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ASYMPTOTIC BEHAVIOR OF SMALL-DATA SOLUTIONS TO A KELLER-SEGEL-NAVIER-STOKES SYSTEM WITH INDIRECT SIGNAL PRODUCTION

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Abstract. We consider the Keller-Segel-Navier-Stokes system

$$\begin{cases} n_t + \mathbf{u}\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla v), & x\in\Omega,\ t>0,\\ v_t + \mathbf{u}\cdot\nabla v = \Delta v - v + w, & x\in\Omega,\ t>0,\\ w_t + \mathbf{u}\cdot\nabla w = \Delta w - w + n, & x\in\Omega,\ t>0,\\ \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = \Delta\mathbf{u} + \nabla P + n\nabla\phi,\ \nabla\cdot\mathbf{u}=0, & x\in\Omega,\ t>0, \end{cases}$$

which is considered in bounded domain $\Omega\subset\mathbb{R}^N$ ($N\in\{2,3\}$) with smooth boundary, where $\phi\in C^{1+\delta}(\overline{\Omega})$ with $\delta\in(0,1)$. We show that if the initial data $\|n_0\|_{L^{N/2}(\Omega)}$, $\|\nabla v_0\|_{L^N(\Omega)}$, $\|\nabla w_0\|_{L^N(\Omega)}$ and $\|\mathbf{u}_0\|_{L^N(\Omega)}$ is small enough, an associated initial-boundary value problem possesses a global classical solution which decays to the constant state $(\bar n_0,\bar n_0,\bar n_0,0)$ exponentially with $\bar n_0 := (1/|\Omega|)\int_\Omega n_0(x)\,\mathrm{d}x$.

Keywords: Keller-Segel-Navier-Stokes; global solution; decay estimate; indirect process

MSC 2020: 35K55, 35B40, 35Q35, 92C17, 35B35

1. Introduction

Chemotaxis is a phenomenon that occurs in both prokaryotic and eukaryotic types of moving cells, including bacteria, protozoa, white blood cells, and tumor cells, where the direction of movement is influenced by the concentration of chemical stimulis. Chemotactic motion is usually directed in the direction of a higher stimulus con-

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centration, so the stimulus is called a *chemoattractant*. In order to describe the movement of cells, especially an aggregation, in 1970s, Keller and Segel proposed the following reaction-diffusion system in which the chemical substance is produced by cells:

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n \nabla v), \\ v_t = \Delta v - v + n, \end{cases} \tag{1.1}$$

where $n = n(x,t)$, $v = v(x,t)$ denote the density of bacteria and concentration of chemical substance, respectively. During the past four decades, system (1.1) and its variants have been paid extensive attentions. For instance, it is a known fact that the homogeneous Neumann problem of (1.1) possesses global and bounded solutions when $N = 1$, see [15]. If $N \geqslant 2$, the remarkable property of system (1.1) is the existence of blow-up solutions in finite or infinite time. Especially, in two-dimensional case, a critical mass phenomenon has been identified and studied in [14]. Namely, beginning with initial mass above it, the solution blows up in finite time. Otherwise, there is a global-in-time solution to problem (1.1). In the case of $N \geqslant 3$, Cao in [2] has proved that under the smallness assumptions of $\|n_0\|_{L^{N/2}(\Omega)}$ and $\|\nabla v_0\|_{L^N(\Omega)}$, global classical solutions exist with the following decay estimate holding: $\|n(\cdot,t) - \bar{n}_0\|_{L^\infty(\Omega)} + \|v(\cdot,t) - \bar{n}_0\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\lambda t}$, where $\bar{n}_0 = |\Omega|^{-1} \int_\Omega n_0(x)$. This kind of model has been widely studied; we can refer to the survey (see [8], [9]) for a broader overview. Besides, the large time behavior of solutions to (1.1) with small initial data can be found in [4], [22].

In nature, migration of cells can be profoundly influenced by environmental changes and vice versa. More commonly, cells tend to live in a viscous fluid, where cells and chemical matrices are transported along with, at the same time, the motion of fluid being influenced by the gravity generated by cell aggregation. To decribe these processes, the following Keller-Segel-(Navier-)Stokes system was proposed and has been studied by many authors:

$$\begin{cases} n_t + \mathbf{u} \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla v), & x \in \Omega,\ t > 0, \\ v_t + \mathbf{u} \cdot \nabla v = \Delta v - v + n, & x \in \Omega,\ t > 0, \\ \mathbf{u}_t + \kappa(\mathbf{u} \cdot \nabla)\mathbf{u} = \Delta \mathbf{u} + \nabla P + n \nabla \phi,\ \nabla \cdot \mathbf{u} = 0, & x \in \Omega,\ t > 0, \\ \nabla n \cdot \nu = \nabla v \cdot \nu = 0,\ \mathbf{u} = 0, & x \in \partial\Omega,\ t > 0, \\ n(x,0) = n_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where ν stands for the outward normal vector on $\partial\Omega$, ϕ is a potential function. We denote the fluid velocity by $\mathbf{u}$ and the associated pressure by P. The symbol κ stands for the coefficient which relats to the strength of nonlinear fluid convection.

The chemotaxis-(Navier)-Stokes model has attracted many researcher's attentions since it was proposed. Different from (1.1), the appearance of fluid brings a lot of substantial difficulties to study the solution of this coupled system. When $\kappa = 0$, which means the fluid flows slowly, system (1.2) is an incompressible chemotaxis-Stokes system. In 2017, authors of [11] obtained global bounded classical solutions if $\|n_0\|_{L^1(\Omega)}$ is small enough in the 2D case. With additional logistic source, (1.2) possesses the global weak solution in two-dimensional setting in [5]. The effort to this 3-D problem is due to the work of Tao and Winkler (see [17]), in which they constructed global bounded classical solution under the explicit condition $\mu \geqslant 23$, where μ is the given parameter from logistic source. In the full chemotaxis-Navier-Stokes system ($\kappa \neq 0$) with additional logistic source concerned, the problem in [18] possesses a global classical solution which is bounded and satisfies $\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|\mathbf{u}(\cdot,t)\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$ while the prescribed function $\int_0^\infty \int_\Omega |g(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \infty$ in the 2D case. When $N = 3$, the problem in [24] possesses at least one globally defined solution in an appropriate generalized sense, and this solution is uniformly bounded with respect to the norm in $L^1(\Omega) \times L^6(\Omega) \times L^2(\Omega; \mathbb{R}^3)$. Very recently, Winkler improved this result and obtained the eventual smooth solutions in [26]. The equality $(1.2)_1$ can be replaced by a matrix-valued sensitivity to get $n_t + \mathbf{u} \cdot \nabla n = \Delta n - \nabla \cdot (nS(x,n,v) \cdot \nabla v)$, where $S \in C^2(\overline{\Omega} \times [0,\infty)^2)^{N \times N}$ reflects rotational chemotactic motion. In particular, under the assumption $|S(x,n,v)| \leqslant C_s(1+n)^{-\alpha}$ with some $C_s > 0$ and $\alpha > 0$, the author of [19] proved the existence of globally bounded classical solution while $\alpha > 0$ in the 2D case. Then Liu and Wang in [12] obtained the global weak solution if $\alpha > \frac{3}{7}$ when $N = 3$. Recently, Wang et al. in [20] further improved this result to the case $\alpha > \frac{1}{3}$. Moreover, by the properties of Neumann heat semigroup and Stokes semigroup, Yu et al. in [27] constructed the global classical solutions of system (1.2) which decay to the constant steady state $(\bar{n}_0, \bar{n}_0, 0)$ exponentially in L^∞-norm with $\bar{n}_0 = (1/|\Omega|)\int_\Omega n_0$, whenever $\|n_0\|_{L^{N/2}(\Omega)}$, $\|\nabla v_0\|_{L^N(\Omega)}$ and $\|\mathbf{u}_0\|_{L^N(\Omega)}$ are small enough. However, without any sensitivity and logistic source, Winkler in [25] constructed the globally defined generalized solution of (1.2) under a smallness assumption merely involving the initial data n_0; furthermore, the following long time behavior holds: $\|n(\cdot,t) - \bar{n}_0\|_{C^2(\overline{\Omega})} + \|v(\cdot,t) - \bar{n}_0\|_{C^2(\overline{\Omega})} + \|\mathbf{u}(\cdot,t)\|_{C^2(\overline{\Omega})} \to 0$ as $t \to \infty$. For more works on corresponding chemotaxis models and their variants, we refer to the survey (see [1]) and the references therein.

Indirect process. The models mentioned above describe a direct effect on the chemical signal. However, the natural environment is more complex. In order to describe how the combination of chemicals might interact to produce aggregation of cells, the authors of [16] proposed the following system based on [13]:

$$
(1.3) \qquad \begin{cases} n_t = \Delta n - \nabla \cdot (n\chi \nabla v) + \nabla \cdot (\xi n \nabla w), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - \beta_1 v + \beta_2 w, & x \in \Omega,\ t > 0, \\ w_t = \Delta w - \gamma w + \delta n, & x \in \Omega,\ t > 0, \\ \nabla n \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ n(x,0) = n_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & x \in \Omega. \end{cases}
$$

The positive parameters χ and ξ are called the *chemosensitivity coefficients*, and $\beta_1, \beta_2, \gamma, \delta > 0$ are chemical production and depredation rates. The symbol w stands for a different chemical substance from v. In contrast to system (1.1), model (1.3) contained two chemical substances and the signal production here occurs in an indirect process. In [16], it is shown that when $N \in \{2,3\}$, solutions of (1.3) exist globally in time with $\xi\gamma > \chi\beta_1, \beta_2 = \delta$ and converge to $(\bar{n}_0, \bar{n}_0\beta_1/\beta_2, \bar{n}_0\gamma/\beta_2)$ exponentially as $t \to \infty$. Further study in [10] shows that as long as $\xi\gamma > \chi\beta_1$, the weak solution to (1.3) exists globally in the 3D case.

