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A REVIEW OF LIE SUPERALGEBRA COHOMOLOGY FOR PSEUDOFORMS

Carlo Alberto Cremonini

Abstract. This note is based on a short talk presented at the "42nd Winter School Geometry and Physics" held in Srni, Czech Republic, January 15th–22nd 2022. We review the notion of Lie superalgebra cohomology and extend it to different form complexes, typical of the superalgebraic setting. In particular, we introduce *pseudoforms* as infinite-dimensional modules related to sub-superalgebras. We then show how to extend the Koszul-Hochschild-Serre spectral sequence for pseudoforms as a computational method to determine the cohomology groups induced by sub-superalgebras. In particular, we show as an example the case of $\mathfrak{osp}(1 \mid 4)$ and choose $\mathfrak{osp}(1 \mid 2) \times \mathfrak{sp}(2)$ as sub-algebra. We finally comment on some physical applications of such new cohomology classes related to super-branes. The note is a compact version of [10].

Introduction

It is more than fifty years that with pioneering and ground-breaking works Berezin introduced the notion of "super" in mathematics. In a few years, the advent of Haag-Łopuszański-Sohnius theorem confirmed the importance of the constructions of Berezin in the Physical realm through the advent of *supersymmetry*.

Supergeometry proved to be a much richer field w.r.t. ordinary geometry; for example, integration theory requires the introduction of a new complex of forms, namely, *integral forms* (or from the original language, *integrable* forms). See, e.g., [4, 5, 8, 28, 29, 31, 32, 38] for an (of course) non-exhaustive list of references where the subject is introduced. In particular, in [38] integral forms are introduced via a very useful distributional realisation (first introduced in [3]) which is inspired by string theory (see, e.g., [2, 19, 37]) that make them very easy to treat computationally. Moreover, inspired by this realisation, in [38] another class of forms is introduced: *pseudoforms*. These forms are defined locally and are related to sub-supermanifolds with non-trivial odd dimension. They were also mentioned as "densities" in this context in [28], but only a few pages of comments are present there.

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The goal of this note is to present the lie algebraic cohomology on these new objects and to hint at some physical applications of it. The study of Lie superalgebras and their cohomology can be found in several works, e.g., [22, 24, 25, 26, 27, 33, 34, 35] and many results are collected in the vast book [20]. We start by reviewing the rigorous construction of pseudoforms on Lie superalgebras as infinite-dimensional modules related to the choice of Lie sub-superalgebras. We then define Chevalley-Eilenberg chain complexes of pseudoforms simply as the usual CE complex but with values in these modules, and its cohomology. We then extend the Koszul-Hochschild-Serre spectral sequence (see [21, 23]) to these complexes as a powerful tool to classify cohomology groups. We then present the example of $\mathfrak{osp}(1 \mid 4)$ and the pseudoforms induced by its sub-superalgebra $\mathfrak{osp}(1 \mid 2) \times \mathfrak{sp}(2)$. Finally, we comment on a possible application of these new cohomology classes, inspired by Free Differential Algebras (or in their dual version, L_∞-algebras), to the classifications of new classes of branes. The rigorous construction of pseudoforms and the related Koszul-Hochschild-Serre spectral sequences were first shown in [10], of which this note is a compact version.

1. Introducing pseudoforms

This section aims to introduce pseudoform modules induced by sub-(super)algebras of a given Lie (super)algebra. We will quickly review some basic definitions regarding Lie (super)algebra cohomology and the Koszul-Hochschild-Serre spectral sequence technique to compute it. We refer the reader to [9, 21, 23, 20] for the original papers on these subjects. We will then review integral forms in the (super)algebraic setting (see [7] for their first-time introduction) and then use them to define pseudoform modules, as done in [10].

Definition 1 (Chevalley-Eilenberg chain complex)**.** Given a (finite dimensional) Lie (super)algebra $\mathfrak{g}$ over the field $\mathbb{K}$ of characteristic zero[1] and a $\mathfrak{g}$-module V, we define *p-chains* of $\mathfrak{g}$ valued in V as (graded) alternating $\mathbb{K}$-linear products

$$C_p\left(\mathfrak{g}, V\right) := \wedge^p \mathfrak{g} \otimes V = \bigoplus_{r=1}^{p} \left(\wedge^r \mathfrak{g}_0 \otimes S^{p-r} \mathfrak{g}_1\right) \otimes V \,. \tag{1}$$

Throughout the paper we will denote with a $\mathfrak{g}$ basis $\{\mathcal{Y}_a\} = \{X_i, \xi_\alpha\}$, where $i = 1, \ldots, \dim \mathfrak{g}_0$ and $\alpha = 1, \ldots, \dim \mathfrak{g}_1$. We can lift (1) to a *chain complex* by introducing the differential $\partial : C_p\left(\mathfrak{g}, V\right) \to C_{p-1}\left(\mathfrak{g}, V\right)$ acting on $f \otimes \left(\mathcal{Y}_{a_1} \wedge \ldots \wedge \mathcal{Y}_{a_p}\right) \in C_p\left(\mathfrak{g}, V\right)$ as

[1] Throughout the paper we will systematically assume $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

$$\begin{aligned}\partial\left[f\otimes\left(\mathcal{Y}_{a_1}\wedge\ldots\wedge\mathcal{Y}_{a_p}\right)\right]=&\sum_{i=1}^{p}(-1)^{\delta_i}\left(\mathcal{Y}_{a_i}f\right)\\&\otimes\left(\mathcal{Y}_{a_1}\wedge\ldots\wedge\hat{\mathcal{Y}}_{a_i}\wedge\ldots\wedge\mathcal{Y}_{a_p}\right)+\sum_{i<j}^{p}(-1)^{\delta_{i,j}}f\\&\otimes\left(\left[\mathcal{Y}_{a_i},\mathcal{Y}_{a_j}\right]\wedge\mathcal{Y}_{a_1}\wedge\ldots\wedge\hat{\mathcal{Y}}_{a_i}\wedge\ldots\wedge\hat{\mathcal{Y}}_{a_j}\wedge\ldots\wedge\mathcal{Y}_{a_p}\right),\end{aligned}\tag{2}$$

where $\delta_i = |\pi\mathcal{Y}_{a_i}|\left(\sum_{p=1}^{i-1}\left|\pi\mathcal{Y}_{a_p}\right| + |f|\right)$, $\delta_{i,j} = |\pi\mathcal{Y}_{a_i}|\sum_{p=1}^{i-1}\left|\pi\mathcal{Y}_{a_p}\right| + \left|\pi\mathcal{Y}_{a_j}\right| \sum_{p=1}^{j-1}\left|\pi\mathcal{Y}_{a_p}\right| - |\pi\mathcal{Y}_{a_i}|\left|\pi\mathcal{Y}_{a_j}\right|$ and where with the symbol $\hat{\mathcal{Y}}$ we indicate that the vector is omitted; the nilpotence of the operator ∂ is a consequence of the (graded) Jacobi identities. The *Chevalley-Eilenberg chain complex* is then denoted as $(C_p(\mathfrak{g},V),\partial)$.

Definition 2 (Chevalley-Eilenberg cochain complex)**.** We define Chevalley-Eilenberg *p-cochains* of $\mathfrak{g}$ valued in V as (graded)alternating $\mathbb{K}$-linear maps

$$C^p(\mathfrak{g},V):=\mathrm{Hom}_{\mathbb{K}}\left(\wedge^p\mathfrak{g},V\right)=\wedge^p\mathfrak{g}^*\otimes V=\bigoplus_{r=1}^{p}\left(\wedge^r\mathfrak{g}_0^*\otimes S^{q-r}\mathfrak{g}_1^*\right)\otimes V\,.\tag{3}$$

We can lift (3) to a *cochain complex* by introducing the differential $d\colon C^p(\mathfrak{g},V)\to C^{p+1}(\mathfrak{g},V)$ defined on $\omega\in C^p(\mathfrak{g},V)$ by

$$\begin{aligned}d\omega(\mathcal{Y}_{a_1},\ldots,\mathcal{Y}_{a_{p+1}})=&\sum_{i=1}^{p+1}(-1)^{\delta_i}\mathcal{Y}_{a_i}\omega\left(\mathcal{Y}_{a_1}\wedge\ldots\wedge\hat{\mathcal{Y}}_{a_i}\wedge\ldots\wedge\mathcal{Y}_{a_{p+1}}\right)\\&+\sum_{i<j}(-1)^{\delta_{i,j}}\omega\left([\mathcal{Y}_{a_i},\mathcal{Y}_{a_j}]\wedge\mathcal{Y}_{a_1}\wedge\ldots\wedge\hat{\mathcal{Y}}_{a_i}\wedge\ldots\wedge\hat{\mathcal{Y}}_{a_j}\wedge\ldots\wedge\mathcal{Y}_{a_{p+1}}\right),\end{aligned}\tag{4}$$

where δ_i and $\delta_{i,j}$ are defined as above; again, the nilpotence of the operator d is a consequence of the (graded) Jacobi identities. The *Chevalley-Eilenberg cochain complex* is then denoted as $(C^p(\mathfrak{g},V),d)$.

