Commentationes Mathematicae Universitatis Carolinae

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Abstract. We deal with a hyperspace selection problem in the setting of connected spaces. We present two solutions of this problem illustrating the difference between selections for the nonempty closed sets, and those for the at most two-point sets. In the first case, we obtain a characterisation of compact orderable spaces. In the latter case — that of selections for at most two-point sets, the same selection property is equivalent to the existence of a ternary relation on the space, known as a cyclic order, and gives a characterisation of the so called weakly cyclically orderable spaces.

Keywords: Vietoris topology; continuous selection; weak selection; weakly orderable space; weakly cyclically orderable space

Classification: 54B20, 54C65, 54D05, 54D30, 54F05, 54F65

1. Introduction

All spaces in this paper are infinite Hausdorff topological spaces. For a space X, let $\mathscr{F}(X)$ be the set of all nonempty closed subsets of X. Usually, we endow $\mathscr{F}(X)$ with the *Vietoris topology* τ_V, and call it the *Vietoris hyperspace* of X. Recall that τ_V is generated by all collections of the form

$$\langle \mathscr{V} \rangle = \left\{ S \in \mathscr{F}(X) \colon S \subset \bigcup \mathscr{V} \text{ and } S \cap V \neq \emptyset, \text{ whenever } V \in \mathscr{V} \right\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of X. In the sequel, any subset $\mathscr{D} \subset \mathscr{F}(X)$ will carry the relative Vietoris topology as a subspace of the hyperspace $(\mathscr{F}(X), \tau_V)$. A map $f \colon \mathscr{D} \to X$ is a *selection* for $\mathscr{D}$ if $f(S) \in S$ for every $S \in \mathscr{D}$. A selection $f \colon \mathscr{D} \to X$ is *continuous* if it is continuous with respect to the relative Vietoris topology on $\mathscr{D}$, and we use $\mathcal{V}_{\mathrm{cs}}[\mathscr{D}]$ to denote the set of all *Vietoris continuous selections* for $\mathscr{D}$.

A space X is *orderable* (or *linearly ordered*) if it is endowed with the open interval topology $\mathscr{T}_{\leq}$ generated by some linear order "$\leq$" on X, called *compatible* for X. Subspaces of orderable spaces are not necessarily orderable, they are termed *suborderable*. A space X is *weakly orderable* if there exists a coarser

orderable topology on X, i.e. if there exists a linear order "$\leq$" on X (called *compatible* for X) such that $\mathscr{T}_\leq \subset \mathscr{T}$, where $\mathscr{T}$ is the topology of X. The weakly orderable spaces were introduced by S. Eilenberg in [8] under the name of "ordered" topological spaces, and are often called "Eilenberg orderable". They were called "weakly orderable" in [1], [2], [6], [24], and in [24] was also proposed to abbreviate them as *KOTS*.

It was shown in [14, Proposition 5.1] that if X is a connected space and $\mathscr{F}(Z)$ has a continuous selection for every connected subset $Z \subset X$ with $|X \setminus Z| \leq 1$, then X is compact and orderable. However, this selection property includes the special case of $Z = X$, i.e. that $\mathscr{F}(X)$ itself has a continuous selection. Here, we are going to show that this result is valid without explicitly requiring that $\mathscr{F}(X)$ has a continuous selection, see Theorem 3.3. Based on this, we will obtain the following characterization of connected compact orderable spaces.

Theorem 1.1. *A connected space X is compact and orderable if and only if $\mathcal{V}_{\mathrm{cs}}[\mathscr{F}(X \setminus \{p\})] \neq \emptyset$ for each $p \in X$.*

Regarding the proper place of Theorem 1.1, let us explicitly remark that if X is a space such that $\mathscr{F}(Z)$ has a continuous selection for every $Z \subset X$ with $|X \setminus Z| \leq 2$, then X is totally disconnected, see [14, Corollary 5.3]. On the other hand, the hypothesis in Theorem 1.1 that X is connected is essential to conclude that it is compact. In fact, we will obtain a natural generalization of the aforementioned result of [14] showing that the hyperspace selection property in Theorem 1.1 is possessed by a natural class of non-compact totally disconnected spaces, see Theorem 4.3 and Corollary 4.4.

Let $\mathscr{F}_2(X) = \{S \in \mathscr{F}(X) \colon |S| \leq 2\}$. A selection $f \colon \mathscr{F}_2(X) \to X$ is commonly called a *weak selection* for X, see the next section for a brief review of such selections. Theorem 1.1 is not valid in the setting of continuous weak selections. In this case, the property is characterizing another class of connected spaces which constitutes the second main result of this paper. In order to state it, we briefly recall some terminology. A ternary relation $\mathbf{C} \subset X^3$ on a set X is called a *cyclic* (or *circular*) *order* on X, see E. V. Huntington in [20], [21] and E. Čech in [4], if the following conditions are satisfied:

$$\left.\begin{array}{l} a \neq b \neq c \neq a \\ (a, b, c) \notin \mathbf{C} \end{array}\right\} \iff (c, b, a) \in \mathbf{C},$$

$$(a, b, c) \in \mathbf{C} \implies (b, c, a) \in \mathbf{C},$$

$$\left.\begin{array}{l} (a, b, c) \in \mathbf{C} \\ (a, c, d) \in \mathbf{C} \end{array}\right\} \implies (a, b, d) \in \mathbf{C}.$$

Several basic properties of a cyclic order can be found in §5 of Chapter I in [4]. For instance, whenever $a \in X$, a cyclic order $\mathbf{C}$ on X defines a (strict) linear order "$<_{\mathbf{C},a}$" on $X \setminus \{a\}$ by $x <_{\mathbf{C},a} y$ if $(a, x, y) \in \mathbf{C}$. The converse is also true, and each linear order "$<$" on $X \setminus \{a\}$ defines a unique cyclic order $\mathbf{C}$ on X with "$<=<_{\mathbf{C},a}$". Furthermore, each linear order "$<$" on X can be extended to a cyclic order $\mathbf{C}_<$ on X, see [22, Proposition 1.6], where $\mathbf{C}_<$ is defined by

$$(a, b, c) \in \mathbf{C}_< \iff \begin{cases} a \neq b \neq c \neq a \quad \text{and} \\ a < b < c \ \text{ or } \ b < c < a \ \text{ or } \ c < a < b. \end{cases}$$

