Dorota Bród; Anetta Szynal-Liana; Iwona Włoch

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ON SOME COMBINATORIAL PROPERTIES OF GENERALIZED COMMUTATIVE JACOBSTHAL QUATERNIONS AND GENERALIZED COMMUTATIVE JACOBSTHAL-LUCAS QUATERNIONS

Dorota Bród, Anetta Szynal-Liana, Iwona Włoch, Rzeszów

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Abstract. We study generalized commutative Jacobsthal quaternions and generalized commutative Jacobsthal-Lucas quaternions. We present some properties of these quaternions and the relations between the generalized commutative Jacobsthal quaternions and generalized commutative Jacobsthal-Lucas quaternions.

Keywords: Jacobsthal number; Jacobsthal-Lucas number; quaternion; generalized quaternion; Binet formula

MSC 2020: 11B37, 11B39

1. Introduction

The Jacobsthal sequence $\{J_n\}$ and Jacobsthal-Lucas sequence $\{j_n\}$ were introduced by Horadam, see [4] and [5]. These sequences are defined by the recurrence relations

$$J_0 = 0, \quad J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2} \quad \text{for } n \geqslant 2,$$
$$j_0 = 2, \quad j_1 = 1, \quad j_n = j_{n-1} + 2j_{n-2} \quad \text{for } n \geqslant 2,$$

respectively. The Binet formulas of these sequences have the form

$$J_n = \frac{2^n - (-1)^n}{3}, \quad j_n = 2^n + (-1)^n.$$

Many interesting properties of the Jacobsthal numbers and Jacobsthal-Lucas numbers are presented in [5]. We give some of them.

$$J_n^2 - J_{n+1}J_{n-1} - = (-2)^{n-1} \quad \text{(Cassini identity)},$$
$$j_n^2 - j_{n-1}j_{n+1} = 9(-1)^n 2^{n-1} \quad \text{(Cassini identity)},$$
$$j_{n+r} + j_{n-r} = 3(J_{n+r} + J_{n-r}) + 4(-1)^{n-r} = 2^{n-r}(2^{2r} + 1) + 2(-1)^{n-r},$$
$$j_{n+r} - j_{n-r} = 3(J_{n+r} - J_{n-r}) = 2^{n-r}(2^{2r} + 1),$$
$$J_n + j_n = 2J_{n+1}, \quad J_{n+1}^2 + 2J_n^2 = J_{2n+1}, \quad J_m J_{n-1} - J_{m-1}J_n = (-1)^n 2^{n-1} J_{m-n},$$
$$J_{n+m} = J_n J_{m+1} + 2J_{n-1}J_m, \quad J_m j_n + J_n j_m = 2J_{m+n},$$
$$J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n}, \quad \sum_{l=0}^{n} J_l = \frac{J_{n+2} - 1}{2}.$$

In this paper, we will use the identities

$$j_{n+1} + j_n = 3(J_{n+1} + J_n) = 3 \cdot 2^n, \tag{1.1}$$
$$j_{n+1} - j_n = 3(J_{n+1} - J_n) + 4(-1)^{n+1} = 2^n + 2(-1)^{n+1}, \tag{1.2}$$
$$J_n + j_n = 2J_{n+1}, \tag{1.3}$$
$$J_{n+2} + 2J_n = j_{n+1}, \tag{1.4}$$
$$3J_n + j_n = 2^{n+1}, \tag{1.5}$$
$$\sum_{l=0}^{n} j_l = \frac{j_{n+2} - 1}{2}. \tag{1.6}$$

The Jacobsthal numbers and Jacobsthal-Lucas numbers are special cases of Horadam numbers W_n defined by the recurrence