Definition 3 (Lie (super)algebra cohomology)**.** Given a Chevalley-Eilenberg cochain complex of a Lie (super)algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V, we define the space of *p-cocycles* (or *closed forms*)

$$Z^p(\mathfrak{g},V):=\{\omega\in C^p(\mathfrak{g},V):d\omega=0\}\,.\tag{5}$$

The space of *p-coboundaries* (or *exact forms*) is defined as

$$B^p(\mathfrak{g},V):=\left\{\omega\in C^p(\mathfrak{g},V):\exists\rho\in C^{p-1}(\mathfrak{g},V):d\rho=\omega\right\}\,.\tag{6}$$

Because of the nilpotence of the operator d, we can consistently define the *p-cohomology group as*

$$H^p(\mathfrak{g},V):=\frac{Z^p(\mathfrak{g},V)}{B^p(\mathfrak{g},V)}\,.\tag{7}$$

Definition 4 (Relative Lie (super)algebra cohomology)**.** Given a Lie (super)algebra $\mathfrak{g}$ and a Lie sub-(super)algebra $\mathfrak{h}$, we define the space of *horizontal p-cochains* with values in a module V as

$$\begin{aligned} C^p_{hor}(\mathfrak{g},\mathfrak{h},V) \equiv C^p(\mathfrak{k},V) := \big\{\omega \in C^p(\mathfrak{g},V) : \omega\left(\mathcal{Y}_a,\mathcal{Y}_{a_1},\ldots,\mathcal{Y}_{a_{p-1}}\right) = 0\,, \\ \forall \mathcal{Y}_a \in \mathfrak{h}, \mathcal{Y}_{a_1},\ldots,\mathcal{Y}_{a_{p-1}} \in \mathfrak{g}\big\}\,. \end{aligned} \tag{8}$$

We define the space of $\mathfrak{h}$-*invariant p-cochains* with values in V as

$$\begin{aligned} (C^p(\mathfrak{g},V))^{\mathfrak{h}} := \Bigg\{\omega \in C^p(\mathfrak{g},V) : \sum_{j=1}^{p}(-1)^{\delta_{i,j}}\omega\big(\mathcal{Y}_{a_1},\ldots, \\ \left[\mathcal{Y}_a,\mathcal{Y}_{a_j}\right],\ldots,\mathcal{Y}_{a_p}\big) = 0, \forall \mathcal{Y}_a \in \mathfrak{h}, \mathcal{Y}_{a_1},\ldots,\mathcal{Y}_{a_p} \in \mathfrak{g}\Bigg\}\,, \end{aligned} \tag{9}$$

where $\delta_{i,j} = |\pi\mathcal{Y}_a| \sum_{r=1}^{j-1} |\pi\mathcal{Y}_{a_r}|$. The forms which are both horizontal and $\mathfrak{h}$-invariant are called *basic* and we will denote them as (we denote $\mathfrak{k} = \mathfrak{g}/\mathfrak{h}$ as a quotient of vector spaces, since $\mathfrak{h}$ in general is not an ideal, or analogously $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$)

$$\big(C^p(\mathfrak{k},V)\big)^{\mathfrak{h}} \equiv \left(C^p(\mathfrak{g}/\mathfrak{h},V)\right)^{\mathfrak{h}}\,. \tag{10}$$

Cocycles and coboundaries are defined by taking the restriction of (5) and (6) to (10); analogously, we can restrict the differential (4) to basic forms, denoting it $d|_{basic} \equiv \nabla_{\mathfrak{k}}$, define closed and exact forms and finally the *cohomology of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$:

$$H^\bullet(\mathfrak{g},\mathfrak{h},V) := \frac{\Big\{\omega \in \left(C^\bullet(\mathfrak{k},V)\right)^{\mathfrak{h}} : \nabla_{\mathfrak{k}}\omega = 0\Big\}}{\Big\{\omega \in \left(C^\bullet(\mathfrak{k},V)\right)^{\mathfrak{h}} : \exists \eta \in \left(C^\bullet(\mathfrak{k},V)\right)^{\mathfrak{h}}, \omega = \nabla_{\mathfrak{k}}\eta\Big\}}\,. \tag{11}$$

By adopting a more modern notation for horizontal and $\mathfrak{h}$-invariant forms, (8) and (9) can be rewritten as

$$\begin{aligned} C^p(\mathfrak{k},V) &:= \left\{\omega \in C^p(\mathfrak{g},V) : \iota_{\mathcal{Y}_a}\omega = 0\,, \forall \mathcal{Y}_a \in \mathfrak{h}\right\}\,, \\ \left(C^p(\mathfrak{g},V)\right)^{\mathfrak{h}} &:= \left\{\omega \in C^p(\mathfrak{g},V) : \mathcal{L}_{\mathcal{Y}_a}\omega = 0\,, \forall \mathcal{Y}_a \in \mathfrak{h}\right\}\,. \end{aligned} \tag{12}$$

respectively, where $\iota_{\mathcal{Y}_a}$ denotes the contraction along the vector $\mathcal{Y}_a$ and $\mathcal{L}_{\mathcal{Y}_a}$ denotes the Lie derivative along the vector $\mathcal{Y}_a$.

In the case $V = \mathbb{K}$, we can use a simpler form for the differential (4): in particular, the first term drops, and, given a $\mathfrak{g}$ basis $\{\mathcal{Y}_a\} = \{X_i, \xi_\alpha\}$ and the (parity changed) dual $\Pi\mathfrak{g}^*$ basis $\{\mathcal{Y}^{*a}\} = \left\{V^i, \psi^\alpha\right\}$, $i = 1,\ldots,\dim\mathfrak{g}_0$, $\alpha = 1,\ldots,\dim\mathfrak{g}_1$, the differential reads

$$d = \sum_{a,b,c} f^a_{bc}\mathcal{Y}^{*b} \wedge \mathcal{Y}^{*c}\iota_{\mathcal{Y}_a}\,, \tag{13}$$

where f^a_{bc} are the structure constants of the Lie (super)algebra $\mathfrak{g}$.

We will now recall the spectral sequence technique to compute Lie (super)algebraic cohomology. The original papers for standard Lie algebras are [23], for the trivial module case, and [21] for the general module case. For the extension to Lie superalgebras see, e.g., [20]. In the following, we will extend this construction to pseudoforms, showing that it allows to greatly simplify the computation of cohomology groups, as it allows to reduce to finite-dimensional spaces (as we will see, pseudoforms are introduced as infinite-dimensional modules).

Definition 5 (Koszul-Hochschild-Serre Spectral Sequence)**.** Given a Lie (super) algebra $\mathfrak{g}$, a sub-(super)algebra $\mathfrak{h}$ and a $\mathfrak{g}$-module V, we define the filtration

$$F^p C^q(\mathfrak{g}, V) = \left\{ \omega \in C^q(\mathfrak{g}, V) : \forall \mathcal{Y}_a \in \mathfrak{h}, \iota_{\mathcal{Y}_{a_1}} \ldots \iota_{\mathcal{Y}_{a_{q+1-p}}} \omega = 0 \right\}, p, q \in \mathbb{Z}, \tag{14}$$

where we denote $C^{q<0}(\mathfrak{g}, V) = 0$. In particular, for $F^p C^q(\mathfrak{g}, V)$ we have

$$\begin{gathered} F^{q+n+1} C^q(\mathfrak{g}, V) = 0, \quad F^{p+1} C^q(\mathfrak{g}, V) \subseteq F^p C^q(\mathfrak{g}, V), \\ dF^p C^q(\mathfrak{g}, V) \subseteq F^p C^{q+1}(\mathfrak{g}, V), \quad \forall n \in \mathbb{N} \cup \{0\}, \ \forall p, q \in \mathbb{Z}. \end{gathered} \tag{15}$$

Associated to the filtration (14), there exist a spectral sequence $\left(E_s^{\bullet,\bullet}, d_s\right)_{s \in \mathbb{N} \cup \{0\}}$ that converges to $H^\bullet(\mathfrak{g}, V)$. More explicitly, this means that we define *page zero* of the spectral sequence as

$$E_0^{p,q} := F^p C^{p+q}(\mathfrak{g}, V) / F^{p+1} C^{p+q}(\mathfrak{g}, V). \tag{16}$$

The (nilpotent) differentials d_s are induced by the Chevalley-Eilenberg differential (4) and each page of the spectral sequence is defined as the cohomology of the previous one:

$$d_s : E_s^{p,q} \to E_s^{p+s, q+1-s}, \ E_{s+1}^{\bullet,\bullet} := \left(E_s^{\bullet,\bullet}, d_s\right), \ E_\infty^{\bullet,\bullet} \cong H^\bullet(\mathfrak{g}, V). \tag{17}$$

If $\exists n \in \mathbb{N} \cup \{0\} : d_s = 0, \forall s \geq n$, then we say that the spectral sequence *converges at page* n and we denote

$$E_n^{\bullet,\bullet} = E_{n+1}^{\bullet,\bullet} = \ldots = E_\infty^{\bullet,\bullet}. \tag{18}$$

Theorem 1 (Hochschild-Serre)**.** *Given a Lie algebra* $\mathfrak{g}$ *over a (characteristic zero) field* $\mathbb{K}$*, a sub-algebra* $\mathfrak{h}$ *reductive in the ambient Lie algebra and a finite-dimensional* $\mathfrak{g}$*-module* V*, the first pages of the spectral sequence read*

$$E_1^{p,q} \cong H^q(\mathfrak{h}, \mathbb{K}) \otimes C^p(\mathfrak{g}/\mathfrak{h}, V)^{\mathfrak{h}}, \tag{19}$$

$$E_2^{p,q} \cong H^q(\mathfrak{h}, \mathbb{K}) \otimes H^p(\mathfrak{g}, \mathfrak{h}, V). \tag{20}$$

One of the main results of [10] was the extension of this theorem to pseudoform cohomology. This is extremely useful, as it immediately gives page two of the spectral sequence in any form complex, in particular, for any pseudoform space induced by (reductive) Lie sub-(super)algebras. In the following, we will recall the main steps that lead to pseudoforms, their cohomology, and the Koszul-Hochschild-Serre spectral sequence extension.