For a cyclic order $\mathbf{C}$ on X and $a, b \in X$, the set

$$(a, b)_{<\mathbf{C}} = \{x \in X : (a, x, b) \in \mathbf{C}\} \subset X \setminus \{a, b\}$$

is called an *interval* from a to b. Evidently, $(a, b)_{<\mathbf{C}} = (a, a)_{<\mathbf{C}} = \emptyset$ provided $a = b$. Otherwise, if $a \neq b$, it was shown in [4] that the linear orders "$<_{\mathbf{C},a}$" and "$<_{\mathbf{C},b}$" coincide on $(a, b)_{<\mathbf{C}}$. A space X is called *weakly cyclically orderable*, see [22], (*cyclically orderable*, in Kok's terminology, see [22]) if there exists a cyclic order $\mathbf{C}$ on X such that all intervals $(a, b)_{<\mathbf{C}}$, $a, b \in X$, are open in X. If, moreover, these intervals form a base for the topology of X, then X is called *cyclically orderable*, see [22], (*strictly cyclically orderable*, in Kok's terminology, see [22]). Each weakly orderable space is weakly cyclically orderable, see [22, Proposition 1.6]. However, an orderable space is not necessarily cyclically orderable. As pointed out in [22], such a space is the half-open interval $[0, 1)$. In the other direction, the plane circle is an example of a connected cyclically orderable space which is not weakly orderable.

If X is weakly cyclically orderable and $p \in X$, then $X \setminus \{p\}$ is weakly orderable; in fact, weakly orderable with respect to the linear order "$<_{\mathbf{C},p}$", see [22, Proposition 1.7]. In particular, $X \setminus \{p\}$ has a continuous weak selection. Evidently, this is a special case of the selection property in Theorem 1.1. We are now ready to state our second main result. Namely, in this paper, we will also prove the following theorem which is complementary to Theorem 1.1.

Theorem 1.2. *A connected space X is weakly cyclically orderable if and only if $X \setminus \{p\}$ has a continuous weak selection for every $p \in X$.*

The paper is organized as follows. The next section contains a brief review of some results about continuous hyperspace selections in the setting of connected spaces. Section 3 contains a special case of Theorem 1.1, see Theorem 3.3, which is based on another weaker interpretation of weak orderability. The proof of Theorem 1.1 is finalized in Section 4, while that of Theorem 1.2 — in Section 5.

2.   Connected weakly orderable spaces

As mentioned in Introduction, the weakly orderable spaces were introduced by
S. Eilenberg in [8]. In the same paper, he obtained the following interesting result
in the setting of connected spaces.

Theorem 2.1 ([8]). *Each connected weakly orderable space has precisely two
compatible orders which are inverse to each other.*

E. Michael was the first to relate linear orders to weak selections. For a set X
and a weak selection $\sigma\colon \mathscr{F}_2(X) \to X$, he defined a natural order-like relation
"$\leq_\sigma$" on X by $x \leq_\sigma y$ if $\sigma(\{x,y\}) = x$, see [23, Definition 7.1]. The relation
"$\leq_\sigma$" is very similar to a linear order on X being both *total* and *antisymmetric*,
but may fail to be *transitive*. For convenience, we write $x <_\sigma y$ provided $x \leq_\sigma y$
and $x \neq y$. For a space X, the strict relation $x <_\sigma y$ plays an important role
in describing continuity of σ. Namely, σ is continuous if and only if for every
$x, y \in X$ with $x <_\sigma y$, there are open sets $U, V \subset X$ such that $x \in U$, $y \in V$
and $s <_\sigma t$ for every $s \in U$ and $t \in V$ [15, Theorem 3.1]. Continuity of a weak
selection σ implies that all $\leq_\sigma$-open intervals $(\leftarrow, x)_{\leq_\sigma} = \{y \in X \colon y <_\sigma x\}$ and
$(x, \rightarrow)_{\leq_\sigma} = \{y \in X \colon x <_\sigma y\}$, $x \in X$, are open in X, see [23], but the converse
is not necessarily true [15, Example 3.6], see also [18, Corollary 4.2 and Exam-
ple 4.3]. For an extended review of (weak) hyperspace selections, the interested
reader is referred to [12]. Finally, let us explicitly remark that if $f \in \mathcal{V}_{\mathrm{cs}}[\mathscr{D}]$ for
some $\mathscr{D} \subset \mathscr{F}(X)$ with $\mathscr{F}_2(X) \subset \mathscr{D}$, then $f \restriction \mathscr{F}_2(X)$ is a continuous weak selec-
tion for X. In this case, the order-like relation generated by $f \restriction \mathscr{F}_2(X)$ will be
simply denoted by "$\leq_f$".

In the setting of connected spaces, using Theorem 2.1, E. Michael gave a com-
plete description of continuity of hyperspace selections. In the one direction, he
showed the following properties of these selections, see [23, Lemmas 7.2 and 7.3].

Theorem 2.2 ([8], [23]). *Let X be a connected space, $\mathscr{F}_2(X) \subset \mathscr{D} \subset \mathscr{F}(X)$ and
$f \in \mathcal{V}_{\mathrm{cs}}[\mathscr{D}]$. Then*

  (1) "$\leq_f$" *is a linear order and X is weakly orderable with respect to* "$\leq_f$".
  (2) $f(S) = \min_{\leq_f} S$ *for every $S \in \mathscr{D}$.*

Moreover, if $g \in \mathcal{V}_{\mathrm{cs}}[\mathscr{D}]$ with $g \neq f$, then $g(S) = \max_{\leq_f} S$ for every $S \in \mathscr{D}$.