In particular, Fujie and Senba studied the following chemotaxis system with Keller-Segel type signal production in [6], [7]:

$$
(1.4) \qquad \begin{cases} n_t = \Delta n - \nabla \cdot (n \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + w, & x \in \Omega,\ t > 0, \\ w_t = \Delta w - w + n, & x \in \Omega,\ t > 0. \end{cases}
$$

They considered the system coupled with homogeneous Neumann boundary conditions or no-flux-Dirichlet-conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ $(N \leqslant 4)$ and obtained the boundedness of classical solution of (1.4) in the 3D case for all reasonably regular initial data. In the four-dimensional setting, the critical mass condition is necessary to derive the boundedness of solution by use of Adam-type inequality and Lyapunov function. As for problem (1.4) with additional logistic term, authors of [29] obtained the decay properties, namely $(n, v, w) \to (1, 1, 1)$ in $L^\infty(\Omega)$ as $t \to \infty$ under some given conditions. In 2020, Yu considered (1.4) in fluid environment by coupling with Stokes equation

$$
(1.5) \qquad \begin{cases} n_t + \mathbf{u} \cdot \nabla n = \Delta n - \nabla \cdot (nS(x,n,v,w)\nabla v), & x \in \Omega,\ t > 0, \\ v_t + \mathbf{u} \cdot \nabla v = \Delta v - v + w, & x \in \Omega,\ t > 0, \\ w_t + \mathbf{u} \cdot \nabla w = \Delta w - w + n, & x \in \Omega,\ t > 0, \\ \mathbf{u}_t = \Delta \mathbf{u} + \nabla P + n \nabla \phi,\ \nabla \cdot \mathbf{u} = 0, & x \in \Omega,\ t > 0, \\ \nabla n \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0,\ \mathbf{u} = 0, & x \in \partial\Omega,\ t > 0, \\ n(x,0) = n_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) & x \in \Omega, \end{cases}
$$

where $S(x,n,v,w)$ is a given chemotactic sensitivity function and satisfies

$$|S(x,n,v,w)| \leqslant C_S(1+n)^{-\alpha}.$$

The author obtained the boundedness of the classical solution of system (1.5) in two-dimensional setting in [28] while $\alpha > 0$. More recently, Wang-Yang in [21] improved the result of [28], they proved that when $N = 2$, only if $\alpha \geqslant 0$, system (1.5) possesses global bounded classical solution. When $N = 3$, the bounded solution of (1.5) can be obtained under the condition that $\alpha > \frac{1}{9}$.

Main result. Inspired by the above works, we pay our attention to the global existence and long-time behavior of the classical solution to the Keller-Segel-Navier-Stokes system

$$\begin{cases} n_t + \mathbf{u}\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla v), & x\in\Omega,\ t>0,\\ v_t + \mathbf{u}\cdot\nabla v = \Delta v - v + w, & x\in\Omega,\ t>0,\\ w_t + \mathbf{u}\cdot\nabla w = \Delta w - w + n, & x\in\Omega,\ t>0,\\ \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = \Delta\mathbf{u} + \nabla P + n\nabla\phi,\ \nabla\cdot\mathbf{u} = 0, & x\in\Omega,\ t>0,\\ \nabla n\cdot\nu = \nabla v\cdot\nu = \nabla w\cdot\nu = 0,\ \mathbf{u} = 0, & x\in\partial\Omega,\ t>0,\\ n(x,0) = n_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) & x\in\Omega, \end{cases} \tag{1.6}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \in \{2,3\}$. Before our proof, some hypotheses will be essential. Suppose that $\phi \in C^{1+\delta}(\overline{\Omega})$ with $\delta \in (0,1)$ and the initial data n_0, v_0, w_0, $\mathbf{u}_0$ satisfy

$$\begin{cases} n_0 \in C^0(\overline{\Omega}),\ n_0 \geqslant 0 \text{ and } n_0 \not\equiv 0 \text{ on } \overline{\Omega},\\ v_0, w_0 \in W^{1,\infty}(\Omega),\ v_0, w_0 \geqslant 0,\\ \mathbf{u}_0 \in D(\mathcal{A}^\beta) \text{ for some } \beta \in \left(\frac{N}{4}, 1\right), \end{cases} \tag{1.7}$$

where $\mathcal{A}$ denotes the realization of the Stokes operator in $L^2(\Omega;\mathbb{R}^N)$, defined on its domain $D(\mathcal{A}) := W^{2,2}(\Omega;\mathbb{R}^N) \cap W_0^{1,2}(\Omega;\mathbb{R}^N) \cap L^2_\sigma(\Omega)$ with $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega;\mathbb{R}^N)\colon\ \nabla\cdot\varphi = 0\}$. Denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary condition by λ_1, and by λ_1' the first eigenvalue of Stokes operator with homogeneous Dirichlet boundary data. Under these assumptions, we construct decay estimates of the classical solutions of (1.6). Our main result is presented as the following theorem:

Theorem 1.1. *Suppose that* $N \in \{2,3\}$, $\beta \in (\frac{1}{4}N, 1)$, $\phi \in C^{1+\delta}(\overline{\Omega})$ *with* $\delta \in (0,1)$. *Then for any* $\alpha_1 \in (0, \lambda_1)$, $\alpha_2 \in (0, \min\{\alpha_1, \lambda_1'\})$ *there exists* $\varepsilon > 0$ *such that if initial data* (1.7) *satisfy*

$$\|n_0\|_{L^{N/2}(\Omega)} + \|\nabla v_0\|_{L^N(\Omega)} + \|\nabla w_0\|_{L^N(\Omega)} + \|\mathbf{u}_0\|_{L^N(\Omega)} \leqslant \varepsilon, \tag{1.8}$$

system (1.6) *possesses a global classical solution* $(n, v, w, \mathbf{u}, P)$, *which enjoys the regularities*

$$\begin{cases} n \in C^0(\overline{\Omega} \times [0,\infty)) \bigcap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ v \in \bigcap\limits_{p>N} C^0([0,\infty); W^{1,p}(\Omega)) \bigcap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ w \in \bigcap\limits_{q>N} C^0([0,\infty); W^{1,q}(\Omega)) \bigcap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ \mathbf{u} \in C^0([0,\infty); D(A^\beta)) \bigcap C^{2,1}(\overline{\Omega} \times [0,\infty); \mathbb{R}^N), \\ P \in C^{1,0}(\overline{\Omega} \times (0,\infty)). \end{cases} \tag{1.9}$$

Moreover, for some $C > 0$ *and all* $t > 0$, *this solution has the property that*

$$\begin{aligned} &\|n(\cdot,t) - \bar{n}_0\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\alpha_1 t}, \qquad \|v(\cdot,t) - \bar{n}_0\|_{W^{1,\infty}(\Omega)} \leqslant C\mathrm{e}^{-\alpha_1 t}, \\ &\|w(\cdot,t) - \bar{n}_0\|_{W^{1,\infty}(\Omega)} \leqslant C\mathrm{e}^{-\alpha_1 t}, \quad \|\mathbf{u}(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\alpha_2 t}, \end{aligned} \tag{1.10}$$

where $\bar{n}_0 := (1/|\Omega|) \int_\Omega n_0(x)\,\mathrm{d}x$.

2. Estimates under the smallness assumption for $\|n_0\|_{L^{q_0}(\Omega)}$ where $\frac{1}{2}N < q_0 < N$

In this section, we obtain some results based on the following smallness hypothesis:

$$\|n_0\|_{L^{q_0}(\Omega)} + \|\nabla v_0\|_{L^N(\Omega)} + \|\nabla w_0\|_{L^N(\Omega)} + \|\mathbf{u}_0\|_{L^N(\Omega)} \leqslant \varepsilon_0 \tag{2.1}$$

for any $q_0 > \frac{1}{2}N$ and some $\varepsilon_0 > 0$. Our main purpose is to prove that the maximal existence time $T_{\max} = \infty$ firstly, and then to deduce the decay estimates listed in (1.10) under hypotheses (2.1). Now, we state the following local existence of classical solutions to (1.6).

Lemma 2.1. *Suppose that* $N \in \{2,3\}$, $\beta \in (\frac{1}{4}N, 1)$, $\phi \in C^{1+\delta}(\overline{\Omega})$ *with* $\delta \in (0,1)$ *and the initial data satisfy* (1.7). *Then:*

(1) *There exist* $\tau = \tau(\beta, \|n_0\|_{L^\infty(\Omega)}, \|v_0\|_{W^{1,\infty}(\Omega)}, \|w_0\|_{W^{1,\infty}(\Omega)}, \|\mathcal{A}^\beta \mathbf{u}_0\|_{L^2(\Omega)})$, $\eta = \eta(\tau) > 0$ *and a classical solution* $(n, v, w, \mathbf{u})$ *of* (1.6) *in* $(0,\tau)$ *such that for all* $t \in [0,\tau]$,

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|\mathcal{A}^\beta \mathbf{u}(\cdot,t)\|_{L^2(\Omega)} \leqslant \eta.$$

(2) *The solution* $(n, v, w, \mathbf{u})$ *of* (1.6) *can be extended to the maximal existence time* $T_{\max} > 0$ *with keeping* n, v, w *positive in* $\overline{\Omega} \times (0, T_{\max})$, *and is unique, up to addition of constants to* P. *Moreover, if* $T_{\max} < \infty$, *then*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|\mathcal{A}^\beta \mathbf{u}(\cdot,t)\|_{L^2(\Omega)} \to \infty$$

as $t \to T_{\max}$.