We start by recalling the definition of integral forms for Lie superalgebras. They are essential objects to deal with when considering integration on supermanifolds; in this context, there is no notion of integration, but integral forms may be used to keep track of some algebraic invariants which are not kept into account among superforms. Moreover, we will leverage on them to infer a rigorous definition of pseudoforms which, on supermanifolds, are now defined only locally. For their introduction on supermanifolds, we refer the reader to, e.g., [28], to [38], where they are introduced with an eye towards some superstring applications and to the recent [29], where many formal aspects and many applications are collected.

Definition 6 (Integral Forms). Given a Lie superalgebra $\mathfrak{g}$, with dimension $\dim \mathfrak{g} = (m|n)$, a basis $\mathcal{Y}_a = \{X_i, \xi_\alpha\}$, its (parity changed) dual $\Pi\mathfrak{g}^*$, with a basis $\mathcal{Y}^{*a} = \left\{V^i, \psi^\alpha\right\}$, $a = 1, \ldots, m$, $\alpha = 1, \ldots, n$, and a $\mathfrak{g}$-module V, we define the *Haar Berezinian* of $\mathfrak{g}$ with values in V as

$$\mathscr{Ber}\left(\mathfrak{g}\right) := V \cdot \left[\bigwedge_{i=1}^{m} V^i \otimes \bigwedge_{\alpha=1}^{n} \xi_\alpha\right] \equiv V \cdot \mathcal{D} \,. \tag{21}$$

We define the integral form cochain complex as the sequence of spaces

$$C^p_{int}\left(\mathfrak{g}, V\right) := \mathscr{Ber}\left(\mathfrak{g}\right) \otimes S^{m-p}\Pi\mathfrak{g} = \mathcal{D} \otimes \left[V \otimes S^{m-p}\Pi\mathfrak{g}\right] \,, \tag{22}$$

equipped with the differential defined in terms of (2)

$$\begin{aligned} \delta \colon C^p_{int}\left(\mathfrak{g}, V\right) &\to C^{p+1}_{int}\left(\mathfrak{g}, V\right) \\ \mathcal{D} \otimes \left[f \otimes \mathcal{Y}_1 \wedge \ldots \wedge \mathcal{Y}_{m-p}\right] &\mapsto \delta\left(\mathcal{D} \otimes \left[f \otimes \mathcal{Y}_1 \wedge \ldots \wedge \mathcal{Y}_{m-p}\right]\right) \\ &= \mathcal{D} \otimes \partial\left[f \otimes \mathcal{Y}_1 \wedge \ldots \wedge \mathcal{Y}_{m-p}\right] \,, \end{aligned} \tag{23}$$

which is clearly nilpotent. Then one can define the Chevalley-Eilenberg cohomology on integral forms in the obvious way. When one considers "basic classical Lie superalgebras", i.e., those classical Lie superalgebras admitting a non-degenerate bilinear form (see [18]), the integral Chevalley-Eilenberg cohomology on the trivial module $V = \mathbb{K}$ $H^p_{int}(\mathfrak{g}, \mathbb{K})$ becomes a *twist* of the $(n-p)$-th homology group by $\mathcal{D}$. Then the following proposition:

Proposition 2 (Berezinian Complement Quasi-Isomorphism). *Given a basic classical Lie (super)algebra* $\mathfrak{g}$ *of dimension* $\dim \mathfrak{g} = (m|n)$*, the map*

$$\begin{aligned} \star \colon C^p(\mathfrak{g}, \mathbb{K}) &\to \Pi^{m+n} C^{m-p}_{int}(\mathfrak{g}, \mathbb{K}) \\ \mathcal{Y}^{*a_1} \wedge \ldots \wedge \mathcal{Y}^{*a_p} &\mapsto \mathcal{D} \otimes \mathcal{Y}_{a_1} \wedge \mathcal{Y}_{a_p} \,, \end{aligned} \tag{24}$$

such that $\left(\mathcal{D} \otimes \mathcal{Y}_{a_1} \wedge \mathcal{Y}_{a_p}\right)\left(\mathcal{Y}^{*b_1} \wedge \ldots \wedge \mathcal{Y}^{*b_p}\right) \propto \mathcal{D}$*, induces a cohomology isomorphism:*

$$\star : H^\bullet\left(\mathfrak{g}, \mathbb{K}\right) \xrightarrow{\cong} \Pi^{m+n} H^{m-\bullet}_{int}\left(\mathfrak{g}, \mathbb{K}\right) \,. \tag{25}$$

Integral forms can be realised in a different way, inspired by string theory (see [19]), in terms of *Dirac distributions* (see [38] where this realisation is discussed extensively): we can realise the Berezinian of $\mathfrak{g}$ with values in V as

$$\mathscr{Ber}\left(\mathcal{SM}\right) := V \cdot \left[\bigwedge_{i=1}^{\dim \mathcal{SM}_0} V^i \wedge \bigwedge_{\alpha=1}^{\dim \mathcal{SM}_1} \delta\left(\psi^\alpha\right)\right] \,, \tag{26}$$

where $\delta(\psi)$ are (formal) Dirac delta distributions. The symbol $\delta(\psi)$ satisfies the following distributional identities

$$\psi\delta(\psi) = 0\,,\ \delta(\lambda\psi) = \frac{1}{\lambda}\delta(\psi)\ ,\ \psi\iota^{(p)}\delta(\psi) = -p\iota^{p-1}\delta(\psi) \equiv -p\delta^{(p-1)}(\psi)\ , \tag{27}$$
$$\delta(\psi^\alpha)\wedge\delta\left(\psi^\beta\right) = -\delta\left(\psi^\beta\right)\wedge\delta(\psi^\alpha)\ ,\ V\wedge\delta(\psi) = -\delta(\psi)\wedge V\,,$$
$$d\delta(\psi) = (d\psi)\wedge\iota\delta(\psi)\ .$$

In this realisation, integral forms are constructed as in (22), but now the parity changed vector fields are represented as contractions acting on (26). This realisation of integral forms suggests the construction of forms with non-maximal and non-zero number of deltas: the *pseudoforms*. A general pseudoform with q Dirac delta's (we call the number of deltas *picture number*, as inspired by string theory) is given by

$$\omega^{(p|q)} = V\cdot\left[V^{i_1}\wedge\ldots\wedge V^{i_r}\wedge\psi^{\alpha_1}\wedge\ldots\wedge\psi^{\alpha_s}\wedge\delta^{(t_1)}\left(\psi^{\beta_1}\right)\wedge\ldots\right. \tag{28}$$
$$\left.\wedge\,\delta^{(t_q)}\left(\psi^{\beta_q}\right)\right],$$

where $\delta^{(i)}(\psi) \equiv (\iota)^i\,\delta(\psi)$. The form number of (28) is $p = r + s - \sum_{i=1}^{q} t_i$, since the contractions carry negative form number, and the number of delta counts the picture number. The form number and the picture number range as $-\infty < p < +\infty$ and $0 \leq q \leq n$. If $q = 0$, we have superforms, if $q = n$ we have integral forms, if $0 < q < n$ we have pseudoforms. One of the advantages of the Dirac delta forms realisation consists in the use of a single differential for superforms, integral forms and pseudoforms: we can use the d on every complex for any picture number, by supporting it with the formal properties in (27). In particular, the action of the CE differential on Dirac delta's (we take $V = \mathbb{K}$) is simply given by

$$d\delta(\psi^\alpha) = \sum_{a,b,c} f^a_{bc}\mathcal{Y}^{*b}\wedge\mathcal{Y}^{*c}\wedge\iota_{\mathcal{Y}_a}\delta(\psi^\alpha) = \sum_{\alpha,b,c} f^\alpha_{bc}\mathcal{Y}^{*b}\wedge\mathcal{Y}^{*c}\wedge\iota_{\xi_\alpha}\delta(\psi^\alpha)\ . \tag{29}$$

This definition, equipped with the rules listed in (27) can be analogously interpreted using the last identity of (27), namely, the chain rule:

$$d\delta(\psi^\alpha) = (d\psi^\alpha)\wedge\iota_{\xi_\alpha}\delta(\psi^\alpha) = \Big(\sum_{a,b,c} f^a_{bc}\mathcal{Y}^{*b}\wedge\mathcal{Y}^{*c}\iota_{\mathcal{Y}_a}\psi^\alpha\Big)\wedge\iota_{\xi_\alpha}\delta(\psi^\alpha)$$
$$= \sum_{b,c} f^\alpha_{bc}\mathcal{Y}^{*b}\wedge\mathcal{Y}^{*c}\wedge\iota_{\xi_\alpha}\delta(\psi^\alpha)\ . \tag{30}$$

In the following, we will give a rigorous definition of pseudoforms without referring to the distributional realisation. Nonetheless, we will underline the dictionary with the distributional realisation, to justify its use, since it leads to (non-trivial) computational advantages.