$$W_n = pW_{n-1} - qW_{n-2} \quad \text{for } n \geqslant 2 \tag{1.7}$$

with $p, q \in \mathbb{Z}$ and fixed real numbers W_0 and W_1. The sequence of Horadam numbers is a certain generalization of other famous sequences such as the Fibonacci sequence $F_n = W_n(0,1;1,-1)$, Lucas sequence $L_n = W_n(2,1;1,-1)$, Pell sequence $P_n = W_n(0,1;2,-1)$, Pell-Lucas sequence $Q_n = W_n(2,2;2,-1)$. In the literature, the sequences defined by (1.7) are called *sequences of the Fibonacci type*. These numbers have many applications in distinct areas of mathematics, also in quaternion theory. In 1963 Horadam in [3] introduced the nth Fibonacci and Lucas quaternions. Many interesting properties of Fibonacci and Lucas quaternions can be found in [2], [6]. Interesting results for Pell quaternions and Pell-Lucas quaternions obtained recently can be found in [1], [8]. In [7], the authors investigated Jacobsthal quaternions. In [10], some generalizations of Jacobsthal and Jacobsthal-Lucas quaternions were studied.

In [9], the generalized commutative quaternions were introduced. Let $\mathbb{H}^c_{\gamma\beta}$ be the set of generalized commutative quaternions $\mathbf{x}$ of the form

$$\mathbf{x} = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3,$$

where the quaternionic units e_1, e_2, e_3 satisfy the equalities

$$e_1^2 = \alpha, \quad e_2^2 = \beta, \quad e_3^2 = \alpha\beta, \tag{1.8}$$

$$e_1 e_2 = e_2 e_1 = e_3, \quad e_2 e_3 = e_3 e_2 = \beta e_1, \quad e_3 e_1 = e_1 e_3 = \alpha e_2 \tag{1.9}$$

and $x_0, x_1, x_2, x_3, \alpha, \beta \in \mathbb{R}$.

The generalized commutative quaternions generalize elliptic quaternions ($\alpha < 0$, $\beta = 1$), parabolic quaternions ($\alpha = 0$, $\beta = 1$), hyperbolic quaternions ($\alpha > 0$, $\beta = 1$), bicomplex numbers ($\alpha = -1$, $\beta = -1$), complex hyperbolic numbers ($\alpha = -1$, $\beta = 1$) and hyperbolic complex numbers ($\alpha = 1$, $\beta = -1$).

In [9], the authors introduced the generalized commutative Horadam quaternions

$$\mathrm{gcH}_n = W_n + W_{n+1} e_1 + W_{n+2} e_2 + W_{n+3} e_3,$$

where W_n is the nth Horadam number and e_1, e_2, e_3 are quaternionic units which satisfy the rules (1.8) and (1.9). The following result has been proved.

Theorem 1.1 ([9], Binet formula for generalized commutative Horadam quaternions)**.** *Let $n \geqslant 0$ be an integer. Then*

$$\mathrm{gcH}_n = A t_1^n \widehat{t_1} + B t_2^n \widehat{t_2},$$

where

$$t_1 = \frac{1}{2}(p - \sqrt{p^2 - 4q}), \;\; t_2 = \frac{1}{2}(p + \sqrt{p^2 - 4q}), \;\; A = \frac{W_1 - W_0 t_2}{t_1 - t_2}, \;\; B = \frac{W_0 t_1 - W_1}{t_1 - t_2},$$
$$\widehat{t_1} = 1 + t_1 e_1 + t_1^2 e_2 + t_1^3 e_3, \;\; \widehat{t_2} = 1 + t_2 e_1 + t_2^2 e_2 + t_2^3 e_3.$$

In this paper, we study generalized commutative Jacobsthal quaternions and generalized commutative Jacobsthal-Lucas quaternions.

2. Generalized commutative Jacobsthal quaternions and generalized commutative Jacobsthal-Lucas quaternions

For $n \geqslant 0$, the nth generalized commutative Jacobsthal quaternion is defined by the relation

$$\mathrm{gcJ}_n = J_n + J_{n+1} e_1 + J_{n+2} e_2 + J_{n+3} e_3, \tag{2.1}$$

where J_n is the nth Jacobsthal number and e_1, e_2, e_3 are quaternionic units which satisfy the rules (1.8) and (1.9).

Analogously, for $n \geqslant 0$ we introduce the nth generalized commutative Jacobsthal-Lucas quaternion

$$\mathrm{gcJL}_n = j_n + j_{n+1}e_1 + j_{n+2}e_2 + j_{n+3}e_3, \tag{2.2}$$

where j_n is the nth Jacobsthal-Lucas number.