Next, he established the following counterpart of Theorem 2.2 in [23, Lem-
ma 7.5.1], for convenience we state it for the special case of $\mathscr{D} = \mathscr{F}(X)$.

Theorem 2.3 ([23]). *Let X be a connected space which is weakly orderable with
respect to a linear order "$\leq$" such that $\min_{\leq} S$ does exist for each $S \in \mathscr{F}(X)$.
Then $\mathscr{F}(X)$ has a continuous selection. In fact, $f(S) = \min_{\leq} S$, $S \in \mathscr{F}(X)$ is
a continuous selection for $\mathscr{F}(X)$.*

Let X be a connected space and $f \in \mathcal{V}_{cs}[\mathscr{F}(X)]$. Then by Theorem 2.2, X is weakly orderable with respect to "$\leq_f$" and $p = f(X)$ is the $\leq_f$-minimal element of X. However, $(p, \rightarrow)_{\leq_f}$ is a connected subset of X being an interval [22, Theorem 1.3]. Accordingly, p has the property that $X \setminus \{p\}$ is connected. Such a point $p \in X$ in a connected space X is called *noncut*. Otherwise, if $X \setminus \{p\}$ is not connected, the point p is called *cut*. Let $\operatorname{nct}(X)$ be the set of all noncut points of X, and $\operatorname{ct}(X)$ — that of all cut points of X. It is well known that each connected weakly orderable space has at most two noncut points. The following further property is an immediate consequence of the above considerations, see [10, Theorem 1.1].

Corollary 2.4. *Let X be a connected space, $f \in \mathcal{V}_{cs}[\mathscr{F}(X)]$ and $p = f(X)$. Then $p \in \operatorname{nct}(X)$ and $f(S) = p$ for every $S \in \mathscr{F}(X)$ with $p \in S$. Moreover, if $q \in X$ with $q \neq p$, then $q \in \operatorname{nct}(X)$ if and only if $f(S) \neq q$ for every $S \in \mathscr{F}(X)$ with $S \neq \{q\}$.*

PROOF: As remarked above, $p \in \operatorname{nct}(X)$. Moreover, since p is the $\leq_f$-minimal element of X, it follows from Theorem 2.2 that $f(S) = p$ for every $S \in \mathscr{F}(X)$ with $p \in S$. Suppose that $q \in X$ with $q \neq p$. Then $X \setminus \{q\}$ is connected precisely when q is the $\leq_f$-maximal element of X, see [11, Corollary 2.7], and also [22, Theorem 1.3]. Therefore, by Theorem 2.2, $q \in \operatorname{nct}(X)$ precisely when $f(S) \neq q$ for every $S \in \mathscr{F}(X)$ with $S \neq \{q\}$. $\qquad\square$

There are connected weakly orderable spaces X such that $|\operatorname{nct}(X)| = 2$, but $\mathscr{F}(X)$ has precisely one continuous selection for $\mathscr{F}(X)$. For instance, such a space is the topologist's sine curve $X = \{(0,0)\} \cup \{(t, \sin 1/t) : 0 < t \leq 1\}$, see [26, Example 8 and Lemma 15]. In this regards, the following natural result was obtained by T. Nogura and D. Shakhmatov in [26].

Theorem 2.5 ([26])**.** *A connected space X is compact and orderable if and only if it has exactly two continuous selections for $\mathscr{F}(X)$.*

3. Selections avoiding noncut points

In this section, X is used to denote a connected space. If $p \in X$ is a cut point of X, then there are disjoint sets $U, V \subset X$ such that $X \setminus \{p\} = U \cup V$ and $\{p\} = \overline{U} \cap \overline{V}$. Following [3], such a pair (U, V) of sets will be called a *p-cut* of X. A point $p \in X$ is said to *separate* $x, y \in X$ if $x \in U$ and $y \in V$ for some p-cut (U, V) of X. If p separates x and y, then p is a cut point of X, and neither x nor y separates the other two points, see [22, Lemma 2.1]. In these terms, X is

called *almost weakly orderable*, see [3, Definition 3.2], if it has finitely many non-cut points and among every three points of X with two of them being cut, there is one which separates the other two.

A subset $E \subset X$ is called an *endset of* X if $X \setminus E$ is connected. Evidently, $p \in \mathrm{nct}(X)$ precisely when the singleton $\{p\}$ is an endset of X. Thus, noncut points are often called *endpoints*. However, a set of endpoints is not necessarily an endset. In contrast, the endpoints of an almost weakly orderable space form an endset, see [3, Corollary 3.4]. Based on this, we have the following alternative interpretation of a special class of almost weakly orderable spaces.

Proposition 3.1. *Let X be a connected space such that $\mathrm{nct}(X)$ is a nonempty finite set. Then X is almost weakly orderable if and only if $\mathrm{nct}(X)$ is an endset of X such that $\{p\} \cup \mathrm{ct}(X)$ is weakly orderable for every $p \in \mathrm{nct}(X)$.*

PROOF: Follows from the definition and the fact that a connected space Z is weakly orderable if and only if among every three points of Z there is one which separates the other two, see [22, Theorem 4.1] (in a footnote of [7], the result was credited to D. Zaremba-Szczepkowicz).   □

If X is almost weakly orderable, then there exists a partial order "$\leq$" on X such that two points of X are $\leq$-comparable precisely when they can be separated, moreover this order is compatible with the topology of X in the sense that all $\leq$-open intervals are open in X, see [3, Corollary 3.7]. Such a partial order on X is called a *separation partial order*, and any two separation partial orderings on X are either identical or inverse to each other, see [3, Proposition 3.8]. Let us explicitly remark that the idea of a separation order induced by cut points goes back to G. T. Whyburn in [27]; the interested reader is also referred to [19], [28], and the more recent monograph [25].