1.1. **Pseudoforms as infinite-dimensional representations.** We start from a Lie (super)algebra $\mathfrak{g}$, $\dim\mathfrak{g} = (m|n)$, a sub-(super)algebra $\mathfrak{h}$, $\dim\mathfrak{h} = (p|q)$ and denote the (super)coset $\mathfrak{k} = \mathfrak{g}/\mathfrak{h}$ (again, as a vector space), $\dim\mathfrak{k} = (m-p|n-q)$.

Associated with $\mathfrak{h}$ and $\mathfrak{k}$ we can define the Berezinian spaces $\mathcal{B}er\left(\mathfrak{h}\right)$ and $\mathcal{B}er\left(\mathfrak{k}\right)$, which can be realised as

$$\mathcal{B}er\left(\mathfrak{h}\right) := \mathbb{K} \cdot \left[\bigwedge_{i=1}^{p} V^i \otimes \bigwedge_{\alpha=1}^{q} \xi_\alpha \right] \equiv \mathbb{K} \cdot \mathcal{D}_{\mathfrak{h}} \ \leftrightsquigarrow \ \mathbb{K} \cdot \left[\bigwedge_{i=1}^{p} V^i \wedge \bigwedge_{\alpha=1}^{q} \delta\left(\psi^\alpha\right) \right] ,$$
$$V^i , \psi^\alpha \in \Pi \mathfrak{h}^* \, , \ \xi_\alpha \in \mathfrak{h} \, ,$$
$$\mathcal{B}er\left(\mathfrak{k}\right) := \mathbb{K} \cdot \left[\bigwedge_{\hat{i}=1}^{m-p} V^{\hat{i}} \otimes \bigwedge_{\hat{\alpha}=1}^{n-q} \xi_{\hat{\alpha}} \right] \equiv \mathbb{K} \cdot \mathcal{D}_{\mathfrak{k}} \ \leftrightsquigarrow \ \mathbb{K} \cdot \left[\bigwedge_{\hat{i}=1}^{m-p} V^{\hat{i}} \wedge \bigwedge_{\hat{\alpha}=1}^{n-q} \delta\left(\psi^{\hat{\alpha}}\right) \right] ,$$
$$V^{\hat{i}} , \psi^{\hat{\alpha}} \in \Pi \mathfrak{k}^* \, , \ \xi_{\hat{\alpha}} \in \mathfrak{k} \, .$$

In [10] we were considering a $\mathfrak{g}$-action on the two spaces above, but we have seen that $\mathcal{B}er\left(\mathfrak{h}\right)$ and $\mathcal{B}er\left(\mathfrak{k}\right)$ *are not* $\mathfrak{g}$*-modules*. However, we managed to complement these two spaces to (infinite-dimensional) $\mathfrak{g}$-modules and we came to the following definition:

Definition 7 (Pseudoform Modules Relative to a Lie sub-(super)algebra)**.** Given a Lie superalgebra $\mathfrak{g}$ of dimension $\dim\left(\mathfrak{g}\right) = (m|n)$, a Lie sub-(super)algebra $\mathfrak{h}$ of dimension $\dim\left(\mathfrak{h}\right) = (p|q)$, and denoting $\mathfrak{k} = \mathfrak{g}/\mathfrak{h}$, we define the *modules of* $(p|q)$*- and* $(m-p|n-q)$*- pseudoforms* as

$$V_{\mathfrak{h}}^{(p|q)} := \bigoplus_{i=0}^{\infty} \left(S^i \Pi \mathfrak{h} \otimes \mathcal{B}er\left(\mathfrak{h}\right) \right) \otimes S^i \Pi \mathfrak{k}^* = \bigoplus_{i=0}^{\infty} C_{int}^{m-i}\left(\mathfrak{h}\right) \otimes C^i\left(\mathfrak{k}\right) \, , \tag{31}$$

$$V_{\mathfrak{k}}^{(m-p|n-q)} := \bigoplus_{i=0}^{\infty} \left(S^i \Pi \mathfrak{k} \otimes \mathcal{B}er\left(\mathfrak{k}\right) \right) \otimes S^i \Pi \mathfrak{h}^* = \bigoplus_{i=0}^{\infty} C_{int}^{m-p-i}\left(\mathfrak{k}\right) \otimes C^i\left(\mathfrak{h}\right) \, . \tag{32}$$

In particular, pseudoforms are constructed as integral forms of $\mathfrak{h}$ (resp., $\mathfrak{k}$) tensored with superforms of $\mathfrak{k}$ (resp., $\mathfrak{h}$). (31) and (32) define $\mathfrak{g}$-modules, where the $\mathfrak{g}$-action is given by

$$\mathcal{L} \colon \mathfrak{g} \times V_\bullet^{(\bullet|\bullet)} \to V_\bullet^{(\bullet|\bullet)}$$
$$\left(\mathcal{Y}_a, \mathcal{D}_\bullet\right) \mapsto \mathcal{L}\left(\mathcal{Y}_a\right) \mathcal{D}_\bullet \equiv \mathcal{L}_{\mathcal{Y}_a} \mathcal{D}_\bullet = \left(d \iota_{\mathcal{Y}_a} + (-1)^{|\mathcal{Y}_a|} \iota_{\mathcal{Y}_a} d \right) \mathcal{D}_\bullet \, , \tag{33}$$

where $\mathcal{L} \colon \mathfrak{g} \to End\left(V_\bullet^{(\bullet|\bullet)}\right)$ by construction satisfies

$$\mathcal{L}\big(\left[\mathcal{Y}_a, \mathcal{Y}_b \right] \big) = \mathcal{L}\left(\mathcal{Y}_a\right) \mathcal{L}\left(\mathcal{Y}_b\right) - (-1)^{|\mathcal{Y}_a||\mathcal{Y}_b|} \mathcal{L}\left(\mathcal{Y}_b\right) \mathcal{L}\left(\mathcal{Y}_a\right) \, . \tag{34}$$

Starting from the two spaces (31) and (32), we can now define *pseudoform cochains* and *pseudoform cohomology*, with respect to a given sub-(super)algebra:

Definition 8 (Pseudoform Cohomology Relative to a sub-(super)algebra)**.** Given a Lie superalgebra $\mathfrak{g}$ of dimension $\dim\left(\mathfrak{g}\right) = (m|n)$, a Lie sub-(super)algebra $\mathfrak{h}$ of dimension $\dim\left(\mathfrak{h}\right) = (p|q)$, and denoting $\mathfrak{k} = \mathfrak{g}/\mathfrak{h}$, and given $s \in \mathbb{N} \cup \{0\}$, we define the spaces of $(p+s|q)$-, $(p-s|q)$-, $(m-p+s|n-q)$*- and* $(m-p-s|n-q)$*- cochains* *as*

$$C^{p+s}\left(\mathfrak{g},V_{\mathfrak{h}}^{(p|q)}\right):=C^{s}\left(\mathfrak{k}\right)\otimes V_{\mathfrak{h}}^{(p|q)}=\bigoplus_{i=0}^{\infty}C_{int}^{p-i}\left(\mathfrak{h}\right)\otimes C^{i+s}\left(\mathfrak{k}\right)\ , \tag{35}$$

$$C^{p-s}\left(\mathfrak{g},V_{\mathfrak{h}}^{(p|q)}\right):=C_{s}\left(\mathfrak{h}\right)\otimes V_{\mathfrak{h}}^{(|q)}=\bigoplus_{i=0}^{\infty}C_{int}^{p-s-i}\left(\mathfrak{h}\right)\otimes C^{i}\left(\mathfrak{k}\right)\ , \tag{36}$$

$$\begin{aligned}C^{m-p+s}\left(\mathfrak{g},V_{\mathfrak{k}}^{(m-p|n-q)}\right)&:=C^{s}\left(\mathfrak{h}\right)\otimes V_{\mathfrak{k}}^{(m-p|n-q)}\\&=\bigoplus_{i=0}^{\infty}C_{int}^{m-p-i}\left(\mathfrak{k}\right)\otimes C^{i+s}\left(\mathfrak{h}\right)\ ,\end{aligned} \tag{37}$$

$$\begin{aligned}C^{m-p-s}\left(\mathfrak{g},V_{\mathfrak{k}}^{(m-p|n-q)}\right)&:=C_{s}\left(\mathfrak{k}\right)\otimes V_{\mathfrak{k}}^{(m-p|n-q)}\\&=\bigoplus_{i=0}^{\infty}C_{int}^{m-p-s-i}\left(\mathfrak{k}\right)\otimes C^{i}\left(\mathfrak{h}\right)\ .\end{aligned} \tag{38}$$

We will adopt the shorter notation

$$\begin{aligned}C^{p\pm s}\left(\mathfrak{g},V_{\mathfrak{h}}^{(p|q)}\right)\equiv C^{(p\pm s|q)}\left(\mathfrak{g}\right)\ ,\ C^{m-p\pm s}\left(\mathfrak{g},V_{\mathfrak{k}}^{(m-p|n-q)}\right)\\\equiv C^{(m-p\pm s|n-q)}\left(\mathfrak{g}\right)\ ,\end{aligned} \tag{39}$$

where the second number in the apex (which we called *picture number* in the previous section), indicates on which module they take values. We can lift $C^{(s|q)}\left(\mathfrak{g}\right)$ and $C^{(s|n-q)}\left(\mathfrak{g}\right)$ to cochain complexes by introducing a Chevalley-Eilenberg differential; in the distributional realisation, this is simply given by

$$\begin{aligned}d\colon C^{(s|\bullet)}\left(\mathfrak{g}\right)&\to C^{(s+1|\bullet)}\left(\mathfrak{g}\right)\\\omega\mapsto d\omega&=\sum_{a,b,c}f_{bc}^{a}\mathcal{Y}^{*b}\wedge\mathcal{Y}^{*c}\wedge\iota_{\mathcal{Y}_{c}}\omega\,.\end{aligned} \tag{40}$$

The translation to polyvectors realisation is again straightforward, but one should use different differentials when acting on forms or vectors, thus making the expressions more cumbersome.