Proof. This result can be derived by a very similar procedure as that of Lemma 2.1 in [23], which is based on the contraction mapping principle and standard regularity theories for the heat equation and the Stokes system. We omit the details here. □

Lemma 2.2. *If the initial data satisfy* (1.7), *then the classical solution of* (1.6) *satisfies*

$$\int_\Omega n(\cdot,t)\,\mathrm{d}x = \int_\Omega n_0(x)\,\mathrm{d}x \quad \forall\, t \in [0, T_{\max}), \tag{2.2}$$

$$\int_\Omega v(\cdot,t)\,\mathrm{d}x \leqslant \max\left\{\int_\Omega v_0(x)\,\mathrm{d}x, \int_\Omega w_0(x)\,\mathrm{d}x, \int_\Omega n_0(x)\,\mathrm{d}x\right\} \quad \forall\, t \in [0, T_{\max}), \tag{2.3}$$

and

$$\int_\Omega w(\cdot,t)\,\mathrm{d}x \leqslant \max\left\{\int_\Omega w_0(x)\,\mathrm{d}x, \int_\Omega n_0(x)\,\mathrm{d}x\right\} \quad \forall\, t \in [0, T_{\max}). \tag{2.4}$$

Proof. Integrating the first equation of (1.6) over Ω, we have

$$\int_\Omega n(\cdot,t)\,\mathrm{d}x = \int_\Omega n_0(x)\,\mathrm{d}x \quad \forall\, t \in [0, T_{\max}) \quad \text{since } \nabla \cdot \mathbf{u} = 0.$$

Similarly, integration on the third equation of (1.6) suggests that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w(\cdot,t)\,\mathrm{d}x + \int_\Omega w(\cdot,t)\,\mathrm{d}x = \int_\Omega n(\cdot,t)\,\mathrm{d}x = \int_\Omega n_0(x)\,\mathrm{d}x \quad \forall\, t \in [0, T_{\max}),$$

which yields (2.4) by taking the time integral. By the same process, we can obtain (2.3) and complete the proof. □

We give the following lemma and it will be frequently used in the rest of the chapters.

Lemma 2.3. *For* $m, n < 1$, $\gamma, \delta > 0$ *with* $\gamma \neq \delta$ *there exists* $C = C(m, n, \delta, \gamma) > 0$ *such that for all* $t > 0$,

$$\int_0^t (1 + (t-s)^{-m})\mathrm{e}^{-\gamma(t-s)}(1 + s^{-n})\mathrm{e}^{-\delta s}\,\mathrm{d}s \leqslant C(1 + t^{\min\{0, 1-m-n\}})\mathrm{e}^{-\min\{\gamma,\delta\}t}.$$

Proof. The proof can be found in [22], Lemma 1.2. □

Fix $\alpha_1 \in (0, \lambda_1)$ and $p_0 \in (N, Nq_0/(N-q_0))$ with $q_0 \in (N/2, N)$. Since $1 - \frac{1}{2} > 0$ and $1 - \frac{1}{2} - N/(2p_0) \geqslant -\frac{1}{2}$, based on Lemma 2.3, there exist $l_1, l_2, l_3 > 0$ such that for all $t > 0$,

$$\int_0^t \mathrm{e}^{-(\lambda_1+1)(t-s)}(1 + s^{-1/2})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \leqslant l_1 \mathrm{e}^{-\alpha_1 t}, \tag{2.5}$$

$$\int_0^t (1 + (t-s)^{-1/2-N/(2p_0)})\mathrm{e}^{-(\lambda_1+1)(t-s)}(1 + s^{-1+N/(2p_0)})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \leqslant l_2(1 + t^{-1/2})\mathrm{e}^{-\alpha_1 t}, \tag{2.6}$$

$$\int_0^t (1 + (t-s)^{-1/2})\mathrm{e}^{-(\lambda_1+1)(t-s)}(1 + s^{-N/(2q_0)})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \leqslant l_3(1 + t^{-1/2})\mathrm{e}^{-\alpha_1 t}. \tag{2.7}$$

Let $\alpha_2 \in (0, \min\{\alpha_1, \lambda_1'\})$, we can choose $\mu \in (\alpha_2, \lambda_1')$ satisfying $\alpha_2 < \min\{\alpha_1, \mu\}$. Due to $1 - \frac{1}{2} - 1 + N/(2p_0) < \frac{1}{2}$ and $1 - \frac{1}{2}N(q_0^{-1} - p_0^{-1}) > 0$, there exist l_4, $l_5 > 0$ such that for all $t > 0$,

$$\int_0^t \mathrm{e}^{-\mu(t-s)}(1 + s^{-N(q_0^{-1}-p_0^{-1})/2})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \leqslant l_4(1 + t^{-1/2+N/(2p_0)})\mathrm{e}^{-\alpha_2 t},$$

$$\int_0^t (t-s)^{-1/2}\mathrm{e}^{-\mu(t-s)}(1 + s^{-1+N/(2p_0)})\mathrm{e}^{-\alpha_2 s}\,\mathrm{d}s \leqslant l_5(1 + t^{-1/2+N/(2p_0)})\mathrm{e}^{-\alpha_2 t}.$$

Since $1 - \frac{1}{2} - \frac{1}{2}N(q_0^{-1} - p_0^{-1}) > 0$ and $\frac{1}{2} + N/(2p_0) < 1$ for some l_6, $l_7 > 0$ we have

$$\int_0^t (t-s)^{-1/2}\mathrm{e}^{-\mu(t-s)}(1 + s^{-N(q_0^{-1}-p_0^{-1})/2})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \leqslant l_6(1 + t^{-1/2})\mathrm{e}^{-\alpha_2 t},$$

$$\int_0^t (t-s)^{-1/2-N/(2p_0)}\mathrm{e}^{-\mu(t-s)}(1 + s^{-1+N/(2p_0)})\mathrm{e}^{-\alpha_2 s}\,\mathrm{d}s \leqslant l_7(1 + t^{-1/2})\mathrm{e}^{-\alpha_2 t}.$$

We can deduce that $\frac{1}{2} + \frac{1}{2}N(p_0^{-1} - \theta^{-1}) < 1$ since $\theta \in [p_0, \infty]$. Once more, we make use of Lemma 2.3, there exist $l_8, l_9 > 0$ such that for all $t > 0$

$$\int_0^t (1 + (t-s)^{-1/2-N(p_0^{-1}-\theta^{-1})/2})\mathrm{e}^{-\lambda_1(t-s)}(1 + s^{-1/2-N(q_0^{-1}-p_0^{-1})/2})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \leqslant l_8(1 + t^{-N(q_0^{-1}-\theta^{-1})/2})\mathrm{e}^{-\alpha_1 t},$$

$$\int_0^t (1 + (t-s)^{-1/2-N(p_0^{-1}-\theta^{-1})/2})\mathrm{e}^{-\lambda_1(t-s)}(1 + s^{-1/2})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \leqslant l_9(1 + t^{-N(q_0^{-1}-\theta^{-1})/2})\mathrm{e}^{-\alpha_1 t}.$$

Now we recall the $L^p - L^q$ estimates for Neumann heat semigroup in bounded domains.

Lemma 2.4. *Let* $(\mathrm{e}^{t\Delta})_{t\geqslant 0}$ *be the Neumann heat semigroup in* Ω *with* $\lambda_1 > 0$ *denoting the first nonzero eigenvalue of* $-\Delta$ *under Neumann boundary condition. Then there exist* $K_1, K_2, K_3, K_4 > 0$ *depending only on* Ω *which have the following properties:*

▷ *If* $1 \leqslant q \leqslant p \leqslant \infty$, *then*

$$\|\mathrm{e}^{t\Delta}\varphi\|_{L^p(\Omega)} \leqslant K_1(1 + t^{-N(q^{-1}-p^{-1})/2})\mathrm{e}^{-\lambda_1 t}\|\varphi\|_{L^q(\Omega)}, \quad t > 0$$

for $\varphi \in L^q(\Omega)$ *satisfying* $\int_\Omega \varphi = 0$.

▷ *If* $1 \leqslant q \leqslant p \leqslant \infty$, *then*

$$\|\nabla\mathrm{e}^{t\Delta}\varphi\|_{L^p(\Omega)} \leqslant K_2(1 + t^{-1/2-N(q^{-1}-p^{-1})/2})\mathrm{e}^{-\lambda_1 t}\|\varphi\|_{L^q(\Omega)}, \quad t > 0$$

for $\varphi \in L^q(\Omega)$.

▷ *If* $2 \leqslant q \leqslant p \leqslant \infty$, *then*

$$\|\nabla\mathrm{e}^{t\Delta}\varphi\|_{L^p(\Omega)} \leqslant K_3(1 + t^{-N(q^{-1}-p^{-1})/2})\mathrm{e}^{-\lambda_1 t}\|\nabla\varphi\|_{L^q(\Omega)}, \quad t > 0$$

for $\varphi \in W^{1,q}(\Omega)$.

▷ *If* $1 < q \leqslant p < \infty$ *or* $1 < q < p = \infty$, *then*

$$\|\mathrm{e}^{t\Delta}\nabla\cdot\varphi\|_{L^p(\Omega)} \leqslant K_4(1 + t^{-1/2-N(q^{-1}-p^{-1})/2})\mathrm{e}^{-\lambda_1 t}\|\varphi\|_{L^q(\Omega)}, \quad t > 0$$

for $\varphi \in (L^q(\Omega))^N$.

Proof. The proof can be found in [22], Lemma 1.3 and [2], Lemma 2.1. □

The following two lemmas could be found in [3], Lemmas 2.2 and 2.3; here we omit the detailed proofs.

Lemma 2.5. *For* $p \in (1,\infty)$, *the Helmholtz projection* $\mathcal{P}$ *defines a bounded linear operator* $\mathcal{P}\colon L^p(\Omega,\mathbb{R}^N) \to L^p_\sigma(\Omega)$, *i.e., there exists a constant* $k_1(p) > 0$ *such that*

$$\|\mathcal{P}w\|_{L^p(\Omega)} \leqslant k_1(p)\|w\|_{L^p(\Omega)}$$

for $w \in L^p(\Omega)$.

Lemma 2.6. *Let* $(\mathrm{e}^{t\mathcal{A}})_{t\geqslant 0}$ *be the analytic semigroup generated by Stokes operator* $\mathcal{A}$ *on* $L^p_\sigma(\Omega)$. *For any fixed* $\mu \in (0,\lambda_1')$ *with* $\lambda_1' := \inf \operatorname{Re}\sigma(\mathcal{A}) > 0$ *denoting the first eigenvalue of* $\mathcal{A}$ *under the homogeneous Dirichlet boundary data, the following estimates hold:*

▷ *If* $1 < p < \infty$ *and* $\gamma \geqslant 0$, *then there exits* $k_2(p,\gamma) > 0$ *such that*

$$\|\mathcal{A}^\gamma \mathrm{e}^{-t\mathcal{A}} w\|_{L^p(\Omega)} \leqslant k_2(p,\gamma) t^{-\gamma} \mathrm{e}^{-\mu t} \|w\|_{L^p(\Omega)}, \quad t > 0$$

for $w \in L^p_\sigma(\Omega)$.