By (2.1) and (2.2), we obtain

$$\begin{aligned} &(2.3) \quad && \mathrm{gcJ}_0 = e_1 + e_2 + 3e_3, && \mathrm{gcJ}_1 = 1 + e_1 + 3e_2 + 5e_3,\\ & && \mathrm{gcJ}_2 = 1 + 3e_1 + 5e_2 + 11e_3, && \mathrm{gcJ}_3 = 3 + 5e_1 + 11e_2 + 21e_3;\\ &(2.4) && \mathrm{gcJL}_0 = 2 + e_1 + 5e_2 + 7e_3, && \mathrm{gcJL}_1 = 1 + 5e_1 + 7e_2 + 17e_3,\\ & && \mathrm{gcJL}_2 = 5 + 7e_1 + 17e_2 + 31e_3, && \mathrm{gcJL}_3 = 7 + 17e_1 + 31e_2 + 65e_3. \end{aligned}$$

By definition of generalized commutative Jacobsthal quaternion, we obtain the following recurrence relations.

Proposition 2.1. *Let $n \geqslant 2$ be an integer. Then*

$$\mathrm{gcJ}_n = \mathrm{gcJ}_{n-1} + 2\mathrm{gcJ}_{n-2},$$

where gcJ_0, gcJ_1 *are given by* (2.3).

Proposition 2.2. *Let $n \geqslant 2$ be an integer. Then*

$$\mathrm{gcJL}_n = \mathrm{gcJL}_{n-1} + 2\mathrm{gcJL}_{n-2},$$

where gcJL_0, gcJL_1 *are given by* (2.4).

Proposition 2.3. *Let $n \geqslant 1$ be an integer. Then*

$$\mathrm{gcJ}_{n+1} + 2\mathrm{gcJ}_{n-1} = \mathrm{gcJL}_n.$$

Proof. By the formulas (2.1) and (1.4) we get

$$\begin{aligned} \mathrm{gcJ}_{n+1} + 2\mathrm{gcJ}_{n-1} &= J_{n+1} + J_{n+2}e_1 + J_{n+3}e_2 + J_{n+4}e_3\\ &\quad + 2J_{n-1} + 2J_ne_1 + 2J_{n+1}e_2 + 2J_{n+2}e_3\\ &= j_n + j_{n+1}e_1 + j_{n+2}e_2 + j_{n+3}e_3 = \mathrm{gcJL}_n, \end{aligned}$$

which ends the proof. □

By the formulas (1.3) and (1.5) we get the next result.

Proposition 2.4. *Let $n \geqslant 0$ be an integer. Then*

(i) $\mathrm{gcJ}_n + \mathrm{gcJL}_n = 2\mathrm{gcJ}_{n+1}$,

(ii) $3\mathrm{gcJ}_n + \mathrm{gcJL}_n = 2^{n+1}(1 + 2e_1 + 4e_2 + 8e_3)$.

By Theorem 1.1, we get the Binet formula for the generalized commutative Jacobsthal quaternions and for the generalized commutative Jacobsthal-Lucas quaternions.

Corollary 2.1. *Let $n \geqslant 0$ be in anteger. Then*

$$\mathrm{gcJ}_n = \frac{1}{3}(2^n(1 + 2e_1 + 4e_2 + 8e_3) - (-1)^n(1 - e_1 + e_2 - e_3)),$$
$$\mathrm{gcJL}_n = 2^n(1 + 2e_1 + 4e_2 + 8e_3) + (-1)^n(1 - e_1 + e_2 - e_3).$$

For simplicity of notation let

$$C = 1 + 2e_1 + 4e_2 + 8e_3, \quad D = 1 - e_1 + e_2 - e_3.$$

Then we get

$$\mathrm{gcJ}_n = \frac{1}{3}(2^n C - (-1)^n D), \tag{2.5}$$
$$\mathrm{gcJL}_n = 2^n C + (-1)^n D. \tag{2.6}$$

3. Properties of the generalized commutative Jacobsthal quaternions and generalized commutative Jacobsthal-Lucas quaternions

In this section, we give some properties of the generalized commutative Jacobsthal quaternions and generalized commutative Jacobsthal-Lucas quaternions.