We can now define closed and exact pseudoforms as in (5) and (6), where we take as module V either $V_{\mathfrak{h}}^{(p|q)}$ or $V_{\mathfrak{k}}^{(m-p|n-q)}$ defined in (31) and (32), respectively:

$$H^{(\bullet|q)}\left(\mathfrak{g}\right):=H^{\bullet}\left(\mathfrak{g},V_{\mathfrak{h}}^{(p|q)}\right)\ , \tag{41}$$

$$H^{(\bullet|n-q)}\left(\mathfrak{g}\right):=H^{\bullet}\left(\mathfrak{g},V_{\mathfrak{k}}^{(m-p|n-q)}\right). \tag{42}$$

The previous definitions can be naturally extended if we consider also a $\mathfrak{g}$-module V: we will define the pseudoform cochains with values in the module V as

$$C^{(\bullet|q)}\left(\mathfrak{g},V\right):=C^{\bullet}\left(\mathfrak{g},V_{\mathfrak{h}}^{(p|q)}\otimes V\right),\ C^{(\bullet|n-q)}\left(\mathfrak{g},V\right):=C^{\bullet}\left(\mathfrak{g},V_{\mathfrak{k}}^{(m-p|n-q)}\otimes V\right), \tag{43}$$

and the cohomology will be defined as usual with respect to the modules $V_{\mathfrak{h}}^{(p|q)} \otimes V$ and $V_{\mathfrak{k}}^{(m-p|n-q)} \otimes V$.

We can now show how the KHS spectral sequence can be treated when dealing with pseudoforms. In particular, this means that given a Lie (super)algebra $\mathfrak{g}$ and chosen a Lie sub-(super)algebra $\mathfrak{h}$, we can define two filtrations and then two sectors of page zero of the KHS spectral sequence, as explained below:

Definition 9 (Generalised KHS Spectral Sequence)**.** Given a Lie (super)algebra $\mathfrak{g}$ and a Lie sub-(super)algebra $\mathfrak{h}$ a $\mathfrak{g}$-module V and denoting $\mathfrak{k} = \mathfrak{g}/\mathfrak{h}$, we define the two inequivalent filtrations

$$F^p C^{(q|l)}(\mathfrak{g}, V) := \left\{ \omega \in C^{(q|l)}(\mathfrak{g}, V) : \forall \mathcal{Y}_a \in \mathfrak{h}, \iota_{\mathcal{Y}_{a_1}} \dots \iota_{\mathcal{Y}_{a_{q+1-p}}} \omega = 0 \right\}, \tag{44}$$

$$\tilde{F}^p C^{(q|l)}(\mathfrak{g}, V) := \left\{ \omega \in C^{(q|l)}(\mathfrak{g}, V) : \forall \mathcal{Y}^{*a} \in \mathfrak{h}^*, \right. \\ \left. \mathcal{Y}^{*a_1} \wedge \dots \wedge \mathcal{Y}^{*a_{q+1-p}} \wedge \omega = 0 \right\}, \tag{45}$$

where $p, q \in \mathbb{Z}$ and $l \in \{0, \dim \mathfrak{h}_1, \dim \mathfrak{k}_1, \dim \mathfrak{g}_1\}$. For each l, associated to (44) and (45), there exist a spectral sequence $(\mathcal{E}_s^{\bullet,\bullet}, d_s)_{s \in \mathbb{N} \cup \{0\}}$ that converges to $H^{(\bullet|l)}(\mathfrak{g}, V)$. In particular, we can define the *total page zero* of the spectral sequence, for each l, as

$$\begin{aligned} \mathcal{E}_0^{m,n} &:= E_0^{m,n} \oplus \tilde{E}_0^{m,n} \\ &:= \frac{F^m C^{(m+n|l)}(\mathfrak{g}, V)}{F^{m+1} C^{(m+n|l)}(\mathfrak{g}, V)} \oplus \frac{\tilde{F}^{m+2n-r} C^{(m+n|l)}(\mathfrak{g}, V)}{\tilde{F}^{m+2n-r+1} C^{(m+n|l)}(\mathfrak{g}, V)} . \end{aligned} \tag{46}$$

The differentials are induced by the Chevalley-Eilenberg differential (4) (adapted to act on pseudoforms as explained in the previous paragraphs) and, for each picture number l, each page of the spectral sequence is defined as the cohomology of the previous one:

$$d_s : \mathcal{E}_s^{p,q} \to \mathcal{E}_s^{p+s,q+1-s}, \ \mathcal{E}_{s+1}^{\bullet,\bullet} := (\mathcal{E}_s^{\bullet,\bullet}, d_s), \ \mathcal{E}_\infty^{\bullet,\bullet} \cong H^{(\bullet|l)}(\mathfrak{g}, V). \tag{47}$$

As usual, if $\exists n \in \mathbb{N} \cup \{0\} : d_s = 0, \forall s \geq n$, then we say that the spectral sequence *converges at page* n and we denote

$$\mathcal{E}_n^{\bullet,\bullet} = \mathcal{E}_{n+1}^{\bullet,\bullet} = \dots = \mathcal{E}_\infty^{\bullet,\bullet} . \tag{48}$$

Remark. Def. 9 is consistent with the cases $l = 0, \dim \mathfrak{g}_1$, namely, with superforms and integral forms. In the former case, $\mathcal{E}_0^{m,n}$ simplifies to $E_0^{m,n}$, and in the latter it simplifies to $\tilde{E}_0^{m,n}$. This in particular shows that integral forms are kept into account when using spectral sequences via the second filtration (45).

The previous definition becomes redundant if one considers a purely even sub-algebra $\mathfrak{h}$; in this case, the filtrations $F^p C^{(q|l)}(\mathfrak{g})$ and $\tilde{F}^p C^{(q|l)}(\mathfrak{g})$ induce

$$E_0^{m,n} \cong \tilde{E}_0^{m,n} = C^n(\mathfrak{h}) \otimes C^{(m|l)}(\mathfrak{k}) \otimes V , \tag{49}$$

for $l = 0, \dim \mathfrak{g}_1$ and $\mathcal{E}_0^{m,n}$ counts the spaces twice. This exactly corresponds to what is done for the calculation of the algebraic superform cohomology for Lie superalgebras as in [20], where one does not need to introduce the second filtration.

It is now easy to extend Thm. 1 to the case of superalgebras with the two filtrations and on pseudoforms with picture number l (we will deal with the trivial module $V = \mathbb{K}$ case only), so we come up with the following proposition:

Proposition 3. *Let* $\mathfrak{g}$ *be a Lie (super)algebra over a (characteristic-zero) field* $\mathbb{K}$ *and let* $\mathfrak{h}$ *be a Lie sub-(super)algebra reductive in* $\mathfrak{g}$, $\dim \mathfrak{h}_1 \neq 0$, *and denote* $\mathfrak{k} = \mathfrak{g}/\mathfrak{h}$. *Then, at picture number* $l = \dim \mathfrak{h}_1, \dim \mathfrak{k}_1$, *if* $\dim \mathfrak{h}_1 \neq \dim \mathfrak{k}_1$, *the first pages of the extended spectral sequence read*

$$
\begin{aligned}
l = \dim \mathfrak{h}_1 \implies \mathcal{E}_1^{m,n} &= \left(C^{(m|0)}(\mathfrak{k})\right)^{\mathfrak{h}} \otimes H^{(n|\dim \mathfrak{h}_1)}(\mathfrak{h}) \, , \\
\mathcal{E}_2^{m,n} &= H^{(m|0)}\left(\mathfrak{g}, \mathfrak{h}\right) \otimes H^{(n|\dim \mathfrak{h}_1)}\left(\mathfrak{h}\right) \, , \\
l = \dim \mathfrak{k}_1 \implies \mathcal{E}_1^{m,n} &= \left(C^{(m|\dim \mathfrak{k}_1)}(\mathfrak{k})\right)^{\mathfrak{h}} \otimes H^{(n|0)}\left(\mathfrak{h}\right) \, , \\
\mathcal{E}_2^{m,n} &= H^{(m|\dim \mathfrak{k}_1)}\left(\mathfrak{g}, \mathfrak{h}\right) \otimes H^{(n|0)}\left(\mathfrak{h}\right) \, .
\end{aligned}
$$

If $\dim \mathfrak{h}_1 = \dim \mathfrak{k}_1 = \dim \mathfrak{g}_1/2$, *then the first two pages at picture number* $l = \dim \mathfrak{g}_1/2$ *read*

$$
\begin{aligned}
\mathcal{E}_1^{m,n} &= \left[\left(C^{(m|0)}(\mathfrak{k})\right)^{\mathfrak{h}} \otimes H^{(n|\dim \mathfrak{g}_1/2)}(\mathfrak{h})\right] \oplus \left[\left(C^{(m|\dim \mathfrak{g}_1/2)}(\mathfrak{k})\right)^{\mathfrak{h}} \otimes H^{(n|0)}\left(\mathfrak{h}\right)\right] , \\
\mathcal{E}_2^{m,n} &= \left[H^{(m|0)}\left(\mathfrak{g}, \mathfrak{h}\right) \otimes H^{(n|\dim \mathfrak{g}_1/2)}\left(\mathfrak{h}\right)\right] \oplus \left[H^{(m|\dim \mathfrak{g}_1/2)}\left(\mathfrak{g}, \mathfrak{h}\right) \otimes H^{(n|0)}\left(\mathfrak{h}\right)\right] .
\end{aligned}
$$

The proof of Prop. 3 can be found in [10].