Let X be almost weakly orderable. It follows from Proposition 3.1, see also [3, Proposition 3.9], that for $p \in \mathrm{nct}(X)$ and a separation partial order "$\leq$" on X, we have either $p < x$ for every $x \in \mathrm{ct}(X)$, or $x < p$ for every $x \in \mathrm{ct}(X)$; in other words, $p < \mathrm{ct}(X)$ or $\mathrm{ct}(X) < p$. In particular, noncut points $p, q \in X$ are not $\leq$-comparable precisely when $\{p, q\} < \mathrm{ct}(X)$ or $\mathrm{ct}(X) < \{p, q\}$. Accordingly, we have the following immediate consequence.

Corollary 3.2. *If an almost weakly orderable space has more than two noncut points, then it has two noncut points which cannot be separated.*

Using these observations, we will prove the following theorem. In the proof of this theorem, for a set Z, a linear order "$\leq$" on Z and $a, b \in Z$ with $a < b$, we let

$$(3.1) \qquad\qquad (a, b)_{\leq} = \{z \in Z : a < z < b\}.$$

Theorem 3.3. *Let X be a connected space which has at least one noncut point. If $\mathscr{F}(X \setminus \{p\})$ has a continuous selection for each $p \in \mathrm{nct}(X)$, then X is compact and orderable.*

PROOF: Take a point $p \in \mathrm{nct}(X)$ and a continuous selection f for $\mathscr{F}(X \setminus \{p\})$. Since $X \setminus \{p\}$ is connected, by Theorem 2.2 and Corollary 2.4, $X \setminus \{p\}$ is weakly orderable with respect to "$\leq_f$" and has a point $q \in X \setminus \{p\}$ with $q \leq_f x$ for every $x \in X \setminus \{p\}$, i.e. a noncut of $X \setminus \{p\}$. Then q is also a noncut point of X and, accordingly, X has at least two noncut points. We will show that these are the only noncut points of X. Contrary to this, assume that X has a noncut point $r \in X \setminus \{p, q\}$. Then $q <_f r$ and there exists a point $y \in X \setminus \{p\}$ with $q <_f y <_f r$. Moreover, both sets $(q, y)_{\leq_f}$ and $(y, r)_{\leq_f}$ are connected being intervals in the connected weakly orderable space $X \setminus \{p\}$, see [22, Theorem 1.3]. Hence, so are the sets $L = \overline{(q, y)_{\leq_f}}$ and $R = \overline{(y, r)_{\leq_f}}$. It is also evident that $q, y \in \mathrm{nct}(L)$ and $y, r \in \mathrm{nct}(R)$. We will show that this is impossible. To this end, let us observe that $p \in \overline{(\leftarrow, r)_{\leq_f}}$ because $p \notin \overline{(\leftarrow, r)_{\leq_f}}$ will imply that $\overline{(\leftarrow, r)_{\leq_f}} \setminus \{r\} = (\leftarrow, r)_{\leq_f}$ is a nonempty clopen proper subset of the connected space $X \setminus \{r\}$. Thus, we have that

$$p \in \overline{(\leftarrow, r)_{\leq_f}} = \overline{(q, y)_{\leq_f} \cup (y, r)_{\leq_f}} = \overline{(q, y)_{\leq_f}} \cup \overline{(y, r)_{\leq_f}} = L \cup R.$$

In particular, $p \in \mathrm{nct}(L)$ or $p \in \mathrm{nct}(R)$. Hence, one of these sets has at least three noncut points. However, both $\mathscr{F}(L)$ and $\mathscr{F}(R)$ have continuous selections because so do $\mathscr{F}(X \setminus \{r\})$ and $\mathscr{F}(X \setminus \{q\})$, while $L \in \mathscr{F}(X \setminus \{r\})$ and $R \in \mathscr{F}(X \setminus \{q\})$. According to Theorem 2.2 and Corollary 2.4, this is impossible because each weakly orderable space has at most two noncut points. Thus, p and q are the only noncut points of X and, in fact, $X \setminus \{p, q\}$ is connected. So, $\mathrm{nct}(X) = \{p, q\}$ is an endset of X. Moreover, by Theorem 2.2, both $X \setminus \{p\}$ and $X \setminus \{q\}$ are weakly orderable. Therefore, by Proposition 3.1, X is almost weakly orderable.

We are also ready to show that X is compact and orderable. To see this, using one of these noncut points, for instance p, take a continuous selection f for $\mathscr{F}(X \setminus \{p\})$. Then by Theorem 2.2, $X \setminus \{p\}$ is weakly orderable with respect to "$\leq_f$" and since $\mathrm{ct}(X) \subset X \setminus \{p\}$, it follows from [3, Corollary 3.7] that X has a separation partial order "$\leq$" which extends the linear order "$\leq_f$". This implies that "$\leq$" is a linear order on X, i.e. that p and q are $\leq$-comparable. Indeed, by Theorem 2.2 and Corollary 2.4, $q \leq x$ for every $x \in X \setminus \{p\}$, i.e. $q < \mathrm{ct}(X)$. If p and q are not $\leq$-comparable, it follows from Corollary 3.2 that $\{p, q\} < \mathrm{ct}(X)$. In this case, take disjoint open sets $U, V \subset X$ with $p \in U$ and $q \in V$, and set $F = X \setminus (V \cup \{p\})$. Then $U \setminus \{p\} \subset F \subset \mathrm{ct}(X)$ which implies that for every $y \in \mathrm{ct}(X)$ there exists $x \in U \setminus \{p\} \subset F$ with $x < y$, because $(\leftarrow, y)_{\leq} \cap U$ is also

a neighbourhood of p. However, by Theorem 2.2, $f(F) = \min_{\leq_f} F \in F \subset \mathrm{ct}(X)$, which is clearly impossible. Thus, X is weakly orderable with respect to "$\leq$" and, in particular, $q < \mathrm{ct}(X) < p$. This implies that f can be extended to a continuous selection h for $\mathscr{F}(X)$. Indeed, now each $S \in \mathscr{F}(X)$ has a $\leq$-minimal element because p is the $\leq$-maximal element of X, so we may define $h(S) = f(S \setminus \{p\}) = \min_{\leq_f} S = \min_{\leq} S$ for every $S \in \mathscr{F}(X)$ with $S \neq \{p\}$. According to Theorem 2.3, h is a continuous selection for $\mathscr{F}(X)$ with $h(X) = q$. Interchanging p and q, the same argument shows that $\mathscr{F}(X)$ also has a continuous selection which assigns to X the point p. Therefore, by Theorem 2.5, X is compact and orderable. $\square$

4. Selections avoiding points

Here, we finalize the proof of Theorem 1.1, which is based on the following special type of hyperspace selections. For a space X and $p \in X$, a selection f for $\mathscr{F}(X)$ is called *p-minimal*, see [10], if $f(S) \neq p$ for every $S \in \mathscr{F}(X)$ with $S \neq \{p\}$. The prototype of the following property can be found in [16, Theorem 3.1].