Theorem 3.1. *Let $n \geqslant 0$ be an integer. Then*

(i) $\mathrm{gcJ}_{n+1} + \mathrm{gcJ}_n = 2^n C$,

(ii) $\mathrm{gcJ}_{n+1} - \mathrm{gcJ}_n = \frac{1}{3}[2^n C + 2(-1)^n D]$,

(iii) $\mathrm{gcJ}_{n+1}^2 + \mathrm{gcJ}_n^2 = \frac{1}{9}(5 \cdot 4^n C^2 + 2(-2)^n CD + 2D^2)$,

where $C = 1 + 2e_1 + 4e_2 + 8e_3$, $D = 1 - e_1 + e_2 - e_3$.

Proof. (i) By the formulas (2.1) and (1.1), we get

$$\begin{aligned}\mathrm{gcJ}_{n+1} + \mathrm{gcJ}_n &= J_{n+1} + J_{n+2}e_1 + J_{n+3}e_2 + J_{n+4}e_3 \\ &\quad + J_n + J_{n+1}e_1 + J_{n+2}e_2 + J_{n+3}e_3 \\ &= 2^n + 2^{n+1}e_1 + 2^{n+2}e_2 + 2^{n+3}e_3 = 2^n(1 + 2e_1 + 4e_2 + 8e_3),\end{aligned}$$

which ends the proof of (i).

(ii) By the formulas (2.1) and (1.2), we have

$$\begin{aligned}\mathrm{gcJ}_{n+1} - \mathrm{gcJ}_n &= J_{n+1} + J_{n+2}e_1 + J_{n+3}e_2 + J_{n+4}e_3 \\ &\quad - J_n - J_{n+1}e_1 - J_{n+2}e_2 - J_{n+3}e_3 \\ &= \frac{1}{3}[2^n + 2(-1)^n + (2^{n+1} + 2(-1)^{n+1})e_1 \\ &\quad + (2^{n+2} + 2(-1)^{n+2})e_2 + (2^{n+3} + 2(-1)^{n+3})e_3] \\ &= \frac{1}{3}[2^n(1 + 2e_1 + 4e_2 + 8e_3) + 2(-1)^n(1 - e_1 + e_2 - e_3)].\end{aligned}$$

(iii) We use the Binet formula (2.5) to get

$$\begin{aligned}\mathrm{gcJ}_{n+1}^2 + \mathrm{gcJ}_n^2 &= \frac{1}{9}(2^{2n+2}C^2 - 2\cdot 2^{n+1}(-1)^{n+1}CD + (-1)^{2n+2}D^2 \\ &\quad + 2^{2n}C^2 - 2\cdot 2^n(-1)^nCD + (-1)^{2n}D^2) \\ &= \frac{1}{9}(5\cdot 4^nC^2 + 2(-2)^nCD + 2D^2).\end{aligned}$$

□

Theorem 3.2. *Let $n \geqslant 1$, $r \geqslant 1$ be integers such that $n \geqslant r$. Then*
(i) $\mathrm{gcJ}_{n+r} + \mathrm{gcJ}_{n-r} = \frac{1}{3}[2^{n-r}(2^{2r} + 1)C - 2(-1)^{n-r}D]$,
(ii) $\mathrm{gcJ}_{n+r} - \mathrm{gcJ}_{n-r} = \frac{1}{3}\cdot 2^{n-r}(2^{2r} - 1)C$,
where $C = 1 + 2e_1 + 4e_2 + 8e_3$, $D = 1 - e_1 + e_2 - e_3$.

P r o o f. (i) We use the Binet formula (2.5) to get

$$\begin{aligned}3(\mathrm{gcJ}_{n+r} + \mathrm{gcJ}_{n-r}) &= 2^{n+r}C - (-1)^{n+r}D + 2^{n-r}C - (-1)^{n-r}D \\ &= 2^{n-r}(2^{2r} + 1)C - (-1)^{n-r}(1 + (-1)^{2r})D \\ &= 2^{n-r}(2^{2r} + 1)C - 2(-1)^{n-r}D,\end{aligned}$$

which completes the proof of (i).