2. An example: $\mathfrak{osp}(1 \mid 4)/\mathfrak{osp}(1 \mid 2) \times \mathfrak{sp}(2)$

In this section we consider the example of $\mathfrak{g} = \mathfrak{osp}\left(1 \mid 4\right)$, with the sub-superalgebra $\mathfrak{h} = \mathfrak{osp}(1 \mid 2) \times \mathfrak{sp}(2)$. We will show that the superform and integral form cohomologies coincide with the usual one obtained by considering the sub-algebra $\mathfrak{sp}(4)$ and that the sub-(super)algebra induces cohomology groups at picture number 2. In the end, we will discuss a curious remark regarding the invariance properties of these cohomology groups; this will be discussed more in detail in the forthcoming [14].

The (super)commutation relations are taken from [18]. It is not difficult to see that one can identify the sub-superalgebra $\mathfrak{h}$, for example, by splitting the index i' as $i' = \{a, \hat{a}\}\, , a = 1, 3, \hat{a} = 2, 4$, so that the generators $E_{(ab)}, \tilde{E}_a$ form the $\mathfrak{osp}(1 \mid 2)$ sub-superalgebra and the generators $E_{(\hat{a}\hat{b})}$ form the $\mathfrak{sp}(2)$ sub-algebra. The (super)commutation relations then read

$$
\begin{aligned}
(50) \qquad & \left[E_{(ab)}, E_{(cd)}\right] = -G_{ad}E_{(bc)} - G_{ac}E_{(bd)} - G_{bd}E_{(ac)} - G_{bc}E_{(ad)} \, , \\
& \left[E_{(ab)}, \tilde{E}_c\right] = -G_{ac}\tilde{E}_b - G_{bc}\tilde{E}_a \, ,
\end{aligned}
$$

(51)
$$\left\{\tilde{E}_a,\tilde{E}_b\right\}=E_{ab}\ ,\ \left[E_{(\hat{a}\hat{b})},E_{(\hat{c}\hat{d})}\right]=-G_{\hat{a}\hat{d}}E_{(\hat{b}\hat{c})}-G_{\hat{a}\hat{c}}E_{(\hat{b}\hat{d})}-G_{\hat{b}\hat{d}}E_{(\hat{a}\hat{c})}-G_{\hat{b}\hat{c}}E_{(\hat{a}\hat{d})}\ ,$$

(52)
$$\left[E_{ab},E_{\hat{a}\hat{b}}\right]=0\,,\ \left[\tilde{E}_a,E_{\hat{a}\hat{b}}\right]=0\,,$$

(53)
$$\left[E_{a\hat{a}},E_{b\hat{b}}\right]=-G_{\hat{a}\hat{b}}E_{ab}-G_{ab}E_{\hat{a}\hat{b}}\ ,\ \left[E_{a\hat{a}},\tilde{E}_{\hat{b}}\right]=-G_{\hat{a}\hat{b}}\tilde{E}_a\ ,\ \left[\tilde{E}_{\hat{a}},\tilde{E}_{\hat{b}}\right]=E_{\hat{a}\hat{b}}\ ,$$

(54)
$$\left[E_{ab},E_{c\hat{a}}\right]=-G_{bc}E_{a\hat{a}}-G_{ac}E_{b\hat{a}}\ ,\ \left[E_{\hat{a}\hat{b}},\tilde{E}_{a\hat{c}}\right]=-G_{\hat{b}\hat{c}}E_{a\hat{a}}-G_{\hat{a}\hat{c}}E_{a\hat{b}}\ ,$$
$$\left[E_{a\hat{a}},\tilde{E}_b\right]=-G_{ab}\tilde{E}_{\hat{a}}\ ,$$

(55)
$$\left[E_{ab},\tilde{E}_{\hat{a}}\right]=0\,,\ \left[E_{\hat{a}\hat{b}},\tilde{E}_{\hat{c}}\right]=-G_{\hat{a}\hat{c}}\tilde{E}_{\hat{b}}-G_{\hat{b}\hat{c}}\tilde{E}_{\hat{a}}\ ,\ \left\{\tilde{E}_a,\tilde{E}_{\hat{a}}\right\}=E_{a\hat{a}}\ ,$$

where $G_{ab}=\epsilon_{ab}$ and $G_{\hat{a}\hat{b}}=\epsilon_{\hat{a}\hat{b}}$. We immediately see that the sub-superalgebra $\mathfrak{h}=\mathfrak{osp}(1\mid 2)\times\mathfrak{sp}(2)$ is reductive in $\mathfrak{g}$ and that the coset $\mathfrak{k}=\mathfrak{g}/\mathfrak{h}$ is homogeneous. We denote the forms in $\Pi\mathfrak{g}^*$ as

$$E_{(ab)}\leftrightsquigarrow V_{ab}\ ,\ \tilde{E}_a\leftrightsquigarrow\psi_a\ ,\ E_{(\hat{a}\hat{b})}\leftrightsquigarrow V_{\hat{a}\hat{b}}\ ,\ E_{a\hat{a}}\leftrightsquigarrow\bar{V}_{a\hat{a}}\ ,\ \tilde{E}_{\hat{a}}\leftrightsquigarrow\bar{\psi}_{\hat{a}}\ . \tag{56}$$

From (54) and (55) we can read the action of the algebraic differential on the forms of $\Pi\mathfrak{k}^*$:

$$d\bar{V}_{a\hat{a}}=-G_{bc}V_{ab}\bar{V}_{c\hat{a}}-G_{\hat{b}\hat{c}}V_{\hat{a}\hat{b}}\bar{V}_{a\hat{c}}+\psi_a\bar{\psi}_{\hat{a}}\ , \tag{57}$$

$$d\bar{\psi}_{\hat{a}}=-G_{ab}\bar{V}_{a\hat{a}}\psi_b-G_{\hat{b}\hat{c}}V_{\hat{a}\hat{b}}\bar{\psi}_{\hat{c}}\ . \tag{58}$$

We can now write page 2 of the spectral sequence as described in Prop.3. Since $\dim\mathfrak{h}_1=2=\dim\mathfrak{k}_1$, we can define a non-trivial page 2 for pseudoforms at picture number 2. First of all we have to compute the cohomology of $\mathfrak{h}$ and $\mathfrak{k}$ at picture numbers $0,2$. The former is given by

$$H^{(p|q)}\left(\mathfrak{osp}(1\mid 2)\times\mathfrak{sp}(2),\mathbb{K}\right)=\begin{cases}\mathbb{K}\,, & \text{if }\ p=0,6\,,\\ (\Pi\mathbb{K})^{\otimes 2}\ , & \text{if }\ p=3\,,\\ \{0\}\ ,\ \text{else}\,, & \end{cases}\qquad q=0,2\,, \tag{59}$$

as obtained from $H^{(\bullet|\bullet)}\left(\mathfrak{osp}(1\mid 2),\mathbb{K}\right)\times H^{\bullet}\left(\mathfrak{sp}(2),\mathbb{K}\right)$. On the other hand, the relative cohomology of $\mathfrak{k}$ reads

$$H^{(p|q)}\left(\mathfrak{osp}(1\mid 4),\mathfrak{osp}(1\mid 2)\times\mathfrak{sp}(2),\mathbb{K}\right)=\begin{cases}\mathbb{K}\,, & \text{if }\ p=0,4\,,\\ \{0\}\ , & \text{else}\,,\end{cases}\qquad q=0,2\,. \tag{60}$$

