(0,0,\frac{1+k}{2},\frac{1-k}{2},-2\Big),$$
$$\Big(2,0,-\frac{k}{2},\frac{k}{2},0\Big),\quad \Big(2,0,-\frac{k}{2},\frac{k}{2},-2\Big),\quad \Big(2,0,-\frac{1+k}{2},-\frac{1-k}{2},0\Big),$$
$$\Big(2,0,-\frac{1+k}{2},-\frac{1-k}{2},2\Big),\quad \Big(1,1,-\frac{1-k}{2},-\frac{1+k}{2},0\Big),\quad \Big(1,1,-\frac{1-k}{2},-\frac{1+k}{2},2\Big),$$
$$\Big(1,1,\frac{k}{2},-\frac{k}{2},0\Big),\quad \Big(1,1,\frac{k}{2},-\frac{k}{2},-2\Big),\quad \Big(1,-1,-\frac{k}{2},\frac{k}{2},0\Big),\quad \Big(1,-1,-\frac{k}{2},\frac{k}{2},2\Big),$$
$$\Big(1,-1,\frac{1-k}{2},\frac{1+k}{2},0\Big),\quad \Big(1,-1,\frac{1-k}{2},\frac{1+k}{2},-2\Big).$$

Meanwhile, we know that $f(xy)$ has 16 possibilities by Lemma 4.2. However, by Lemma 4.3, one can check that the pair of $f(y)$ and $f(xy)$ only have 8 possibilities, see Table 1.

$f(1,y) = \frac{k}{2}y - \frac{k}{2}xy$	$f(1,xy) = -\frac{k}{2}y + \frac{k}{2}xy$
$f(2,y) = \frac{k}{2}y - \frac{k}{2}xy + 2z$	$f(2,xy) = -\frac{k}{2}y + \frac{k}{2}xy + 2z$
$f(3,y) = \frac{1+k}{2}y + \frac{1-k}{2}xy$	$f(3,xy) = \frac{1-k}{2}y + \frac{1+k}{2}xy$
$f(4,y) = \frac{1+k}{2}y + \frac{1-k}{2}xy - 2z$	$f(4,xy) = \frac{1-k}{2}y + \frac{1+k}{2}xy - 2z$
$f(5,y) = 1 + x - \frac{1-k}{2}y - \frac{1+k}{2}xy$	$f(5,xy) = 1 + x - \frac{1+k}{2}y - \frac{1-k}{2}xy$
$f(6,y) = 1 + x - \frac{1-k}{2}y - \frac{1+k}{2}xy + 2z$	$f(6,xy) = 1 + x - \frac{1+k}{2}y - \frac{1-k}{2}xy + 2z$
$f(7,y) = 1 + x + \frac{k}{2}y - \frac{k}{2}xy$	$f(7,xy) = 1 + x - \frac{k}{2}y + \frac{k}{2}xy$
$f(8,y) = 1 + x + \frac{k}{2}y - \frac{k}{2}xy - 2z$	$f(8,xy) = 1 + x - \frac{k}{2}y + \frac{k}{2}xy - 2z$

Table 1.

By direct computation, we list all possibilities of $f(x)$ and $f(z)$ meeting the relation $f(xz) = f(x)f(z) = f(z)$ in Table 2.

$f(1,x)$	$f(2,z)$, $f(4,z)$, $f(5,z)$, $f(8,z)$, $f(9,z)$, $f(11,z)$, $f(14,z)$
$f(2,x)$	$f(5,z)$, $f(9,z)$, $f(11,z)$
$f(3,x)$	$f(3,z)$, $f(5,z)$, $f(6,z)$, $f(9,z)$, $f(10,z)$, $f(11,z)$, $f(12,z)$
$f(4,x)$	$f(3,z)$
$f(5,x)$	$f(1,z)$, $f(2,z)$, $f(3,z)$
$f(6,x)$	$f(2,z)$
$f(7,x)$	$f(1,z)$, $f(2,z)$, $f(3,z)$, $f(7,z)$, $f(8,z)$, $f(9,z)$, $f(10,z)$
$f(8,x)$	$f(2,z)$, $f(8,z)$, $f(9,z)$
$f(9,x)$	$f(9,z)$
$f(10,x)$	$f(3,z)$, $f(9,z)$, $f(10,z)$
$f(11,x)$	$f(11,z)$
$f(12,x)$	$f(3,z)$, $f(11,z)$, $f(12,z)$
$f(13,x)$	$f(1,z)$, $f(2,z)$, $f(3,z)$, $f(11,z)$, $f(12,z)$, $f(13,z)$, $f(14,z)$
$f(14,x)$	$f(2,z)$, $f(11,z)$, $f(14,z)$

Table 2.