(ii) In the same way we prove the formula (ii),

$$\begin{aligned}3(\mathrm{gcJ}_{n+r} - \mathrm{gcJ}_{n-r}) &= 2^{n+r}C - (-1)^{n+r}D - 2^{n-r}C + (-1)^{n-r}D \\ &= 2^{n-r}(2^{2r} - 1)C + (-1)^{n-r}(1 - (-1)^{2r})D = 2^{n-r}(2^{2r} - 1)C,\end{aligned}$$

which ends the proof. □

In the same manner we can prove next results.

Theorem 3.3. *Let $n \geqslant 0$ be an integer. Then*

$$\mathrm{gcJL}_{n+1} + \mathrm{gcJL}_n = 3\cdot 2^nC, \quad \mathrm{gcJL}_{n+1} - \mathrm{gcJL}_n = 2^nC - 2(-1)^nD,$$

where $C = 1 + 2e_1 + 4e_2 + 8e_3$, $D = 1 - e_1 + e_2 - e_3$.

Theorem 3.4. *Let $n \geqslant 1$, $r \geqslant 1$ be integers such that $n \geqslant r$. Then*

$$\mathrm{gcJL}_{n+r} + \mathrm{gcJL}_{n-r} = 2^{n-r}(2^{2r}+1)C + 2(-1)^{n-r}D,$$
$$\mathrm{gcJL}_{n+r} - \mathrm{gcJL}_{n-r} = 2^{n-r}(2^{2r}-1)C,$$

where $C = 1 + 2e_1 + 4e_2 + 8e_3$, $D = 1 - e_1 + e_2 - e_3$.

Theorem 3.5. *Let $n \geqslant 1$, $m \geqslant 1$ be integers. Then*

$$\mathrm{gcJ}_m \cdot \mathrm{gcJ}_{n-1} - \mathrm{gcJ}_{m-1} \cdot \mathrm{gcJ}_n = \frac{1}{6}(-2)^n CD(2^{m-n} - (-1)^{m-n}),$$

where $C = 1 + 2e_1 + 4e_2 + 8e_3$, $D = 1 - e_1 + e_2 - e_3$.

P r o o f. By simple calculations we get

$$\begin{aligned}\mathrm{gcJ}_m \cdot \mathrm{gcJ}_{n-1} - \mathrm{gcJ}_{m-1} \cdot \mathrm{gcJ}_n &= \frac{1}{9}CD\Big(\frac{3}{2}\cdot 2^m(-1)^n - \frac{3}{2}\cdot 2^n(-1)^m\Big)\\ &= \frac{1}{6}CD(2^{m-n}(-1)^n 2^n - (-1)^{m-n}2^n(-1)^n)\\ &= \frac{1}{6}(-2)^n CD(2^{m-n} - (-1)^{m-n}).\end{aligned}$$

□

Theorem 3.6. *Let $n \geqslant 1$, $m \geqslant 1$ be integers. Then*

(i) $\mathrm{gcJ}_m \cdot \mathrm{gcJL}_n - \mathrm{gcJ}_n \cdot \mathrm{gcJL}_m = \frac{2}{3}(-2)^n CD(2^{m-n} - (-1)^{m-n})$,
(ii) $\mathrm{gcJ}_m \cdot \mathrm{gcJL}_n + \mathrm{gcJ}_n \cdot \mathrm{gcJL}_m = \frac{2}{3}(2^{m+n}C^2 - (-1)^{m+n}D^2)$,
(iii) $\mathrm{gcJ}_n \cdot \mathrm{gcJL}_n = \frac{1}{3}(4^n C^2 - D^2)$,
(iv) $\mathrm{gcJL}_n \cdot \mathrm{gcJ}_{m+1} + 2\mathrm{gcJL}_{n-1} \cdot \mathrm{gcJ}_m = 2^{n+m}C^2 + (-1)^{n+m}D^2$,

where $C = 1 + 2e_1 + 4e_2 + 8e_3$, $D = 1 - e_1 + e_2 - e_3$.