▷ *If* $1 < q \leqslant p < \infty$, *then there exits* $k_3(p,q) > 0$ *such that*

$$\|\mathrm{e}^{-t\mathcal{A}} w\|_{L^p(\Omega)} \leqslant k_3(q,p) t^{-N(q^{-1}-p^{-1})/2} \mathrm{e}^{-\mu t} \|w\|_{L^q(\Omega)}, \quad t > 0$$

for $w \in L^q_\sigma(\Omega)$.

▷ *If* $1 < q \leqslant p < \infty$, *then there exits* $k_4(p,q) > 0$ *such that*

$$\|\nabla \mathrm{e}^{-t\mathcal{A}} w\|_{L^p(\Omega)} \leqslant k_4(q,p) t^{-1/2-N(q^{-1}-p^{-1})/2} \mathrm{e}^{-\mu t} \|w\|_{L^q(\Omega)}, \quad t > 0$$

for $w \in L^q_\sigma(\Omega)$.

▷ *If* $\gamma \geqslant 0$ *and* $1 < q < p < \infty$ *with* $2\gamma - N/q \geqslant 1 - N/p$, *then there exits* $k_5(\gamma,p,q) > 0$ *such that*

$$\|w\|_{W^{1,p}(\Omega)} \leqslant k_5(\gamma,q,p) \|\mathcal{A}^\gamma w\|_{L^q(\Omega)}, \quad t > 0$$

for $w \in \mathcal{D}_q(\mathcal{A}^\gamma)$.

Next, we define some constants which will be used in the following lemmas. The constants $l_i > 0$ $(i = 1,\dots,9)$ are introduced after Lemma 2.3. The inequalities $K_j > 0$ $(j = 1,\dots,4)$ come from Lemma 2.4, k_r $(r = 1,\dots,5) > 0$ are constants in Lemmas 2.5 and 2.6. With these constants, we have the following lemma.

Lemma 2.7. *For any* $p_0 \in (N, Nq_0/(N-q_0))$ *with* $q_0 \in (\frac{1}{2}N, N)$, *there exist positive constants* C_1, C_2, C_3, C_4 *such that*

(2.8) $$C_1 \geqslant \frac{2K_3 + 2K_3C_2l_1}{1 - 6K_2C_3l_2\varepsilon_0},$$

(2.9) $$C_2 \geqslant \frac{2K_3 + 2K_2(1+2K_1)l_3}{1 - 6K_2C_3l_2\varepsilon_0},$$

$$C_3 \geqslant \frac{2k_3(p_0,N) + 2k_3(p_0,p_0)k_1(p_0)(1+2K_1)\|\nabla\phi\|_{L^\infty(\Omega)}l_4}{1 - 6k_3(\Upsilon,p_0)k_1(\Upsilon)C_4l_5\varepsilon_0},$$

$$C_4 \geqslant \frac{2k_4(N,N) + 2k_4(N,N)k_1(N)(1+2K_1)\|\nabla\phi\|_{L^\infty(\Omega)}l_6}{1 - 6k_4(\Upsilon,N)k_1(\Upsilon)C_3l_7\varepsilon_0},$$

(2.10) $$(3K_4(1+2K_1)C_1l_8 + K_4C_1|\Omega|^{p_0^{-1}-q_0^{-1}}l_9 + 3K_4(1+2K_1)C_3l_8)\varepsilon_0 \leqslant \frac{1}{2},$$

where $\Upsilon = Np_0/(N+p_0)$,

(2.11) $$\varepsilon_0 := \min\Big\{\frac{1}{12K_2C_3l_2}, \frac{1}{12k_3(\Upsilon,p_0)k_1(\Upsilon)C_4l_5}, \frac{1}{12k_4(\Upsilon,N)k_1(\Upsilon)C_3l_7}, \frac{1}{6K_4(1+2K_1)C_1l_8 + 2K_4C_1|\Omega|^{p_0^{-1}-q_0^{-1}}l_9 + 6K_4(1+2K_1)C_3l_8}\Big\}.$$

P r o o f. Fix

$$C_1 = 4K_3 + 16K_3^2 l_1 + 16K_2K_3(1+2K_1)l_3l_1, \tag{2.12}$$

$$C_2 = 4K_3 + 4K_2(1+2K_1)l_3, \tag{2.13}$$

$$C_3 = 4k_3(p_0,N) + 4k_3(p_0,p_0)k_1(p_0)(1+2K_1)\|\nabla\phi\|_{L^\infty(\Omega)}l_4,$$

$$C_4 = 4k_4(N,N) + 4k_4(N,N)k_1(N)(1+2K_1)\|\nabla\phi\|_{L^\infty(\Omega)}l_6. \tag{2.14}$$

We give the proof for (2.8) and (2.9). Using simple calculation with (2.12), (2.13) and (2.11), we can obtain that

$$\begin{aligned}\frac{2K_3 + 2K_3C_2l_1}{1 - 6K_2C_3l_2\varepsilon_0} &\leqslant \frac{2K_3 + 8K_3^2l_1 + 8K_2K_3(1+2K_1)l_3l_1}{1 - 6K_2C_3l_2/(12K_2C_3l_2)} \\ &\leqslant 4K_3 + 16K_3^2l_1 + 16K_2K_3(1+2K_1)l_3l_1 = C_1\end{aligned}$$

and

$$\begin{aligned}\frac{2K_3 + 2K_2(1+2K_1)l_3}{1 - 6K_2C_3l_2\varepsilon_0} &= \frac{C_2}{2 - 12K_2C_3l_2\varepsilon_0} \\ &\leqslant \frac{C_2}{2 - 12K_2C_3l_2/(12K_2C_3l_2)} = C_2.\end{aligned}$$

By the same procedure, we can deduce that

$$\begin{aligned}&\frac{2k_3(p_0,N) + 2k_3(p_0,p_0)k_1(p_0)(1+2K_1)\|\nabla\phi\|_{L^\infty(\Omega)}l_4}{1 - 6k_3(\Upsilon,p_0)k_1(\Upsilon)C_4l_5\varepsilon_0} \\ &\quad\leqslant \frac{2k_3(p_0,N) + 2k_3(p_0,p_0)k_1(p_0)(1+2K_1)\|\nabla\phi\|_{L^\infty(\Omega)}l_4}{1 - 6k_3(\Upsilon,p_0)k_1(\Upsilon)C_4l_5/(12k_3(\Upsilon,p_0)k_1(\Upsilon)C_4l_5)} = C_3\end{aligned}$$

and

$$\begin{aligned}&\frac{2k_4(N,N) + 2k_4(N,N)k_1(N)(1+2K_1)\|\nabla\phi\|_{L^\infty(\Omega)}l_6}{1 - 6k_4(\Upsilon,N)k_1(\Upsilon)C_3l_7\varepsilon_0} \\ &\quad\leqslant \frac{2k_4(N,N) + 2k_4(N,N)k_1(N)(1+2K_1)\|\nabla\phi\|_{L^\infty(\Omega)}l_6}{1 - 6k_4(\Upsilon,N)k_1(\Upsilon)C_3l_7/(12k_4(\Upsilon,N)k_1(\Upsilon)C_3l_7)} = C_4.\end{aligned}$$

Thus, we conclude the lemma. □

Lemma 2.1 asserts that there is a classical solution to (1.6), which is defined on an interval $[0, T_{\max})$. In order to prove $T_{\max} = \infty$, we need the following definition.

Definition 2.8. Assume that $N \in \{2,3\}$, $\phi \in C^{1+\delta}(\overline{\Omega})$ with $\delta \in (0,1)$. Let $\alpha_1 \in (0, \lambda_1)$, $\alpha_2 \in (0, \min\{\alpha_1, \lambda_1'\})$, $p_0 \in (N, Nq_0/(N-q_0))$ with $q_0 \in (\frac{1}{2}N, N)$ and C_i

$(i=1,2,3,4)$, $\varepsilon_0>0$ are taken from Lemma 2.7. By use of these constants, we define

$$
\begin{aligned}
(2.15)\qquad T:=\sup\{\widetilde{T}\in(0,T_{\max})\colon\ & \|n(\cdot,t)-\mathrm{e}^{t\Delta}n_0\|_{L^\theta(\Omega)}\leqslant\varepsilon_0(1+t^{-N(q_0^{-1}-\theta^{-1})/2})\mathrm{e}^{-\alpha_1 t}\ \forall\,\theta\in[p_0,\infty],\\
&\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}\leqslant C_1\varepsilon_0(1+t^{-1/2})\mathrm{e}^{-\alpha_1 t},\\
&\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)}\leqslant C_2\varepsilon_0(1+t^{-1/2})\mathrm{e}^{-\alpha_1 t},\\
&\|\mathbf{u}(\cdot,t)\|_{L^{p_0}(\Omega)}\leqslant C_3\varepsilon_0(1+t^{-1/2+N/2p_0})\mathrm{e}^{-\alpha_2 t},\\
&\|\nabla\mathbf{u}(\cdot,t)\|_{L^N(\Omega)}\leqslant C_4\varepsilon_0(1+t^{-1/2})\mathrm{e}^{-\alpha_2 t}\ \forall\,t\in[0,\widetilde{T}).\}
\end{aligned}
$$

Next, with the help of Lemma 2.7 and Definition 2.8, we will re-examine the decay properties showed in (2.15) to confirm $T=T_{\max}=\infty$ by the following Lemmas 2.9–2.11. Since some of the following estimates have been proved in [27], Lemmas 3.3–3.6, we will omit their proofs.