So far, $f(x)$, $f(y)$, $f(xy)$ and $f(z)$ satisfy the relations $(f(x))^2 = 1$, $(f(z))^2 = f(z)$, $f(xz) = f(z)$ and $(f(y))^2 = f(y) + f(xy)$. Now, it is left to check that they satisfy the relations $f(x)f(y) = f(xy)$ and $f(y)f(z) = 2f(z)$.

In Table 3, all possible cases of $f(y)$ and $f(z)$ meeting the relation $f(y)f(z) = 2f(z)$ are listed.

$f(2,y)$	$f(3,z)$
$f(3,y)$	$f(1,z)$, $f(2,z)$, $f(3,z)$
$f(4,y)$	$f(2,z)$
$f(4,x)$	$f(3,z)$
$f(5,y)$	$f(9,z)$
$f(6,y)$	$f(3,z)$, $f(9,z)$, $f(10,z)$
$f(7,y)$	$f(1,z)$, $f(2,z)$, $f(3,z)$, $f(7,z)$, $f(8,z)$, $f(9,z)$, $f(10,z)$
$f(8,y)$	$f(2,z)$, $f(8,z)$, $f(9,z)$

Table 3.

At last, we check the relation $f(x)f(y) = f(xy)$. As an example, we consider $f(2,y)$ and $f(3,z)$. By Table 2, in this case $f(x)$ could be $f(3,x)$, $f(4,x)$, $f(5,x)$, $f(7,x)$, $f(10,x)$, $f(12,x)$ and $f(13,x)$.

(1) By $f(3,x)f(2,y) = ky/2 - kxy/2 + 2z = f(2,xy)$, we have $k = 0$, which gives $f(2,y) = f(2,xy) = 2z$, a contradiction.

(2) $f(4,x)f(2,y) = f(2,xy)$. Then

$$|A_f| = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{k}{2} & -\frac{k}{2} & 0 \\ 0 & 0 & -\frac{k}{2} & \frac{k}{2} & 0 \\ 0 & 2 & 2 & 2 & 1 \end{vmatrix} = 0,$$

which is contrary to $|A_f| \neq 0$.

(3) $f(5,x)f(2,y) = f(2,xy)$. Then

$$|A_f| = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{k}{2} & -\frac{k}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{k}{2} & \frac{k}{2} & 0 \\ 0 & 0 & 2 & 2 & 1 \end{vmatrix} = 0,$$

a contradiction.

(4) $f(7,x)f(2,y) = f(2,xy)$. Then

$$|A_f| = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{k}{2} & -\frac{k}{2} & 0 \\ 0 & 0 & -\frac{k}{2} & \frac{k}{2} & 0 \\ 0 & 0 & 2 & 2 & 1 \end{vmatrix} = 0,$$

a contradiction.

(5) $f(10,x)f(2,y) = f(2,xy)$. Then

$$|A_f| = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{k}{2} & -\frac{k}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{k}{2} & \frac{k}{2} & 0 \\ 0 & 2 & 2 & 2 & 1 \end{vmatrix} = 0,$$

a contradiction.

(6) By $f(12,x)f(2,y) = ky/2 - kxy/2 + 2z = f(2,xy)$, we have $k = 0$. Then $f(y) = f(xy) = 2z$, a contradiction.

(7) By $f(13,x)f(2,y) = ky/2 - kxy/2 + 2z = f(2,xy)$, we have $f(y) = f(xy) = 2z$, a contradiction.

In this manner, one can check that f indeed has and only has 6 classes conditions C_i, $i = 1,\ldots,6$ given at the beginning of Section 3.

Remark 4.4. The multiplication formulas of $f(i,\text{-})$, $i = 1,\ldots,6$ are listed in Table 4.