P r o o f. By the formulas (2.5) and (2.6) we get

(i) $\mathrm{gcJ}_m \cdot \mathrm{gcJL}_n - \mathrm{gcJ}_n \cdot \mathrm{gcJL}_m$

$$\begin{aligned}&= \frac{1}{3}(2^m C - (-1)^m D)(2^n C + (-1)^n D) - \frac{1}{3}(2^n C - (-1)^n D)(2^m C + (-1)^m D)\\ &= \frac{2}{3}CD(2^m(-1)^n - 2^n(-1)^m) = \frac{2}{3}CD(-2)^n(2^{m-n} - (-1)^{m-n}),\end{aligned}$$

(ii) $\mathrm{gcJ}_m \cdot \mathrm{gcJL}_n + \mathrm{gcJ}_n \cdot \mathrm{gcJL}_m$

$$\begin{aligned}&= \frac{1}{3}(2^m C - (-1)^m D)(2^n C + (-1)^n D) + \frac{1}{3}(2^n C - (-1)^n D)(2^m C + (-1)^m D\\ &= \frac{2}{3}(2^{m+n}C^2 - (-1)^{m+n}D^2),\end{aligned}$$

(iii)
$$\begin{aligned}\mathrm{gcJ}_n \cdot \mathrm{gcJL}_n &= \frac{1}{3}(2^n C - (-1)^n D)(2^n C + (-1)^n D)\\ &= \frac{1}{3}(2^{2n}C^2 - (-1)^{2n}D^2) = \frac{1}{3}(4^n C^2 - D^2),\end{aligned}$$

(iv) $\mathrm{gcJL}_n \cdot \mathrm{gcJ}_{m+1} + 2\mathrm{gcJL}_{n-1} \cdot \mathrm{gcJ}_m$

$$= \frac{1}{3}(2^{n+m+1}C^2 - 2^n(-1)^{m+1}CD + 2^{m+1}(-1)^nCD - (-1)^{n+m+1}D^2)$$
$$+ \frac{2}{3}(2^{n+m-1}C^2 - 2^{n-1}(-1)^mCD + 2^m(-1)^{n-1}CD - (-1)^{n+m-1}D^2))$$
$$= \frac{1}{3}(2^{n+m}3C^2 - 2^n[(-1)^{m+1} + (-1)^m]CD$$
$$+ 2^{m+1}[(-1)^n + (-1)^{n-1}]CD - (-1)^{n+m-1}3D^2)$$
$$= 2^{n+m}C^2 + (-1)^{n+m}D^2,$$

which completes the proof. □

4. Some identities for the generalized commutative Jacobsthal-Lucas quaternions

Now, we give some identities for the generalized commutative Jacobsthal-Lucas quaternions.

Theorem 4.1 (General bilinear index-reduction formula for the generalized commutative Jacobsthal-Lucas quaternions). *Let $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$, $d \geqslant 0$ be integers such that $a+b=c+d$. Then*

$$\mathrm{gcJL}_a \cdot \mathrm{gcJL}_b - \mathrm{gcJL}_c \cdot \mathrm{gcJL}_d = CD(2^a(-1)^b + 2^b(-1)^a - 2^c(-1)^d - 2^d(-1)^c),$$

where $C = 1 + 2e_1 + 4e_2 + 8e_3$, $D = 1 - e_1 + e_2 - e_3$.

Proof. By the formula (2.6), we get

$$\mathrm{gcJL}_a \cdot \mathrm{gcJL}_b - \mathrm{gcJL}_c \cdot \mathrm{gcJL}_d$$
$$= (2^aC + (-1)^aD)(2^bC + (-1)^bD) - (2^cC + (-1)^cD)(2^dC + (-1)^dD)$$
$$= C^2(2^{a+b} - 2^{c+d}) + D^2((-1)^{a+b} - (-1)^{c+d})$$
$$+ CD(2^a(-1)^b - 2^c(-1)^d + 2^b(-1)^a - 2^d(-1)^c).$$

Using the fact that $a+b=c+d$, we get the result. □

It is easily seen that for special values of a, b, c, d, by Theorem 4.1, we get new identities for generalized commutative Jacobsthal-Lucas quaternions:

- ▷ Catalan identity (for $a=b=n$, $c=n-m$ and $d=n+m$),
- ▷ Cassini identity (for $a=b=n$, $c=n-1$ and $d=n+1$),
- ▷ d'Ocagne identity (for $a=n$, $b=m+1$, $c=n+1$ and $d=m$),
- ▷ Vajda identity (for $a=m+k$, $b=n-k$, $c=m$ and $d=n$).