Proposition 4.1. *Let X be a space which has a p-minimal selection*

$$f \in \mathscr{V}_{\mathrm{cs}}[\mathscr{F}(X)]$$

for some point $p \in X$. Then $f(S \cup \{p\}) \in S$, whenever $S \subset X$ is a nonempty subset with $S \cup \{p\} \in \mathscr{F}(X)$. In particular, $\mathscr{V}_{\mathrm{cs}}[\mathscr{F}(X \setminus \{p\})] \neq \emptyset$.

PROOF: Let $S \subset X$ be a nonempty set with $S \cup \{p\} \in \mathscr{F}(X)$. Then $f(S \cup \{p\}) = p$ precisely when $S \cup \{p\} = \{p\}$, i.e. $S = \{p\}$, because f is p-minimal. This is equivalent to $f(S \cup \{p\}) \in S$. For the second part of this proposition, let us observe that $\overline{S} \subset S \cup \{p\}$, whenever $S \in \mathscr{F}(X \setminus \{p\})$. Hence, we may define a map $\varphi \colon \mathscr{F}(X \setminus \{p\}) \to \mathscr{F}(X)$ by

$$\varphi(S) = S \cup \{p\} \qquad \text{for every } S \in \mathscr{F}(X \setminus \{p\}).$$

Take $S \in \mathscr{F}(X \setminus \{p\})$ and a finite family $\mathscr{V}$ of open subsets of X with $\varphi(S) \in \langle \mathscr{V} \rangle$. Next, take another finite family $\mathscr{U}$ of nonempty open subsets of X such that $\mathscr{U}$ refines $\mathscr{V}$ and $\bigcup \mathscr{U} = \bigcup \mathscr{V} \setminus \{p\}$. Then $S \in \langle \mathscr{U} \rangle$ and $\varphi(\langle \mathscr{U} \rangle) \subset \langle \mathscr{V} \rangle$, so φ is continuous with respect to the Vietoris topology on these hyperspaces. We may now define a continuous selection g for $\mathscr{F}(X \setminus \{p\})$ by $g = f \circ \varphi$. $\square$

PROOF OF THEOREM 1.1: Assume that X is a compact connected space which is orderable. Next, take a compatible linear order "$\leq$" on X and a point $p \in X$.

If p is a noncut point of X, then by Corollary 2.4 and Theorem 2.5, $\mathscr{F}(X)$ has a continuous p-minimal selection. Hence, by Proposition 4.1, $\mathcal{V}_{\mathrm{cs}}[\mathscr{F}(X\setminus\{p\})] \neq \emptyset$. If $p \in \mathrm{ct}(X)$, then p is a noncut point for both intervals

$$Y = \{x \in X : x \leq p\} \qquad \text{and} \qquad Z = \{x \in X : p \leq x\}.$$

Moreover, these intervals are infinite, compact and orderable. Hence, for the same reason as before, $\mathcal{V}_{\mathrm{cs}}[\mathscr{F}(Y\setminus\{p\})] \neq \emptyset \neq \mathcal{V}_{\mathrm{cs}}[\mathscr{F}(Z\setminus\{p\})]$. Since $Y\setminus\{p\}$ and $Z\setminus\{p\}$ form a clopen partition of $X \setminus \{p\}$, we also have that $\mathcal{V}_{\mathrm{cs}}[\mathscr{F}(X \setminus \{p\})] \neq \emptyset$.

To see the converse, by Theorem 3.3, it suffices to show that X has a noncut point provided $\mathcal{V}_{\mathrm{cs}}[\mathscr{F}(X \setminus \{p\})] \neq \emptyset$ for each $p \in X$. To this end, take cut points $y, z \in \mathrm{ct}(X)$. Next, let (A, B) be a y-cut of X, and (U, V) be a z-cut of X. Then z does not belong to one of the sets A or B, say $z \notin A$. Since $S = A \cup \{y\}$ is a connected subset of X, it is a subset of U or V, for instance $S \subset U$. Moreover, it is closed in X. This implies that S has a noncut point $p \in S$ with $p \neq y$. Indeed, by Corollary 2.4, S has a noncut point because $\mathscr{F}(X \setminus \{z\})$ has a continuous selection, and hence so does $\mathscr{F}(S)$. If this point is y, then $A = S \setminus \{y\}$ is connected and $\mathscr{F}(A)$ has a continuous selection because so does $\mathscr{F}(X\setminus\{y\})$. Therefore, A has a noncut point $p \in A$. Since S is weakly orderable, p is also a noncut point of S. Thus, $p \neq y$ and $S \setminus \{p\}$ is connected. However, $B \cup \{y\}$ is also connected and $y \in S \setminus \{p\}$, so $X \setminus \{p\} = (S \setminus \{p\}) \cup B \cup \{y\}$ is connected as well, i.e. $p \in \mathrm{nct}(X)$. We may now apply Theorem 3.3 to complete the proof. $\qquad\square$

We conclude this section with some observations regarding the proper place of Theorem 1.1. To this end, we will first establish the following partial inverse of Proposition 4.1.