$\cdot$	$f(1,\text{-})$	$f(2,\text{-})$	$f(3,\text{-})$	$f(4,\text{-})$	$f(5,\text{-})$	$f(6,\text{-})$
$f(1,\text{-})$	$f(2,\text{-}\cdot\text{-})$	$f(1,\text{-}\cdot\text{-})$	$f(5,\text{-}\cdot\text{-})$	$f(6,\text{-}\cdot\text{-})$	$f(3,\text{-}\cdot\text{-})$	$f(4,\text{-}\cdot\text{-})$
$f(2,\text{-})$	$f(1,\text{-}\cdot\text{-})$	$f(2,\text{-}\cdot\text{-})$	$f(3,\text{-}\cdot\text{-})$	$f(4,\text{-}\cdot\text{-})$	$f(5,\text{-}\cdot\text{-})$	$f(6,\text{-}\cdot\text{-})$
$f(3,\text{-})$	$f(4,\text{-}\cdot\text{-})$	$f(3,\text{-}\cdot\text{-})$	$f(6,\text{-}\cdot\text{-})$	$f(5,\text{-}\cdot\text{-})$	$f(1,\text{-}\cdot\text{-})$	$f(2,\text{-}\cdot\text{-})$
$f(4,\text{-})$	$f(3,\text{-}\cdot\text{-})$	$f(4,\text{-}\cdot\text{-})$	$f(1,\text{-}\cdot\text{-})$	$f(2,\text{-}\cdot\text{-})$	$f(6,\text{-}\cdot\text{-})$	$f(5,\text{-}\cdot\text{-})$
$f(5,\text{-})$	$f(6,\text{-}\cdot\text{-})$	$f(5,\text{-}\cdot\text{-})$	$f(4,\text{-}\cdot\text{-})$	$f(3,\text{-}\cdot\text{-})$	$f(2,\text{-}\cdot\text{-})$	$f(1,\text{-}\cdot\text{-})$
$f(6,\text{-})$	$f(5,\text{-}\cdot\text{-})$	$f(6,\text{-}\cdot\text{-})$	$f(2,\text{-}\cdot\text{-})$	$f(1,\text{-}\cdot\text{-})$	$f(4,\text{-}\cdot\text{-})$	$f(3,\text{-}\cdot\text{-})$

Table 4.

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References

[1] *N. Aizawa, P. S. Isaac*: Weak Hopf algebras corresponding to $U_q[sl_n]$. J. Math. Phys. *44* (2003), 5250–5267. zbl MR doi

[2] *V. G. Bardakov, M. V. Neshchadim, Y. V. Sosnovsky*: Groups of triangular automorphisms of a free associative algebra and a polynomial algebra. J. Algebra *362* (2012), 201–220. zbl MR doi

[3] *M. Beattie, S. Dăscălescu, L. Grünenfelder*: Constructing pointed Hopf algebras by Ore extensions. J. Algebra *225* (2000), 743–770. zbl MR doi

[4] *H. Chen, F. Van Oystaeyen, Y. Zhang*: The Green rings of Taft algebras. Proc. Am. Math. Soc. *142* (2014), 765–775. zbl MR doi

[5] *S. Dăscălescu*: On the dimension of the space of integrals for finite dimensional bialgebras. Stud. Sci. Math. Hung. *45* (2008), 411–417. zbl MR doi

[6] *W. Dicks*: Automorphisms of the polynomial ring in two variables. Publ., Secc. Mat., Univ. Autòn. Barc. *27* (1983), 155–162. zbl MR

[7] *V. Drensky, J.-T. Yu*: Coordinates and automorphisms of polynomial and free associative algebra of rank three. Front. Math. China *2* (2007), 13–46. zbl MR doi

[8] *J. A. Green*: The modular representation algebra of a finite group. Ill. J. Math. *6* (1962), 607–619. zbl MR doi

[9] *T. Jia, R. Zhao, L. Li*: Automorphism group of Green ring of Sweedler Hopf algebra. Front. Math. China *11* (2016), 921–932. zbl MR doi

[10] *F. Li*: Weak Hopf algebras and new solutions of the quantum Yang-Baxter equation. J. Algebra *208* (1998), 72–100. zbl MR doi

[11] *L. Li, Y. Zhang*: The Green rings of the generalized Taft Hopf algebras. Hopf Algebras and Tensor Categories. Contemporary Mathematics 585. AMS, Providence, 2013, pp. 275–288. zbl MR doi
[12] *J. H. McKay, S. S.-S. Wang*: An elementary proof of the automorphism theorem for the polynomial ring in two variables. J. Pure Appl. Algebra *52* (1988), 91–102. zbl MR doi
[13] *A. Perepechko*: On solvability of the automorphism group of a finite-dimensional algebra. J. Algebra *403* (2014), 455–458. zbl MR doi
[14] *I. P. Shestakov, U. U. Umirbaev*: The tame and the wild automorphisms of polynomial rings in three variables. J. Am. Math. Soc. *17* (2004), 197–227. zbl MR doi
[15] *D. Su, S. Yang*: Automorphism group of representation ring of the weak Hopf algebra $\widetilde{H}_8$. Czech. Math. J. *68* (2018), 1131–1148. zbl MR doi
[16] *D. Su, S. Yang*: Green rings of weak Hopf algebras based on generalized Taft algebras. Period. Math. Hung. *76* (2018), 229–242. zbl MR doi
[17] *E. J. Taft*: The order of the antipode of finite-dimensional Hopf algebra. Proc. Natl. Acad. Sci. USA *68* (1971), 2631–2633. zbl MR doi
[18] *R. Zhao, C. Yuan, L. Li*: The automorphism group of Green algebra of 9-dimensional Taft Hopf algebra. Algebra Colloq. *27* (2020), 767–798. zbl MR doi

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