Lemma 2.9. *Under the conditions of Definition* 2.8 *and* (2.1), *the estimates*

$$(2.16)\qquad \|n(\cdot,t)-\bar{n}_0\|_{L^\theta(\Omega)}\leqslant(1+2K_1)\varepsilon_0(1+t^{-N(q_0^{-1}-\theta^{-1})/2})\mathrm{e}^{-\alpha_1 t}\quad\forall\,\theta\in[p_0,\infty],$$

$$(2.17)\qquad \|\nabla w(\cdot,t)\|_{L^\infty(\Omega)}\leqslant\frac{C_2}{2}\varepsilon_0(1+t^{-1/2})\mathrm{e}^{-\alpha_1 t},$$

$$(2.18)\qquad \|\mathbf{u}(\cdot,t)\|_{L^{p_0}(\Omega)}\leqslant\frac{C_3}{2}\varepsilon_0(1+t^{-1/2+N/(2p_0)})\mathrm{e}^{-\alpha_2 t},$$

$$(2.19)\qquad \|\nabla\mathbf{u}(\cdot,t)\|_{L^N(\Omega)}\leqslant\frac{C_4}{2}\varepsilon_0(1+t^{-1/2})\mathrm{e}^{-\alpha_2 t},$$

hold for all $t\in(0,T)$, *where* T *is defined as in Definition* 2.8.

P r o o f. The proof can be found in [27], Lemmas 3.3–3.6. The verification of (2.17) used the fact that the equation satisfied by w in our present case is the same as that of c in the model of [27]. □

Now we give the proof of the estimate of $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}$ in Lemma 2.10.

Lemma 2.10. *Under the conditions of Definition* 2.8 *and* (2.1), *the estimate*

$$(2.20)\qquad \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}\leqslant\frac{C_1}{2}\varepsilon_0(1+t^{-1/2})\mathrm{e}^{-\alpha_1 t}$$

holds for all $t\in(0,T)$, *where* T *is defined as in Definition* 2.8.

P r o o f. According to $(1.6)_2$ we have the following representation formula for all $t\in(0,T)$,

$$v(\cdot,t)=\mathrm{e}^{t(\Delta-1)}v_0+\int_0^t\mathrm{e}^{(t-s)(\Delta-1)}(w-\mathbf{u}\cdot\nabla v)(\cdot,s)\,\mathrm{d}s.$$

Applying Lemma 2.4, we have $K_2, K_3 > 0$ such that

$$
\begin{aligned}
(2.21)\quad \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} &\leqslant K_3(1+t^{-1/2})\mathrm{e}^{-(\lambda_1+1)t}\|\nabla v_0\|_{L^N(\Omega)} \\
&\quad + \int_0^t \|\nabla \mathrm{e}^{(t-s)(\Delta-1)} w(\cdot,s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\
&\quad + \int_0^t \|\nabla \mathrm{e}^{(t-s)(\Delta-1)} \mathbf{u}(\cdot,s)\cdot\nabla v(\cdot,s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\
&=: K_3(1+t^{-1/2})\mathrm{e}^{-(\lambda_1+1)t}\|\nabla v_0\|_{L^N(\Omega)} + I_1 + I_2, \quad t\in(0,T).
\end{aligned}
$$

Next we focus on estimating I_1, I_2. According to $(2.15)_3$ and (2.5) and Lemma 2.4,

$$
\begin{aligned}
(2.22)\qquad I_1 &\leqslant \int_0^t K_3 \mathrm{e}^{-(\lambda_1+1)(t-s)}\|\nabla w(\cdot,s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\
&\leqslant \int_0^t K_3 C_2\varepsilon_0 \mathrm{e}^{-(\lambda_1+1)(t-s)}(1+s^{-1/2})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \\
&\leqslant K_3 C_2 l_1 \varepsilon_0 \mathrm{e}^{-\alpha_1 t}, \quad t\in(0,T).
\end{aligned}
$$

At this point, using $(2.15)_{2,4}$ (2.6), Lemma 2.4 and Hölder's inequality, we derive that

(2.23)

$$
\begin{aligned}
I_2 &\leqslant \int_0^t K_2(1+(t-s)^{-1/2-N/2p_0})\mathrm{e}^{-(\lambda_1+1)(t-s)}\|\mathbf{u}(\cdot,s)\cdot\nabla v(\cdot,s)\|_{L^{p_0}(\Omega)}\,\mathrm{d}s \\
&\leqslant \int_0^t K_2(1+(t-s)^{-1/2-N/2p_0})\mathrm{e}^{-(\lambda_1+1)(t-s)}\|\mathbf{u}(\cdot,s)\|_{L^{p_0}(\Omega)}\cdot\|\nabla v(\cdot,s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\
&\leqslant \int_0^t K_2(1+(t-s)^{-1/2-N/2p_0})\mathrm{e}^{-(\lambda_1+1)(t-s)} C_1 C_3 (1+s^{-1/2+N/(2p_0)}) \\
&\quad \times (1+s^{-1/2})\mathrm{e}^{-(\alpha_1+\alpha_2)s}\varepsilon_0^2\,\mathrm{d}s \\
&\leqslant 3K_2C_1C_3\varepsilon_0^2 \int_0^t (1+(t-s)^{-1/2-N/2p_0})\mathrm{e}^{-(\lambda_1+1)(t-s)}(1+s^{-1+N/2p_0})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \\
&\leqslant 3K_2C_1C_3 l_2 (1+t^{-1/2})\mathrm{e}^{-\alpha_1 t}\varepsilon_0^2, \quad t\in(0,T).
\end{aligned}
$$

Collecting (2.21)–(2.23) and (2.1), we obtain

$$
\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} &\leqslant (K_3 + K_3C_2l_1 + 3K_2C_1C_3l_2\varepsilon_0)(1+t^{-1/2})\mathrm{e}^{-\alpha_1 t}\varepsilon_0 \\
&\leqslant \frac{C_1}{2}\varepsilon_0(1+t^{-1/2})\mathrm{e}^{-\alpha_1 t} \quad \forall\, t\in(0,T).
\end{aligned}
$$

□

With lemmas above in hand, we can furthermore obtain the following estimates for n and $\mathbf{u}$ from [27], Lemma 3.7 and [3], Lemma 4.9.

Lemma 2.11. *Under the conditions of Definition* 2.8 *and* (2.1), $\beta \in (\frac{1}{4}N, 1)$, *the estimates*

$$\|n(\cdot,t) - \mathrm{e}^{t\Delta}n_0\|_{L^\theta(\Omega)} \leqslant \frac{\varepsilon_0}{2}(1 + t^{-N(q_0^{-1}-\theta^{-1})/2})\mathrm{e}^{-\alpha_1 t} \quad \forall\,\theta \in [p_0, \infty], \tag{2.24}$$

$$\|\mathcal{A}^\beta \mathbf{u}(\cdot,t)\|_{L^2(\Omega)} \leqslant C_5 \mathrm{e}^{-\alpha_2 t} \tag{2.25}$$

hold for all $t \in (0,T)$, *where* T *is defined as in Definition* 2.8 *and* $C_5 = C_5(\beta, \varepsilon_0) > 0$.

Next, we prove $T = T_{\max} = \infty$ using the lemmas we have obtained above to state that the classical solution of (1.6) is global.

Lemma 2.12. *Under the conditions of Definition* 2.8 *and the smallness hypothesis* (2.1) *of initial data, the solution of* (1.6) *exists globally in time with the decay properties listed in* (1.10).

P r o o f. Firstly, we prove $T = T_{\max} = \infty$. If $T_{\max} > T$, according to (2.18)–(2.20) and (2.24) we have

$$\begin{aligned}
\|n(\cdot,T) - \mathrm{e}^{t\Delta}n_0\|_{L^\theta(\Omega)} &\leqslant \frac{\varepsilon_0}{2}(1 + T^{-N(q_0^{-1}-\theta^{-1})/2})\mathrm{e}^{-\alpha_1 T} \quad \forall\,\theta \in [p_0, \infty],\\
\|\nabla v(\cdot,T)\|_{L^\infty(\Omega)} &\leqslant \frac{C_1}{2}\varepsilon_0(1 + T^{-1/2})\mathrm{e}^{-\alpha_1 T},\\
\|\nabla w(\cdot,T)\|_{L^\infty(\Omega)} &\leqslant \frac{C_2}{2}\varepsilon_0(1 + T^{-1/2})\mathrm{e}^{-\alpha_1 T},\\
\|\mathbf{u}(\cdot,T)\|_{L^{p_0}(\Omega)} &\leqslant \frac{C_3}{2}\varepsilon_0(1 + T^{-1/2+N/2p_0})\mathrm{e}^{-\alpha_2 T},\\
\|\nabla \mathbf{u}(\cdot,T)\|_{L^N(\Omega)} &\leqslant \frac{C_4}{2}\varepsilon_0(1 + T^{-1/2})\mathrm{e}^{-\alpha_2 T}
\end{aligned}$$

due to the fact that solutions of (1.6) continuously depend on t. This contradicts the definition of T, see Definition 2.8.

If $T_{\max} = T < \infty$, from Lemma 2.1 and (2.25) we know that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \to \infty \quad \text{as } t \to T_{\max}. \tag{2.26}$$

Since $\|n(\cdot,t)\|_{L^\infty(\Omega)}$, $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)}$ are finite for $t \in (0, T_{\max})$ due to $(2.15)_{1,2,3}$, we can derive the following estimates for $\|v(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|w(\cdot,t)\|_{L^\infty(\Omega)}$ by use of Poincaré's inequality and Lemma 2.2. Namely, there exist some $C_{vp}, C_{wp} > 0$ such that for all $t \in (0,T)$,

$$\begin{aligned}
\Big\|w(\cdot,t) - \frac{1}{|\Omega|}\int_\Omega w_0(x)\,\mathrm{d}x\Big\|_{L^\infty(\Omega)} &\leqslant C_{wp}\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)},\\
\Big\|v(\cdot,t) - \frac{1}{|\Omega|}\int_\Omega v_0(x)\,\mathrm{d}x\Big\|_{L^\infty(\Omega)} &\leqslant C_{vp}\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}.
\end{aligned}$$

Hence, for some positive constants

$$C_w := \frac{\max\{\int_\Omega w_0, \int_\Omega n_0\}}{|\Omega|}, \quad C_v := \frac{\max\{\int_\Omega v_0, \int_\Omega w_0, \int_\Omega n_0\}}{|\Omega|},$$

for all $t \in (0,T)$ we have

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C_{wp}\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} + C_w, \tag{2.27}$$

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C_{vp}\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} + C_v. \tag{2.28}$$

Combining (2.27), (2.28) with the finiteness of $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)}$, we can finally confirm the finiteness of $\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ and $\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}$. This contradicts (2.26). Therefore, $T_{\max} = T = \infty$.