Proposition 4.2. *Let X be a space and $p \in X$ be a point which is a countable intersection of clopen sets. Then the following are equivalent:*

   (a) $\mathcal{V}_{\mathrm{cs}}[\mathscr{F}(X)] \neq \emptyset$;
   (b) $\mathscr{F}(X)$ *has a continuous p-minimal selection;*
   (c) $\mathcal{V}_{\mathrm{cs}}[\mathscr{F}(X \setminus \{p\})] \neq \emptyset$.

PROOF: The implication (a) $\Longrightarrow$ (b) follows from [17, Proposition 3.6], while that of (b) $\Longrightarrow$ (c) is a consequence of Proposition 4.1. The implication (c) $\Longrightarrow$ (a) is implicitly contained in the proof of [17, Proposition 3.6]. Briefly, take a selection $f \in \mathcal{V}_{\mathrm{cs}}[\mathscr{F}(X \setminus \{p\})]$ and a decreasing clopen family $\{V_n : n < \omega\}$ with $V_0 = X$ and $\{p\} = \bigcap_{n<\omega} V_n$. Set $S_n = V_n \setminus V_{n+1}$, $n < \omega$, and for every $F \in \mathscr{F}(X)$ with $F \neq \{p\}$, let $n(F) = \min\{n < \omega : F \cap S_n \neq \emptyset\}$. Finally, define a p-minimal selection $g \colon \mathscr{F}(X) \to X$ by $g(F) = f\big(F \cap S_{n(F)}\big)$, whenever $F \in \mathscr{F}(X)$ with $F \neq \{p\}$; this is correct because $F \cap S_{n(F)} \in \mathscr{F}(X \setminus \{p\})$. Moreover, g is

continuous at $\{p\}$ because each selection is continuous on the singletons. Take $F \in \mathscr{F}(X)$ with $F \neq \{p\}$, and a neighbourhood U of $g(F)$. Since $f \upharpoonright \mathscr{F}(S_{n(F)})$ is continuous, there exists a finite family $\mathscr{W}_0$ of nonempty open subsets of $S_{n(F)}$ such that $F \cap S_{n(F)} \in \langle \mathscr{W}_0 \rangle \subset f^{-1}(U)$. Set $W_0 = V_{n(F)+1} \cup \left(\bigcup \mathscr{W}_0 \right)$ and $\mathscr{W} = \{W_0\} \cup \mathscr{W}_0$. Then $\langle \mathscr{W} \rangle$ is a τ_V-neighbourhood of F with $g(\langle \mathscr{W} \rangle) \subset U$. Indeed, $T \in \langle \mathscr{W} \rangle$ implies $T \cap S_{n(F)} \neq \emptyset$ and $T \subset V_{n(F)}$, so $n(T) = n(F)$. On the other hand, $T \cap S_{n(F)} \in \langle \mathscr{W}_0 \rangle$ and therefore $g(T) = f(T \cap S_{n(T)}) \in U$. The proof is complete.   □

Based on this, we have the following theorem which is complementary to Theorem 1.1. In this theorem, a space X is *totally disconnected* if each singleton of X is an intersection of clopen subsets of X. Also, let us recall that a space X is of *countable tightness* if for each $A \subset X$ and $x \in \bar{A}$, there exists a countable set $B \subset A$ with $x \in \bar{B}$.

Theorem 4.3. *Let X be a totally disconnected space which is of countable tightness. Then $\mathscr{V}_{cs}[\mathscr{F}(X)] \neq \emptyset$ if and only if $\mathscr{V}_{cs}[\mathscr{F}(X \setminus \{p\})] \neq \emptyset$ for every $p \in X$.*

PROOF: Take points $p, q \in X$ and a clopen set $Y \subset X$ with $p \in Y$ and $q \notin Y$. If $\mathscr{F}(X)$ has a continuous selection, then so does $\mathscr{F}(Y)$ because $Y \in \mathscr{F}(X)$. Similarly, $\mathscr{F}(Y)$ has a continuous selection provided so does $\mathscr{F}(X \setminus \{q\})$. The proof now consists of showing that if $\mathscr{V}_{cs}[\mathscr{F}(Y)] \neq \emptyset$, then $\mathscr{V}_{cs}[\mathscr{F}(X)] \neq \emptyset$ precisely when $\mathscr{V}_{cs}[\mathscr{F}(X \setminus \{p\})] \neq \emptyset$. To this end, by Proposition 4.2, it suffices to show that p is a countable intersection of clopen sets of Y. So, take a selection $f \in \mathscr{V}_{cs}[\mathscr{F}(Y)]$. Then the above property is reduced to show that p is a countable intersection of relatively clopen sets in each one of the $\leq_f$-intervals

$$(\leftarrow, p]_{\leq_f} = \{y \in Y : y \leq_f p\} \quad \text{and} \quad [p, \rightarrow)_{\leq_f} = \{y \in Y : p \leq_f y\}.$$

If p is a non-isolated point of $(\leftarrow, p]_{\leq_f}$, using that X is of countable tightness, there is a countable set $A \subset (\leftarrow, p)_{\leq_f}$ with $p \in \bar{A}$. Therefore, $\bigcap_{x \in A}(x, p]_{\leq_f} = \{p\}$, see [10, Theorem 4.1] and [13, Remark 3.5]. According to [18, Proposition 5.6], this implies that p is a countable intersection of clopen subsets of $(\leftarrow, p]_{\leq_f}$ because X is totally disconnected. Similarly, p is also a countable intersection of clopen subsets of $[p, \rightarrow)_{\leq_f}$. The proof is complete.   □

According to [14, Corollary 5.3], a space X is totally disconnected provided $\mathscr{F}(Z)$ has a continuous selection for every $Z \subset X$ with $|X \setminus Z| \leq 2$. Applying twice Theorem 4.3, this gives the following immediate consequence.

Corollary 4.4. *For a space X which is of countable tightness, the following conditions are equivalent:*

(a) X is totally disconnected and $\mathscr{V}_{cs}[\mathscr{F}(X)] \neq \emptyset$.