This means that among superforms, in addition to constants, there is a $(4\mid 0)$-superform $\omega^{(4|0)}$, representative of the cohomology group $H^{(4|0)}$ $\left(\mathfrak{osp}(1\mid 4),\mathfrak{osp}(1\mid 2)\times\mathfrak{sp}(2),\mathbb{K}\right)$; the integral form cohomology is simply evaluated via Berezinian complement duality. Here we show how to construct this $(4\mid 0)$-form. Since the coset is homogeneous, it follows that $H^{(p|q)}\left(\mathfrak{osp}(1\mid 4),\mathfrak{osp}(1\mid 2)\times\mathfrak{sp}(2),\mathbb{K}\right)$

is given by $\mathfrak{h}$-invariant objects. By restricting to forms $\omega^{(\bullet|\bullet)}$ constructed with $\bar{V}_{a\hat{a}}$ and $\bar{\psi}_{\hat{a}}$ only (i.e., *horizontal* forms), the requirement of being basic simplifies as

$$\mathcal{L}_\xi \omega^{(\bullet|\bullet)} = 0 \iff \iota_\xi d\omega^{(\bullet|\bullet)} = 0\,,\ \forall \xi \in \mathfrak{h}\,, \tag{61}$$

i.e., $d\omega^{(\bullet|\bullet)}$ must be horizontal. Actually, since $\mathfrak{k}$ is homogeneous (i.e., the differential $\mathcal{Y}^*_{\mathfrak{k}} \wedge \mathcal{Y}^*_{\mathfrak{k}} \iota_{\mathcal{Y}_{\mathfrak{k}}} = 0$), the CE differential acting on horizontal forms of $\Pi\mathfrak{k}^*$ formally reads $\mathcal{Y}^*_{\mathfrak{h}} \mathcal{Y}^*_{\mathfrak{k}} \iota_{\mathcal{Y}_{\mathfrak{k}}}$, hence it always includes forms in $\Pi\mathfrak{h}^*$; thus, a form $\omega^{(\bullet|\bullet)}$ satisfying (61) must satisfy $d\omega^{(\bullet|\bullet)} = 0$. The $(4 \mid 0)$-superform $\omega^{(4|0)} \in H^{(4|0)}\left(\mathfrak{osp}(1 \mid 4), \mathfrak{osp}(1 \mid 2) \times \mathfrak{sp}(2), \mathbb{K}\right)$ must then satisfy $\iota_\xi \omega^{(4|0)} = 0, \forall \xi \in \mathfrak{h}$ and $d\omega^{(4|0)} = 0$. Such superform is given by (we omit the wedge symbols)

$$\omega^{(4|0)} = \bar{V}_{a\hat{a}} G_{ab} \bar{V}_{b\hat{b}} G_{\hat{b}\hat{d}} \bar{V}_{c\hat{c}} G_{cd} \bar{V}_{d\hat{d}} G_{\hat{a}\hat{c}} - 2 G_{ab} \bar{V}_{a\hat{a}} \bar{V}_{b\hat{b}} G_{\hat{a}\hat{c}} G_{\hat{b}\hat{d}} \bar{\psi}_{\hat{c}} \bar{\psi}_{\hat{d}}\,. \tag{62}$$

The horizontality is directly satisfied, and the d-closure can be verified explicitly by using (57) and (58):

$$\begin{aligned} d\omega^{(4||0)} = &-4 G_{mn} V_{am} \bar{V}_{n\hat{a}} G_{ab} \bar{V}_{b\hat{b}} G_{\hat{b}\hat{d}} \bar{V}_{c\hat{c}} G_{cd} \bar{V}_{d\hat{d}} G_{\hat{a}\hat{c}} \\ &- 4 G_{\hat{m}\hat{n}} V_{\hat{a}\hat{m}} \bar{V}_{a\hat{n}} G_{ab} \bar{V}_{b\hat{b}} G_{\hat{b}\hat{d}} \bar{V}_{c\hat{c}} G_{cd} \bar{V}_{d\hat{d}} G_{\hat{a}\hat{c}} \\ &+ 4 G_{ab} \bar{V}_{b\hat{b}} G_{\hat{b}\hat{d}} \bar{V}_{c\hat{c}} G_{cd} \bar{V}_{d\hat{d}} G_{\hat{a}\hat{c}} \psi_a \bar{\psi}_{\hat{a}} + 4 G_{ab} G_{cd} V_{ac} \bar{V}_{d\hat{a}} \bar{V}_{b\hat{b}} G_{\hat{a}\hat{c}} G_{\hat{b}\hat{d}} \bar{\psi}_{\hat{c}} \bar{\psi}_{\hat{d}} \\ &+ 4 G_{ab} G_{\hat{m}\hat{n}} V_{\hat{a}\hat{m}} \bar{V}_{a\hat{n}} \bar{V}_{b\hat{b}} G_{\hat{a}\hat{c}} G_{\hat{b}\hat{d}} \bar{\psi}_{\hat{c}} \bar{\psi}_{\hat{d}} - 4 G_{ab} \bar{V}_{b\hat{b}} G_{\hat{a}\hat{c}} G_{\hat{b}\hat{d}} \bar{\psi}_{\hat{c}} \bar{\psi}_{\hat{d}} \psi_a \bar{\psi}_{\hat{a}} \\ &+ 4 G_{ab} \bar{V}_{a\hat{a}} \bar{V}_{b\hat{b}} G_{\hat{a}\hat{c}} G_{\hat{b}\hat{d}} G_{mn} \bar{V}_{m\hat{c}} \psi_n \bar{\psi}_{\hat{d}} + 4 G_{ab} \bar{V}_{a\hat{a}} \bar{V}_{b\hat{b}} G_{\hat{a}\hat{c}} G_{\hat{b}\hat{d}} G_{\hat{m}\hat{n}} V_{\hat{c}\hat{m}} \bar{\psi}_{\hat{n}} \bar{\psi}_{\hat{d}}\,. \end{aligned} \tag{63}$$

One can directly verify that, due to their symmetries, only the third and seventh terms are non-zero, but they sum to zero. One can wonder whether it is possible to construct higher forms out of $\omega^{(4|0)}$ by means of the ring structure of superforms; for example, one could define $\omega^{(8|0)} = \omega^{(4|0)} \wedge \omega^{(4|0)}$, but actually it is zero:

$$\begin{aligned} \omega^{(8|0)} &= \omega^{(4|0)} \wedge \omega^{(4|0)} = \left(G_{ab} \bar{V}_{a\hat{a}} \bar{V}_{b\hat{b}} G_{\hat{a}\hat{c}} G_{\hat{b}\hat{d}} \bar{\psi}_{\hat{c}} \bar{\psi}_{\hat{d}}\right) \left(G_{mn} \bar{V}_{m\hat{m}} \bar{V}_{n\hat{n}} G_{\hat{m}\hat{r}} G_{\hat{n}\hat{s}} \bar{\psi}_{\hat{r}} \bar{\psi}_{\hat{s}}\right) \\ &= 16 \left(\bar{V}_{12} \bar{V}_{32} \bar{\psi}_4^2 - \bar{V}_{12} \bar{V}_{34} \bar{\psi}_2 \bar{\psi}_4 - \bar{V}_{14} \bar{V}_{32} \bar{\psi}_2 \bar{\psi}_4 + \bar{V}_{14} \bar{V}_{34} \bar{\psi}_2^2\right)^2 \\ &= 32 \left(\bar{V}_{12} \bar{V}_{32} \bar{V}_{14} \bar{V}_{34} \bar{\psi}_2^2 \bar{\psi}_2^4 + \bar{V}_{12} \bar{V}_{34} \bar{V}_{14} \bar{V}_{32} \bar{\psi}_2^2 \bar{\psi}_4^2\right) = 0\,. \end{aligned} \tag{64}$$

Hence the cohomology of $\mathfrak{k}$ at picture numbers 0 and 2 reads as in (60). We can now write page 2 of the spectral sequence; we begin with superform, hence picture number zero, as displayed in Table (1). We immediately see that the differentials d_2 and d_3 are zero, as they move by two steps vertically, one left and three steps vertically, two left, hence we have $\mathcal{E}_4^{m,n} \equiv \mathcal{E}_3^{m,n} \equiv \mathcal{E}_2^{m,n}$. On the other hand, the differential d_4, moving three steps on the left and four to the top, is non-trivial. Actually, d_4 trades a $(3 \mid 0)$-form in $H^{(3|0)}(\mathfrak{h})$ for a $(4 \mid 0)$-form in $H^{(4|0)}(\mathfrak{k})$ and its net action is to lead to the convergence of the spectral sequence. For example, given that $H^{(3|0)}(\mathfrak{h})$ is generated by two forms $\omega_{\mathfrak{h}}^{(3|0)}$ and $\tilde{\omega}_{\mathfrak{h}}^{(3|0)}$ and that $H^{(4|0)}(\mathfrak{k})$ is generated by a form $\omega_{\mathfrak{k}}^{(4|0)}$, we could formally write d_4 as

$$d_4 = \omega_{\mathfrak{k}}^{(4|0)} \iota_{\omega_{\mathfrak{h}}^{(3|0)} + \tilde{\omega}_{\mathfrak{h}}^{(3|0)}}\,. \tag{65}$$

...	...	⋮	...	...	...	...	...	...	...	...	...
...	0	5	0	0	0	0	0	0	0	0	...
...	0	4	$H^{(0\|0)}(\mathfrak{h}) \otimes H^{(4\|0)}(\mathfrak{k})$	0	0	$H^{(3\|0)}(\mathfrak{h}) \otimes H^{(4\|0)}(\mathfrak{k})$	0	0	$H^{(6\|0)}(\mathfrak{h}) \otimes H^{(4\|0)}(\mathfrak{k})$	0	...
...	0	3	0	0	0	0	0	0	0	0	...
...	0	2	0	0	0	0	0	0	0	0	...
...	0	1	0	0	0	0	0	0	0	0	...
...	0	0	$H^{(0\|0)}(\mathfrak{h}) \otimes H^{(0\|0)}(\mathfrak{k})$	0	0	$H^{(3\|0)}(\mathfrak{h}) \otimes H^{(0\|0)}(\mathfrak{k})$	0	0	$H^{(6\|0)}(\mathfrak{h}) \otimes H^{(0\|0)}(\mathfrak{k})$	0	...
...	-1		0	1	2	3	4	5	6	7	...
...	0	-1	0	0	0	0	0	0	0	0	...
...	...	⋮	...	...	...	...	...	...	...	...	...