Now, we give the proof of the decaying properties listed in (1.10).

From (2.25) and embedding property with $D(\mathcal{A}^\beta) \hookrightarrow L^\infty(\Omega)$, $\beta \in (N/4, 1)$, there exists $C_6 > 0$ such that

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \|\mathcal{A}^\beta \mathbf{u}(\cdot,t)\|_{L^2(\Omega)} \leqslant C_6 \mathrm{e}^{-\alpha_2 t}, \quad t > 0.$$

As for the estimates for n, v and w, we consider them in two time intervals: $(0,t_1)$ and $[t_1,\infty)$, where $t_1 \in (0,\tau]$. Firstly we focus on $(0,t_1)$. Lemma 2.1 guarantees that $\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leqslant \eta$, hence by Minkowski's inequality for all $t \in (0,t_1)$ we have

$$\|n(\cdot,t) - \bar{n}_0\|_{L^\infty(\Omega)} \leqslant \|n(\cdot,t)\|_{L^\infty(\Omega)} + \|\bar{n}_0\|_{L^\infty(\Omega)} \leqslant (\eta + \bar{n}_0)\mathrm{e}^{\alpha_1 t_1}\mathrm{e}^{-\alpha_1 t}. \tag{2.29}$$

Similarly, we can obtain

$$\|v(\cdot,t) - \bar{n}_0\|_{W^{1,\infty}(\Omega)} \leqslant \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|\bar{n}_0\|_{L^\infty(\Omega)} \leqslant (\eta + \bar{n}_0)\mathrm{e}^{\alpha_1 t_1}\mathrm{e}^{-\alpha_1 t}, \tag{2.30}$$

$$\|w(\cdot,t) - \bar{n}_0\|_{W^{1,\infty}(\Omega)} \leqslant \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|\bar{n}_0\|_{L^\infty(\Omega)} \leqslant (\eta + \bar{n}_0)\mathrm{e}^{\alpha_1 t_1}\mathrm{e}^{-\alpha_1 t} \tag{2.31}$$

for all $t \in (0,t_1)$.

Secondly, we consider $t \geqslant t_1$. According to (2.16), we can deduce that

$$\begin{aligned} \|n(\cdot,t) - \bar{n}_0\|_{L^\infty(\Omega)} &\leqslant (1+2K_1)\varepsilon_0(1 + t^{-N/2q_0})\mathrm{e}^{-\alpha_1 t} \\ &\leqslant (1+2K_1)\varepsilon_0(1 + t_1^{-N/2q_0})\mathrm{e}^{-\alpha_1 t} \end{aligned} \tag{2.32}$$

for all $t \geqslant t_1$.

From (2.29) and (2.32) we can conclude that for all $t > 0$, $\|n(\cdot,t) - \bar{n}_0\|_{L^\infty(\Omega)} \leqslant C_7 \mathrm{e}^{-\alpha_1 t}$, where $C_7 := \max\{(\eta + \bar{n}_0)\mathrm{e}^{\alpha_1 t_1}, (1+2K_1)\varepsilon_0(1 + t_1^{-N/2q_0})\}$.

Now we estimate the stabilization of $w(\cdot,t)$. Since

$$\begin{aligned} w(\cdot,t)-\bar{n}_0 &= \mathrm{e}^{t(\Delta-1)}w_0+\int_0^t \mathrm{e}^{(t-s)(\Delta-1)}(n(\cdot,s)-\bar{n}_0)\,\mathrm{d}s \\ &\quad+\int_0^t \mathrm{e}^{(t-s)(\Delta-1)}\mathbf{u}(\cdot,s)\cdot\nabla w(\cdot,s)\,\mathrm{d}s-\mathrm{e}^{-t}\bar{n}_0 \\ &\leqslant \mathrm{e}^{t(\Delta-1)}w_0+\int_0^t \mathrm{e}^{(t-s)(\Delta-1)}(n(\cdot,s)-\bar{n}_0)\,\mathrm{d}s \\ &\quad+\int_0^t \mathrm{e}^{(t-s)(\Delta-1)}\mathbf{u}(\cdot,s)\cdot\nabla w(\cdot,s)\,\mathrm{d}s \end{aligned}$$

hold for all $t>0$. Applying (2.17), Lemmas 2.4 and 2.3, we can find $C_8>0$ such that

$$\begin{aligned} &\|w(\cdot,t)-\bar{n}_0\|_{L^\infty(\Omega)} \\ &\leqslant \|\mathrm{e}^{t(\Delta-1)}w_0\|_{L^\infty(\Omega)}+\int_0^t \|\mathrm{e}^{(t-s)(\Delta-1)}(n(\cdot,s)-\bar{n}_0)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\ &\quad+\int_0^t \|\mathrm{e}^{(t-s)(\Delta-1)}\mathbf{u}(\cdot,s)\cdot\nabla w(\cdot,s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\ &\leqslant 2K_1\mathrm{e}^{-(\lambda_1+1)t}\|w_0\|_{L^\infty(\Omega)} \\ &\quad+\int_0^t K_1(1+2K_1)\varepsilon_0\mathrm{e}^{-(\lambda_1+1)(t-s)}(1+s^{-N/(2q_0)})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \\ &\quad+\int_0^t \mathrm{e}^{-(\lambda_1+1)(t-s)}C_6\mathrm{e}^{-\alpha_2 s}C_2\varepsilon_0(1+s^{-1/2})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \\ &\leqslant C_8\mathrm{e}^{-\alpha_1 t},\quad t\geqslant t_1. \end{aligned} \tag{2.33}$$

According to $\nabla\mathrm{e}^{(t-s)(\Delta-1)}\bar{n}_0=0$, there exists $C_9>0$ such that

$$\begin{aligned} &\|\nabla(w(\cdot,t)-\bar{n}_0)\|_{L^\infty(\Omega)} \\ &\leqslant \|\nabla\mathrm{e}^{t(\Delta-1)}w_0\|_{L^\infty(\Omega)}+\int_0^t \|\nabla\mathrm{e}^{(t-s)(\Delta-1)}(n(\cdot,s)-\bar{n}_0)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\ &\quad+\int_0^t \|\nabla\mathrm{e}^{(t-s)(\Delta-1)}\mathbf{u}(\cdot,s)\cdot\nabla w(\cdot,s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\ &\leqslant K_2(1+{t_0}^{-1/2})\mathrm{e}^{-(\lambda_1+1)t}\|w_0\|_{L^\infty(\Omega)} \\ &\quad+K_2(1+2K_1)\varepsilon_0\int_0^t(1+(t-s)^{-1/2})\mathrm{e}^{-(\lambda_1+1)(t-s)}(1+s^{-N/(2q_0)})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \\ &\quad+K_2\int_0^t(1+(t-s)^{-1/2})\mathrm{e}^{-(\lambda_1+1)(t-s)}C_6\mathrm{e}^{-\alpha_2 s}C_2\varepsilon_0(1+s^{-1/2})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \\ &\leqslant C_9\mathrm{e}^{-\alpha_1 t},\quad t\geqslant t_1. \end{aligned} \tag{2.34}$$

Collecting (2.31), (2.33) and (2.34) we can find $C_{10} > 0$ such that for all $t > 0$,

$$\|w(\cdot,t) - \bar{n}_0\|_{W^{1,\infty}(\Omega)} \leqslant C_{10}\mathrm{e}^{-\alpha_1 t}.$$

As for the estimates for v, according to the smooth properties of the Neumann heat semigroup with (2.33), (2.18) and Lemma 2.10, there exists constant $C_{11} > 0$ such that

(2.35)

$$\begin{aligned}
&\|v(\cdot,t) - \bar{n}_0\|_{L^\infty(\Omega)} \\
&\quad \leqslant \|\mathrm{e}^{t(\Delta-1)}v_0\|_{L^\infty(\Omega)} + \int_0^t \|\mathrm{e}^{(t-s)(\Delta-1)}(w(\cdot,s) - \bar{n}_0)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\
&\qquad + \int_0^t \|\mathrm{e}^{(t-s)(\Delta-1)}\mathbf{u}(\cdot,s)\cdot\nabla v(\cdot,s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\
&\quad \leqslant 2K_1\mathrm{e}^{-(\lambda_1+1)t}\|v_0\|_{L^\infty(\Omega)} + \int_0^t 2C_{10}K_1\mathrm{e}^{-(\lambda_1+1)(t-s)}\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \\
&\qquad + K_2\int_0^t (1+(t-s)^{-1/2})\mathrm{e}^{-(\lambda_1+1)(t-s)}C_6\mathrm{e}^{-\alpha_2 s}C_1\varepsilon_0(1+s^{-1/2})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \\
&\quad \leqslant C_{11}\mathrm{e}^{-\alpha_1 t}, \quad t \geqslant t_1.
\end{aligned}$$

Following similar procedure as (2.33)–(2.34), one can find $C_{12} > 0$ such that

$$\|v(\cdot,t) - \bar{n}_0\|_{W^{1,\infty}(\Omega)} \leqslant C_{12}\mathrm{e}^{-\alpha_1 t} \quad \forall\, t \geqslant t_1. \tag{2.36}$$

Due to the fact that (2.30), (2.35) and (2.36) hold, we can finally complete the proof. □

3. Proof of Theorem 1.1

In this section, we improve the smallness assumption about initial data in (2.1). Namely, under the condition that

$$\|n_0\|_{L^{N/2}(\Omega)} + \|\nabla v_0\|_{L^N(\Omega)} + \|\nabla w_0\|_{L^N(\Omega)} + \|\mathbf{u}_0\|_{L^N(\Omega)} \leqslant \tilde{\varepsilon}_0 \tag{3.1}$$

for some $\tilde{\varepsilon}_0 > 0$, we will establish the final decay estimates in Theorem 1.1. Another definition will be needed.

Definition 3.1. Assume that $N \in \{2,3\}$, $\phi \in C^{1+\delta}(\overline{\Omega})$ with $\delta \in (0,1)$. Let $\alpha_1 \in (0,\lambda_1)$, $\alpha_2 \in (0,\min\{\alpha_1,\lambda_1'\})$, $\tilde{p}_0 \in (2\tilde{q}_0, N\tilde{q}_0/(N-\tilde{q}_0))$ with $\tilde{q}_0 \in (N/2,N)$.