(b) $\mathscr{V}_{cs}[\mathscr{F}(X \setminus S)] \neq \emptyset$ for every $S \in \mathscr{F}_2(X)$.

Evidently, by a finite induction, (b) of Corollary 4.4 can be extended to all nonempty finite subsets $S \subset X$. Regarding this, let us remark that if X is a regular space such that $\mathscr{F}(Z)$ has a continuous selection for every nonempty open $Z \subset X$, then X has a clopen π-base [14, Theorem 5.4]. Here, a family $\mathscr{P}$ of nonempty open subsets of X is a π-*base* for X, or a *pseudobase*, if each nonempty open subset of X contains an element of $\mathscr{P}$.

The hypothesis in Theorem 1.1 that X is connected is essential to conclude that X is compact. Namely, each completely metrizable space X which has a covering dimension zero, i.e. being *strongly zero-dimensional*, has a continuous selection for $\mathscr{F}(X)$, see [5], [9]. Hence, by Theorem 4.3, if X is a strongly zero-dimensional completely metrizable space, then $\mathscr{V}_{cs}[\mathscr{F}(X \setminus \{p\})] \neq \emptyset$ for every $p \in X$. In fact, Theorem 4.3 is not so relevant in this case. It is well known that complete metrizability is inherited on G_δ-sets. Moreover, such sets remain strongly zero-dimensional provided so is X. Thus, in the setting of a strongly zero-dimensional completely metrizable space X, the hyperspace $\mathscr{F}(Z)$ has a continuous selection for every nonempty G_δ-subset $Z \subset X$. Based on this, the following question was posed in [14, Question 5], it is still open.

Question 1 ([14])**.** Let X be a (completely) metrizable space with the property that $\mathscr{F}(Z)$ has a continuous selection for every nonempty G_δ-subset $Z \subset X$. Then, is it true that X is strongly zero-dimensional?

Going back to the selection property that " $\mathscr{V}_{cs}[\mathscr{F}(X \setminus \{p\})] \neq \emptyset$ for every point $p \in X$", let us remark that it always implies the existence of a continuous selection for the nonempty closed subsets of X. Namely, answering a question in a previous version of this paper, the following observation was communicated to the author by J. A. C. Chapital.

Proposition 4.5. Let X be a space such that $\mathscr{V}_{cs}[\mathscr{F}(X \setminus \{p\})] \neq \emptyset$ for every $p \in X$. Then $\mathscr{V}_{cs}[\mathscr{F}(X)] \neq \emptyset$.

PROOF: If X is connected, this follows from Theorem 1.1. Otherwise, if X is not connected, then it has a nonempty clopen proper subset $A \subset X$. Taking points $p \in A$ and $q \in B = X \setminus A$, it follows that $A \in \mathscr{F}(X \setminus \{q\})$ and $B \in \mathscr{F}(X \setminus \{p\})$, therefore $\mathscr{V}_{cs}[\mathscr{F}(A)] \neq \emptyset$ and $\mathscr{V}_{cs}[\mathscr{F}(B)] \neq \emptyset$. Since the sets A and B form a clopen partition of X, we also have that $\mathscr{V}_{cs}[\mathscr{F}(X)] \neq \emptyset$. $\square$

The condition in Proposition 4.5 that $\mathscr{F}(X \setminus \{p\})$ has a continuous selection for every $p \in X$ is important. Indeed, one can easily construct examples of compact connected metrizable spaces which have a continuous selection for $\mathscr{F}(X \setminus \{p\})$ for

some point $p \in X$, but $\mathcal{V}_{cs}[\mathscr{F}(X)] = \emptyset$. For instance, take any simple triod X, i.e. the union of 3 arcs having a common endpoint $p \in X$ and being mutually disjoint except at that point.

5.   Weak selections avoiding points

In this section, we will prove Theorem 1.2, which is based on known results and the following special case of this theorem, compare with [22, Theorem 3.11].

Lemma 5.1. Let X be a connected space such that $\mathcal{V}_{cs}[\mathscr{F}_2(X \setminus \{p\})] \neq \emptyset$ for every $p \in X$. If X has a cut point, then it is weakly orderable.

PROOF: Suppose that X has a cut point $q \in X$, and take a q-cut (U, V) of X. Then $C = U \cup \{q\}$ is a connected set. Hence, by Theorem 2.2, it is weakly orderable because $C \subset X \setminus \{p\}$ for every (some) point $p \in V$, and $X \setminus \{p\}$ has a continuous weak selection. We are going to show that q is a noncut point of C. To this end, suppose that q is a cut point of C, and take another cut point $r \in U$ of $C = U \cup \{q\}$, also an r-cut (A, B) of C with $q \in A$. Since C is weakly orderable, A is connected and q is a cut point of A as well. Moreover, A is also weakly orderable. Let "$\leq_A$" be a compatible linear order on A and $a, b \in A$ be such that

(5.1) $a <_A q <_A b$.

Next, for convenience, set $E = (a, b)_{\leq_A} \subset A$, see (3.1), which is a connected set being an interval, see [22, Theorem 1.3]. Finally, let

$$D = V \cup \{q\} \subset X \setminus \{r\} \qquad \text{and} \qquad Y = A \cup V = A \cup D.$$

Then Y is a connected subset of $X \setminus \{r\}$ because $q \in A \cap D$ and D is connected. Hence, for the same reason as before, Y is weakly orderable. Since both A and Y are connected, by Theorem 2.1, Y has a compatible linear order "$\leq$" with $\leq \restriction A = \leq_A$. We now have that $E = (a, b)_{\leq} = \{y \in Y : a < y < b\}$ because $y \in Y \setminus E$ implies that $E \subset (\leftarrow, y)_{\leq}$ or $E \subset (y, \rightarrow)_{\leq}$, see e.g. [11, Proposition 2.6]. Thus, E is an open subset of Y and $V \subset Y \setminus E$, therefore $D = \overline{V} \subset Y \setminus E$. However, this is impossible because $q \in E \cap D$. A contradiction!