Corollary 4.1 (Catalan identity for generalized commutative Jacobsthal-Lucas quaternions)**.** *Let $n \geqslant 0$, $m \geqslant 0$ be integers such that $n \geqslant m$. Then*

$$\mathrm{gcJL}_n^2 - \mathrm{gcJL}_{n-m} \cdot \mathrm{gcJL}_{n+m} = (-2)^n CD\Big(2 - \Big(-\frac{1}{2}\Big)^m - (-2)^m\Big).$$

Corollary 4.2 (Cassini identity for generalized commutative Jacobsthal-Lucas quaternions)**.** *Let $n \geqslant 1$ be an integer. Then*

$$\mathrm{gcJL}_n^2 - \mathrm{gcJL}_{n-1} \cdot \mathrm{gcJL}_{n+1} = \frac{9}{2}(-2)^n CD.$$

Corollary 4.3 (d'Ocagne identity for the generalized commutative Jacobsthal-Lucas quaternions)**.** *Let $n \geqslant 0$, $m \geqslant 0$ be integers. Then*

$$\mathrm{gcJL}_n \cdot \mathrm{gcJL}_{m+1} - \mathrm{gcJL}_{n+1} \cdot \mathrm{gcJL}_m = 3CD(-2)^m((-1)^{n-m} - 2^{n-m}).$$

Corollary 4.4 (Vajda identity for the generalized commutative Jacobsthal-Lucas quaternions)**.** *Let $n \geqslant 0$, $m \geqslant 0$, $k \geqslant 0$ be integers such that $n \geqslant k$. Then*

$$\begin{aligned}&\mathrm{gcJL}_{m+k} \cdot \mathrm{gcJL}_{n-k} - \mathrm{gcJL}_m \cdot \mathrm{gcJL}_n \\ &\qquad = (-2)^m CD\Big(2^{n-m}\Big[\Big(-\frac{1}{2}\Big)^k - 1\Big] + (-1)^{n-m}[(-2)^k - 1]\Big).\end{aligned}$$

The next theorem presents a summation formula for the generalized commutative Jacobsthal-Lucas quaternions.

Theorem 4.2. *Let $n \geqslant 0$ be an integer. Then*

$$\sum_{l=0}^{n} \mathrm{gcJL}_l = \frac{\mathrm{gcJL}_{n+2} - \mathrm{gcJL}_1}{2}.$$

P r o o f. Using formula (1.6), we get

$$\begin{aligned}\sum_{l=0}^{n} \mathrm{gcJL}_l &= \sum_{l=0}^{n} (j_l + j_{l+1}e_1 + j_{l+2}e_2 + j_{l+3}e_3) \\ &= \sum_{l=0}^{n} j_l + \sum_{l=0}^{n} j_{l+1}e_1 + \sum_{l=0}^{n} j_{l+2}e_2 + \sum_{l=0}^{n} j_{l+3}e_3 \\ &= \frac{1}{2}[j_{n+2} - 1 + (j_{n+3} - 1 - 4)e_1 + (j_{n+4} - 1 - 4 - 2)e_2 + (j_{n+5} - 1 - 4 - 2 - 10)e_3] \\ &= \frac{1}{2}[j_{n+2} + j_{n+3}e_1 + j_{n+4}e_2 + j_{n+5}e_3 - (1 + 5e_1 + 7e_2 + 17e_3)] \\ &= \frac{1}{2}(\mathrm{gcJL}_{n+2} - \mathrm{gcJL}_1).\end{aligned}$$

□

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Authors' address: Dorota Bród (corresponding author), Anetta Szynal-Liana, Iwona Włoch, Rzeszów University of Technology, Aleja Powstańców Warszawy 12, 35-959 Rzeszów, Poland, e-mail: `dorotab@prz.edu.pl`, `aszynal@prz.edu.pl`, `iwloch@prz.edu.pl`.