We define

$$T := \sup\{\widetilde{T} \in (0, T_{\max})\colon \tag{3.2}$$
$$\|n(\cdot,t) - \mathrm{e}^{t\Delta} n_0\|_{L^\theta(\Omega)} \leqslant \tilde{\varepsilon}_0(1 + t^{-1+N/2\theta})\mathrm{e}^{-\alpha_1 t}, \quad \theta \in [\tilde{q}_0, \tilde{p}_0],$$
$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \widetilde{C}_1 \tilde{\varepsilon}_0(1 + t^{-1/2})\mathrm{e}^{-\alpha_1 t},$$
$$\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \widetilde{C}_2 \tilde{\varepsilon}_0(1 + t^{-1/2})\mathrm{e}^{-\alpha_1 t},$$
$$\|\mathbf{u}(\cdot,t)\|_{L^{\tilde{p}_0}(\Omega)} \leqslant \widetilde{C}_3 \tilde{\varepsilon}_0(1 + t^{-1/2+N/2\tilde{p}_0})\mathrm{e}^{-\alpha_2 t},$$
$$\|\nabla \mathbf{u}(\cdot,t)\|_{L^N(\Omega)} \leqslant \widetilde{C}_4 \tilde{\varepsilon}_0(1 + t^{-1/2})\mathrm{e}^{-\alpha_2 t}\ \forall\, t \in [0, \widetilde{T})\}$$

with $\tilde{\varepsilon}_0, \widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \widetilde{C}_4 > 0$ satisfying

$$\widetilde{C}_1 \geqslant 4K_3 + 4K_3\widetilde{C}_2\tilde{l}_1,$$
$$\widetilde{C}_2 \geqslant 4K_3 + 4K_2(1 + 2K_1)\tilde{l}_3,$$
$$\widetilde{C}_3 \geqslant 4k_3(\tilde{p}_0, N) + 4k_3(\tilde{p}_0, \tilde{p}_0)k_1(\tilde{p}_0)(1 + 2K_1)\|\nabla\phi\|_{L^\infty(\Omega)}\tilde{l}_4,$$
$$\widetilde{C}_4 \geqslant 2k_4(N, N) + 2k_4(N, N)k_1(N)|\Omega|^{(\tilde{p}_0 - N)/(\tilde{p}_0 N)}(1 + 2K_1)\|\nabla\phi\|_{L^\infty(\Omega)}\tilde{l}_6,$$
$$\tilde{\varepsilon}_0 \leqslant \min\Big\{\frac{1}{12K_2\widetilde{C}_3\tilde{l}_2}, \frac{1}{12k_3(\widetilde{\Upsilon}, \tilde{p}_0)k_1(\widetilde{\Upsilon})\widetilde{C}_4\tilde{l}_5}, \frac{1}{12k_4(\widetilde{\Upsilon}, N)k_1(\widetilde{\Upsilon})\widetilde{C}_3\tilde{l}_7},$$
$$\frac{1}{6K_4(1 + 2K_1)\widetilde{C}_1\tilde{l}_8 + 2K_4\widetilde{C}_1|\Omega|^{1/\tilde{q}_0 - 2/N}\tilde{l}_9 + 6K_4(1 + 2K_1)\widetilde{C}_3\tilde{l}_8}\Big\},$$

where $\widetilde{\Upsilon} = N\tilde{p}_0/(N + \tilde{p}_0)$.

The quantities $\tilde{l}_1$, $\tilde{l}_2$ and $\tilde{l}_4$, $\tilde{l}_5$, $\tilde{l}_6$, $\tilde{l}_7$ are analogous to l_1, l_2 and l_4, l_5, l_6, l_7, respectively, which arise in Lemma 2.7 with p_0, q_0 replaced by $\tilde{p}_0$, $\tilde{q}_0$. The quantities $\tilde{l}_3$, $\tilde{l}_8$, $\tilde{l}_9$ are constants which appear in the inequalities

$$\int_0^t (1 + (t-s)^{-1/2-N/2\tilde{p}_0})\mathrm{e}^{-\lambda_1(t-s)}(1 + s^{-1+N/2\tilde{p}_0})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \leqslant \tilde{l}_3(1 + t^{-1/2})\mathrm{e}^{-\alpha_1 t},$$
$$\int_0^t (1 + (t-s)^{-1/2-N/2(N/\tilde{q}_0 - 1/\theta)})\mathrm{e}^{-\lambda_1(t-s)}(1 + s^{-3/2+N/2\tilde{q}_0})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \leqslant \tilde{l}_8(1 + t^{-1+N/2\theta})\mathrm{e}^{-\alpha_1 t},$$
$$\int_0^t (1 + (t-s)^{-1/2-N/2(N/\tilde{q}_0 - 1/\theta)})\mathrm{e}^{-\lambda_1(t-s)}(1 + s^{-1/2})\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s \leqslant \tilde{l}_9(1 + t^{-1+N/2\theta})\mathrm{e}^{-\alpha_1 t},$$

by Lemma 2.3 since $\frac{1}{2} + N/(2\tilde{p}_0) < 1$, $\frac{3}{2} - N/(2\tilde{q}_0) < 1$ and $\frac{1}{2} + \frac{1}{2}N(N/\tilde{q}_0 + 1/\theta) < 1$.

It is also necessary to re-check the decay estimates in (3.2) to show that $T = T_{\max} = \infty$. Similarly to Lemma 2.9, we can obtain some further conclusions by the same methods. We state them in the following lemma.

Lemma 3.2. *Under the conditions of Definition* 3.1 *and* (3.1), *the estimates*

$$\|n(\cdot,t)-\bar{n}_0\|_{L^\theta(\Omega)} \leqslant (1+2K_1)\tilde{\varepsilon}_0(1+t^{-1+N/2\theta})\mathrm{e}^{-\alpha_1 t}, \quad \theta\in[\tilde{q}_0,\tilde{p}_0],$$

$$\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \frac{\widetilde{C}_2}{2}\tilde{\varepsilon}_0(1+t^{-1/2})\mathrm{e}^{-\alpha_1 t}, \tag{3.3}$$

$$\|\mathbf{u}(\cdot,t)\|_{L^{\tilde{p}_0}(\Omega)} \leqslant \frac{\widetilde{C}_3}{2}\tilde{\varepsilon}_0(1+t^{-1/2+N/2\tilde{p}_0})\mathrm{e}^{-\alpha_2 t}, \tag{3.4}$$

$$\|\nabla\mathbf{u}(\cdot,t)\|_{L^N(\Omega)} \leqslant \frac{\widetilde{C}_4}{2}\tilde{\varepsilon}_0(1+t^{-1/2})\mathrm{e}^{-\alpha_2 t} \tag{3.5}$$

hold for all $t\in(0,T)$, *where* T *is defined as in Definition* 3.1.

Proof. This lemma can be concluded by using the procedures as that in the proof of [27], Lemmas 3.9–3.12. □

Based on Lemma 3.2, we have the estimate for ∇v in Lemma 3.3. Since the procedure is analogous to the proof of Lemma 2.10, we omit the details.

Lemma 3.3. *Under the conditions of Definition* 3.1 *and* (3.1), *the estimate*

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \frac{\widetilde{C}_1}{2}\tilde{\varepsilon}_0(1+t^{-1/2})\mathrm{e}^{-\alpha_1 t} \tag{3.6}$$

holds for all $t\in(0,T)$, *where* T *is defined as in Definition* 3.1.

Finally, from Lemma 3.2 and (3.6) we have

$$\|n(\cdot,t)-\mathrm{e}^{t\Delta}n_0\|_{L^\theta(\Omega)} \leqslant \frac{\tilde{\varepsilon}_0}{2}(1+t^{-1+N/2\theta})\mathrm{e}^{-\alpha_1 t}, \quad \theta\in[\tilde{q}_0,\tilde{p}_0], \tag{3.7}$$

$$\|\mathcal{A}^\beta\mathbf{u}(\cdot,t)\|_{L^2(\Omega)} \leqslant \widetilde{C}_5\mathrm{e}^{-\alpha_2 t} \tag{3.8}$$

for all $t\in(0,T)$, where T is defined as in Definition 3.1. More details about the proof can be found in [27], Lemma 3.13 and [3], Lemma 4.9.

Lemma 3.4. *Under the conditions of Definition* 2.8 *and smallness hypothesis* (3.1) *of initial data, system* (1.6) *admits a global classical solution. Moreover, for all* $t>0$,

$$\begin{aligned}
\|n(\cdot,t)-\bar{n}_0\|_{L^\theta(\Omega)} &\leqslant \widetilde{C}_0\tilde{\varepsilon}_0(1+t^{-1+N/2\theta})\mathrm{e}^{-\alpha_1 t}, \quad \theta\in[\tilde{q}_0,\tilde{p}_0],\\
\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} &\leqslant \widetilde{C}_0\tilde{\varepsilon}_0(1+t^{-1/2})\mathrm{e}^{-\alpha_1 t},\\
\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} &\leqslant \widetilde{C}_0\tilde{\varepsilon}_0(1+t^{-1/2})\mathrm{e}^{-\alpha_1 t},\\
\|\mathbf{u}(\cdot,t)\|_{L^\infty(\Omega)} &\leqslant \overline{C}_0\mathrm{e}^{-\alpha_2 t}
\end{aligned}$$

with $\widetilde{C}_0>0$ *independent of* $\tilde{\varepsilon}_0$ *and* $\overline{C}_0=\overline{C}_0(\tilde{\varepsilon}_0)>0$.

P r o o f. Firstly, we focus on deducing $T = T_{\max} = \infty$. In the case $T = T_{\max} < \infty$, we can derive that $\|n(\cdot,t)\|_{L^\infty(\Omega)} < \infty$ if $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded by applying the properties of Neumann heat semigroup. In the case $T_{\max} > T$, from (3.4)–(3.7), we know that the definition of T in (3.2) is failed. According to Lemma 2.1 we can finally conclude $T = T_{\max} = \infty$. The equations (3.3)–(3.8) could yield the required decay estimates in Lemma 3.4. □

Now, we are in the position to prove Theorem 1.1.