Evidently, the same reasoning applies to show that q is also a noncut point of D. Since C and D are weakly orderable, so is the space $X = C \cup D$.   □

PROOF OF THEOREM 1.2: If X is weakly cyclically orderable and $p \in X$, then $X \setminus \{p\}$ is weakly orderable [22, Proposition 1.7]. Accordingly, $X \setminus \{p\}$ has a continuous weak selection. Conversely, suppose that $X \setminus \{p\}$ has a continuous weak selection for each $p \in X$. To show that X is weakly cyclically orderable, we

distinguish the following two cases. If $\mathrm{nct}(X) = X$, take any point $p \in X$ and a nonempty connected subset $Y \subset X \setminus \{p\}$. By hypothesis, $X \setminus \{p\}$ has a continuous weak selection, hence so does Y. Accordingly, by Theorem 2.2, Y is weakly orderable which implies that it has at most two noncut points, see e.g. [22, Theorem 3.5]. Therefore, by [22, Theorem 3.18], X is weakly cyclically orderable. If X has a cut point, it follows from Lemma 5.1 that X is weakly orderable, hence it is weakly cyclically orderable as well [22, Proposition 1.6]. □

We conclude this paper with the following consequence of Theorem 1.2, which provides a natural generalization of [22, Proposition 1.7].

Corollary 5.2. *A connected space X is weakly cyclically orderable if and only if $X \setminus \{p\}$ is weakly orderable for every $p \in X$.*

PROOF: If X is weakly cyclically orderable, then by [22, Proposition 1.7], $X \setminus \{p\}$ is weakly orderable for each point $p \in X$. If $X \setminus \{p\}$ is weakly orderable for each point $p \in X$, then each $X \setminus \{p\}$, $p \in X$, has a continuous weak selection. Hence, by Theorem 1.2, X is weakly cyclically orderable. □

REFERENCES

[1] Brouwer A. E., *A characterization of connected (weakly) orderable spaces*, Math. Centrum, Amsterdam, Afd. zuivere Wisk. ZW 10/71 (1971), 7 pages.

[2] Brouwer A. E., *On the topological characterization of the real line*, Math. Centrum, Amsterdam, Afd. zuivere Wisk. ZW 8/71 (1971), 6 pages.

[3] Buhagiar D., Gutev V., *Selections and deleted symmetric products*, Tsukuba J. Math. **41** (2017), no. 1, 1–20.

[4] Čech E., *Topological Spaces*, Publishing House of the Czechoslovak Academy of Sciences, Praha, Interscience Publishers John Wiley & Sons, London, 1966.

[5] Čoban M. M., *Many-valued mappings and Borel sets*, Trans. Mosc. Math. Soc. **22** (1970), 258–280.

[6] van Dalen J., Wattel E., *A topological characterization of ordered spaces*, General Topology and Appl. **3** (1973), 347–354.

[7] Duda R., *On ordered topological spaces*, Fund. Math. **63** (1968), 295–309.

[8] Eilenberg S., *Ordered topological spaces*, Amer. J. Math. **63** (1941), 39–45.

[9] Engelking R., Heath R. W., Michael E., *Topological well-ordering and continuous selections*, Invent. Math. **6** (1968), 150–158.

[10] García-Ferreira S., Gutev V., Nogura T., Sanchis M., Tomita A., *Extreme selections for hyperspaces of topological spaces*, Proc. of the International Conf. on Topology and Its Applications, Yokohama, 1999, Topology Appl. **122** (2002), no. 1–2, 157–181.

[11] Gutev V., *Weak orderability of second countable spaces*, Fund. Math. **196** (2007), no. 3, 275–287.

[12] Gutev V., *Selections and hyperspaces*, Recent Progress in General Topology, III, Atlantis Press, Paris, 2014, pages 535–579.

[13] Gutev V., *Selections and approaching points in products*, Comment. Math. Univ. Carolin. **57** (2016), no. 1, 89–95.

[14] Gutev V., *Scattered spaces and selections*, Topology Appl. **231** (2017), 306–315.

[15] Gutev V., Nogura T., *Selections and order-like relations*, Appl. Gen. Topol. **2** (2001), no. 2, 205–218.

[16] Gutev V., Nogura T., *Fell continuous selections and topologically well-orderable spaces*, Mathematika **51** (2004), no. 1–2, 163–169.

[17] Gutev V., Nogura T., *Set-maximal selections*, Topology Appl. **157** (2010), no. 1, 53–61.

[18] Gutev V., Nogura T., *Weak orderability of topological spaces*, Topology Appl. **157** (2010), no. 8, 1249–1274.

[19] Hocking J. G., Young G. S., *Topology*, Addison–Wesley Publishing, London, 1961.

[20] Huntington E. V., *A set of independent postulates for cyclic order*, Proc. Natl. Acad. Sci. USA **2** (1916), 630–631.

[21] Huntington E. V., *Sets of completely independent postulates for cyclic order*, Proc. Natl. Acad. Sci. USA **10** (1924), 74–78.

[22] Kok H., *Connected Orderable Spaces*, Mathematical Centre Tracts, 49, Mathematisch Centrum, Amsterdam, 1973.

[23] Michael E., *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.

[24] van Mill J., Wattel E., *Selections and orderability*, Proc. Amer. Math. Soc. **83** (1981), no. 3, 601–605.

[25] Nadler S. B., Jr., *Continuum Theory*, An Introduction, Monographs and Textbooks in Pure and Applied Mathematics, 158, Marcel Dekker, New York, 1992.

[26] Nogura T., Shakhmatov D., *Characterizations of intervals via continuous selections*, Rend. Circ. Mat. Palermo (2) **46** (1997), no. 2, 317–328.

[27] Whyburn G. T., *Concerning the cut points of continua*, Trans. Amer. Math. Soc. **30** (1928), no. 3, 597–609.

[28] Willard S., *General Topology*, Addison–Wesley Publishing, London, 1970.

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