TAB. 1: $E_2^{m,n}$ as defined in Prop. 3. The integers n and m are spanned on the horizontal and vertical axes, respectively.

It follows that one of the combinations of $\omega_{\mathfrak{h}}^{(3|0)}$ and $\tilde{\omega}_{\mathfrak{h}}^{(3|0)}$ is mapped to $\omega_{\mathfrak{k}}^{(4|0)}$, thus annihilating in cohomology $H^{(4|0)}(\mathfrak{k})$ and decreasing by one the dimension of $H^{(3|0)}(\mathfrak{h})$. Another example is given by the space $H^{(6|0)}(\mathfrak{h})$: it is generated by $\omega_{\mathfrak{h}}^{(3|0)} \wedge \tilde{\omega}_{\mathfrak{h}}^{(3|0)}$ and (65) maps it into (the "Leibnitz rule" is trivially verified thanks to the factorisation of the form and the differential)

$$d_4 \omega_{\mathfrak{h}}^{(3|0)} \wedge \tilde{\omega}_{\mathfrak{h}}^{(3|0)} = \omega_{\mathfrak{k}}^{(4|0)} \wedge \left(\omega_{\mathfrak{h}}^{(3|0)} - \tilde{\omega}_{\mathfrak{h}}^{(3|0)} \right) \in H^{(3|0)}(\mathfrak{h}) \otimes H^{(4|0)}(\mathfrak{k}) \,, \tag{66}$$

where $H^{(3|0)}(\mathfrak{h}) \otimes H^{(4|0)}(\mathfrak{k})$ is generated by $\omega_{\mathfrak{h}}^{(3|0)} \wedge \omega_{\mathfrak{k}}^{(4|0)}$ and $\tilde{\omega}_{\mathfrak{h}}^{(3|0)} \wedge \omega_{\mathfrak{k}}^{(4|0)}$. Hence the combination of $H^{(3|0)}(\mathfrak{h}) \otimes H^{(4|0)}(\mathfrak{k})$ in (66) is d_4-exact. The other combination is d_4 closed, because of (64):

$$d_4 \omega_{\mathfrak{k}}^{(4|0)} \wedge \left(\omega_{\mathfrak{h}}^{(3|0)} + \tilde{\omega}_{\mathfrak{h}}^{(3|0)} \right) = 2 \omega_{\mathfrak{k}}^{(4|0)} \wedge \omega_{\mathfrak{k}}^{(4|0)} = 0 \,. \tag{67}$$

Hence, page five of the spectral sequence, where the spectral sequence converges as all the higher differentials are trivial, reads

$$H^{(p|0)}(\mathfrak{osp}(1 \mid 4), \mathbb{K}) = \begin{cases} \mathbb{K} \,, & \text{if } \ p = 0, 10 \,, \\ \Pi\mathbb{K} \,, & \text{if } \ p = 3, 7 \,, \\ \{0\} \,, & \text{else.} \end{cases} \tag{68}$$

An analogous computation can be done for the integral form cohomology, since as we can read from (59) and (60) the integral form cohomology of $\mathfrak{h}$ and $\mathfrak{k}$ are copies of the superform ones. The computation is different for pseudoforms, since page two of the spectral sequence is built out of two terms:

$$\mathcal{E}_2^{m,n} = \left[H^{(m|2)}(\mathfrak{g}, \mathfrak{h}, \mathbb{K}) \otimes H^{(n|0)}(\mathfrak{h}, \mathbb{K}) \right] \oplus \left[H^{(m|0)}(\mathfrak{g}, \mathfrak{h}, \mathbb{K}) \otimes H^{(n|2)}(\mathfrak{h}, \mathbb{K}) \right] . \tag{69}$$

In particular, we observe that there is a doubling of the cohomology groups since picture-2 forms can be obtained both from $\mathfrak{h}$ and from $\mathfrak{k}$. Now the convergence of the spectral sequence follows the same arguments described above: the differentials d_2 and d_3 are trivial, while the differential d_4 leads to page five which coincides with the picture-2 cohomology of $\mathfrak{osp}(1 \mid 4)$, relative to the sub-superalgebra

$\mathfrak{osp}(1 \mid 2) \times \mathfrak{sp}(2)$:

$$H^{(p|2)}\left(\mathfrak{osp}(1 \mid 4), \mathbb{K}\right) = \begin{cases} \mathbb{K}^{\otimes 2}\,, & \text{if } p = 0, 10\,, \\ (\Pi\mathbb{K})^{\otimes 2}\,, & \text{if } p = 3, 7\,, \\ \{0\}\,, & \text{else.} \end{cases} \tag{70}$$

Remark. We want to stress a property of the cohomology classes we found: in [9] (see also [20]), the authors proved that when $\mathfrak{g}$ is a semi-simple Lie algebra and we consider a finite-dimensional module V, then $H^p(\mathfrak{g}, V) = H^p(\mathfrak{g}, V^{\mathfrak{g}})$. Moreover, in [9] it is shown that given a semi-simple Lie algebra $\mathfrak{g}$ with values over a characteristic zero field $\mathbb{K}$, every cohomology class $H^q(\mathfrak{g}, \mathbb{K})$ contains a $\mathfrak{g}$-invariant cocycle. This extends to the superalgebraic setting in the sectors of superforms and integral forms. This however fails when dealing with pseudoforms, because of the infinite dimensionality of the module they are defined through. In particular, it is possible to show that the cohomology groups of (70) *do not* contain $\mathfrak{osp}(1 \mid 4)$-invariant cocycles, but only (by construction) $\mathfrak{osp}(1 \mid 2) \times \mathfrak{sp}(2)$-invariant ones. In [**?**] we will deeply comment on this by supporting the spectral sequence technique with another powerful computational algorithm: the Molien-Weyl integral formula. We will explicitly show that the requirement of some invariances, which correspond to the choice of the sub-(super)algebra, will switch on or off some pseudoform sectors.

3. A fugacious glance at super-branes

One of the driving reasons to study Lie superalgebra cohomology in pseudoform sectors comes from Physics. Inspired by the pioneering work of Sullivan [36], the authors of [6] (but not only them) used the notion of *free differential algebra* (FDA) to show how to introduce higher form fields in supergravity. The procedure consists in “trivialising" cohomology classes of the Lie (super)algebra under examination with new formal generators; these generators correspond to p-superforms of supergravity multiplets that can be used to construct form Lagrangians in the framework of “rheonomy" (see [6]). In those books and the following literature, it was shown how these forms correspond to *super-branes* that were classified in various dimensions e.g. in [1]. Recently, the interest in branes has been revived e.g. in [17] and other papers by the same authors. In [17] they “complete" the classification of branes (the *old brane scan*) in the context of *homotopy algebras*. In particular, they highlight the correspondence between the FDA approach long known in Physics with the homotopy-algebraic approach long known in Mathematics. They then use the extended Lie algebras to construct Wess-Zumino-Witten like models, but with higher forms. Another very recent discussion on the new brane scan in various relevant supergravity backgrounds can be found in [15].

After this brief historical contextualisation, we come to the motivation for studying cohomology in pseudoform complexes: we believe it could be interesting to investigate if these new cohomology classes can be related to different kinds of super-branes. In particular, given a (physically relevant, see [15]) Lie superalgebra $\mathfrak{g}$, we have shown how to introduce and compute pseudoform cohomology groups related to sub-superalgebras; given $\omega^{(p|q)} \in H^{(p|q)}(\mathfrak{g})$, we could follow Sullivan's procedure and introduce a new generator $\eta^{(p-1|q)}$ s.t.

$$d\eta^{(p-1|q)} = \omega^{(p|q)} . \tag{71}$$

One could use this new generator to construct WZW like models, but now, one will have to integrate on a surface $\Sigma^{(p-1|q)}$ with non-trivial odd dimension, i.e., a supermanifold itself. This would be the "generalised super-brane" we are looking for. Notice that, as we emphasised at the end of the description of the previous example, pseudoform cohomology classes may in general not contain $\mathfrak{g}$-invariant representatives, but are invariant w.r.t. the sub-superalgebra only; it could be interesting to investigate how this would relate to the symmetries, preserved or broken or maybe non-linearly realised, the super-branes deriving from such objects would have.

It is still not clear whether this project could work, but there are still some hints that pseudoforms can give rise to unexplored non-trivial objects: in [11], it was shown in a super Chern-Simons toy model, and in its BV extension in [12], how pseudoforms are naturally related to homotopy algebras, in analogy to what was shown in open superstring field theory (see [16]). In [13] it was also argued that pseudoforms may be related to chiral/self-dual fields on supermanifolds, as pseudoforms allow to define stable complex w.r.t. the Hodge operator on supermanifolds. In the future we will investigate this direction, to see if pseudoforms can lead to new, completely unexplored classes of branes that will enlarge the zoology of supergravity and superstring theory.

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