P r o o f of Theorem 1.1. Let $q_0 \in [\tilde{q}_0, N)$ and choose the

$$\varepsilon := \min\Big\{\tilde{\varepsilon}_0, \frac{\varepsilon_0}{2|\Omega|^{1/q_0-2/N}}\Big\}, \tag{3.9}$$

where ε_0, $\tilde{\varepsilon}_0$ are determined in Definitions 2.8 and 3.1, respectively. If

$$\|n_0\|_{L^{N/2}(\Omega)} + \|\nabla v_0\|_{L^N(\Omega)} + \|\nabla w_0\|_{L^N(\Omega)} + \|\mathbf{u}_0\|_{L^N(\Omega)} \leqslant \varepsilon,$$

from Lemma 3.4 we can obtain that for all $t > 0$,

$$\begin{gathered}\|n(\cdot,t) - \bar{n}_0\|_{L^{q_0}(\Omega)} \leqslant \widetilde{C}_0\tilde{\varepsilon}_0(1 + t^{-1+N/2q_0})\mathrm{e}^{-\alpha_1 t},\\ \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \widetilde{C}_0\tilde{\varepsilon}_0(1 + t^{-1/2})\mathrm{e}^{-\alpha_1 t},\\ \|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \widetilde{C}_0\tilde{\varepsilon}_0(1 + t^{-1/2})\mathrm{e}^{-\alpha_1 t}, \quad \|\mathbf{u}(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \overline{C}_0\mathrm{e}^{-\alpha_2 t}.\end{gathered}$$

In order to use Lemma 2.12, we obtain the following estimates by using the above inequalities.

$$\begin{aligned}\|n(\cdot,t)\|_{L^{q_0}(\Omega)} &\leqslant \|n(\cdot,t) - \bar{n}_0\|_{L^{q_0}(\Omega)} + \|\bar{n}_0\|_{L^{q_0}(\Omega)}\\ &\leqslant \widetilde{C}_0\tilde{\varepsilon}_0(1 + t^{-1+N/2q_0})\mathrm{e}^{-\alpha_1 t} + \varepsilon|\Omega|^{1/q_0-2/N},\\ \|\nabla v(\cdot,t)\|_{L^N(\Omega)} &\leqslant |\Omega|^{1/N}\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \leqslant |\Omega|^{1/N}\widetilde{C}_0\tilde{\varepsilon}_0(1 + t^{-1/2})\mathrm{e}^{-\alpha_1 t},\\ \|\nabla w(\cdot,t)\|_{L^N(\Omega)} &\leqslant |\Omega|^{1/N}\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \leqslant |\Omega|^{1/N}\widetilde{C}_0\tilde{\varepsilon}_0(1 + t^{-1/2})\mathrm{e}^{-\alpha_1 t},\\ \|\mathbf{u}(\cdot,t)\|_{L^N(\Omega)} &\leqslant |\Omega|^{1/N}\|\mathbf{u}(\cdot,t)\|_{L^\infty(\Omega)} \leqslant |\Omega|^{1/N}\overline{C}_0\mathrm{e}^{-\alpha_2 t}.\end{aligned}$$

Noticing that $-1 + N/2q_0 < 0$ with $q_0 \geqslant \tilde{q}_0 > \frac{1}{2}N$, we know that there exists $t_0 > 0$ such that

$$\begin{gathered}\widetilde{C}_0\tilde{\varepsilon}_0(1 + t_0{}^{-1+N/2q_0})\mathrm{e}^{-\alpha_1 t_0} \leqslant \frac{\varepsilon_0}{2}, \quad |\Omega|^{1/N}\widetilde{C}_0\tilde{\varepsilon}_0(1 + t_0{}^{-1/2})\mathrm{e}^{-\alpha_1 t_0} \leqslant \varepsilon_0,\\ |\Omega|^{1/N}\overline{C}_0\mathrm{e}^{-\alpha_2 t_0} \leqslant \varepsilon_0.\end{gathered}$$

From (3.9), we know that

$$\varepsilon|\Omega|^{1/q_0-2/N} \leqslant \frac{\varepsilon_0}{2}. \tag{3.10}$$

Consequently, $(n(x,t_0), v(x,t_0), w(x,t_0), \mathbf{u}(x,t_0))$ satisfies the condition of the initial data demanded in Lemma 2.12. This proves Theorem 1.1. □

References

[1] *N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler*: Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues. Math. Models Methods Appl. Sci. *25* (2015), 1663–1763. zbl MR doi

[2] *X. Cao*: Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces. Discrete Contin. Dyn. Syst. *35* (2015), 1891–1904. zbl MR doi

[3] *X. Cao, J. Lankeit*: Global classical small-data solutions for a three-dimensional chemotaxis Navier-Stokes system involving matrix-valued sensitivities. Calc. Var. Partial Differ. Equ. *55* (2016), Article ID 107, 39 pages. zbl MR doi

[4] *L. Corrias, B. Perthame*: Asymptotic decay for the solutions of the parabolic-parabolic Keller-Segel chemotaxis system in critical spaces. Math. Comput. Modelling *47* (2008), 755–764. zbl MR doi

[5] *E. Espejo, T. Suzuki*: Reaction terms avoiding aggregation in slow fluids. Nonlinear Anal., Real World Appl. *21* (2015), 110–126. zbl MR doi

[6] *K. Fujie, T. Senba*: Application of an Adams type inequality to a two-chemical substances chemotaxis system. J. Differ. Equations *263* (2017), 88–148. zbl MR doi

[7] *K. Fujie, T. Senba*: Blowup of solutions to a two-chemical substances chemotaxis system in the critical dimension. J. Differ. Equations *266* (2019), 942–976. zbl MR doi

[8] *T. Hillen, K. J. Painter*: A user's guide to PDE models for chemotaxis. J. Math. Biol. *58* (2009), 183–217. zbl MR doi

[9] *D. Horstmann*: From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. I. Jahresber. Dtsch. Math.-Ver. *105* (2003), 103–165. zbl MR

[10] *H.-Y. Jin*: Boundedness of the attraction-repulsion Keller-Segel system. J. Math. Anal. Appl. *422* (2015), 1463–1478. zbl MR doi

[11] *X. Li, Y. Xiao*: Global existence and boundedness in a 2D Keller-Segel-Stokes system. Nonlinear Anal., Real World Appl. *37* (2017), 14–30. zbl MR doi

[12] *J. Liu, Y. Wang*: Global weak solutions in a three-dimensional Keller-Segel-Navier-Stokes system involving a tensor-valued sensitivity with saturation. J. Differ. Equations *262* (2017), 5271–5305. zbl MR doi

[13] *M. Luca, A. Chavez-Ross, L. Edelstein-Keshet, A. Mogilner*: Chemotactic signaling, microglia, and Alzheimer's disease senile plagues: Is there a connection? Bull. Math. Biol. *65* (2003), 693–730. zbl doi

[14] *T. Nagai, T. Senba, K. Yoshida*: Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis. Funkc. Ekvacioj, Ser. Int. *40* (1997), 411–433. zbl MR

[15] *K. Osaki, A. Yagi*: Finite dimensional attractor for one-dimensional Keller-Segel equations. Funkc. Ekvacioj, Ser. Int. *44* (2001), 441–469. zbl MR

[16] *Y. Tao, Z.-A. Wang*: Competing effects of attraction vs. repulsion in chemotaxis. Math. Models Methods Appl. Sci. *23* (2013), 1–36. zbl MR doi

[17] *Y. Tao, M. Winkler*: Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system. Z. Angew. Math. Phys. *66* (2015), 2555–2573. zbl MR doi

[18] *Y. Tao, M. Winkler*: Blow-up prevention by quadratic degradation in a two-dimensional Keller-Segel-Navier-Stokes system. Z. Angew. Math. Phys. *67* (2016), Article ID 138, 23 pages. zbl MR doi

[19] *Y. Wang*: Global weak solutions in a three-dimensional Keller-Segel-Navier-Stokes system with subcritical sensitivity. Math. Models Methods Appl. Sci. *27* (2017), 2745–2780. zbl MR doi

[20] *Y. Wang, M. Winkler, Z. Xiang*: Global classical solutions in a two-dimensional chemotaxis-Navier-Stokes system with subcritical sensitivity. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) *18* (2018), 421–466. zbl MR doi

[21] *Y. Wang, L. Yang*: Boundedness in a chemotaxis-fluid system involving a saturated sensitivity and indirect signal production mechanism. J. Differ. Equations *287* (2021), 460–490. zbl MR doi

[22] *M. Winkler*: Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model. J. Differ. Equations *248* (2010), 2889–2905. zbl MR doi

[23] *M. Winkler*: Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops. Commun. Partial Differ. Equations *37* (2012), 319–351. zbl MR doi

[24] *M. Winkler*: A three-dimensional Keller-Segel-Navier-Stokes system with logistic source: Global weak solutions and asymptotic stabilization. J. Funct. Anal. *276* (2019), 1339–1401. zbl MR doi

[25] *M. Winkler*: Small-mass solutions in the two-dimensional Keller-Segel system coupled to Navier-Stokes equations. SIAM J. Math. Anal. *52* (2020), 2041–2080. zbl MR doi

[26] *M. Winkler*: Reaction-driven relaxation in three-dimensional Keller-Segel-Navier-Stokes interaction. Commun. Math. Phys. *389* (2022), 439–489. zbl MR doi

[27] *H. Yu, W. Wang, S. Zheng*: Global classical solutions to the Keller-Segel-Navier-Stokes system with matrix-valued sensitivity. J. Math. Anal. Appl. *461* (2018), 1748–1770. zbl MR doi

[28] *P. Yu*: Blow-up prevention by saturated chemotactic sensitivity in a 2D Keller-Segel-Stokes system. Acta Appl. Math. *169* (2020), 475–497. zbl MR doi

[29] *W. Zhang, P. Niu, S. Liu*: Large time behavior in a chemotaxis model with logistic growth and indirect signal production. Nonlinear Anal., Real World Appl. *50* (2019), 484–497. zbl MR doi

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