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CERTAIN ADDITIVE DECOMPOSITIONS IN A NONCOMMUTATIVE RING

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Abstract. We determine when an element in a noncommutative ring is the sum of an idempotent and a radical element that commute. We prove that a 2×2 matrix A over a projective-free ring R is strongly J-clean if and only if $A\in J(M_2(R))$, or $I_2-A\in J(M_2(R))$, or A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda\in J(R)$, $\mu\in 1+J(R)$, and the equation $x^2-x\mu-\lambda=0$ has a root in $J(R)$ and a root in $1+J(R)$. We further prove that $f(x)\in R[[x]]$ is strongly J-clean if $f(0)\in R$ be optimally J-clean.

Keywords: idempotent matrix; nilpotent matrix; projective-free ring; quadratic equation; power series

MSC 2020: 15A09, 16E50, 16U60

1. Introduction

Let R be an associative ring with identity. An element $a\in R$ is called *strongly J-clean* if a is the sum of an idempotent and a radical element that commute. Every strongly J-clean element is clean, i.e., it is the sum of an idempotent and a unit, see [1], [5], [6], [10], [11], [12]. But the converse is not true. It is of interest to investigate when an element in a ring is strongly J-clean. Recently, strong J-cleanness in a commutative ring has been studied by many authors, see [2], [3], [4], [9]. The motivation of this paper is to explore when an element in a noncommutative ring is the sum of idempotent and radical element that commute.

A ring R is a projective-free ring if every generated projective right R-module is free. For instance, every local ring and every principal ideal ring (may not be

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commutative) is projective-free. In Section 2, we investigate strongly J-clean matrices over a noncommutative projective-free rings. For a projective-free ring R, we prove that $A \in M_2(R)$ is strongly J-clean if and only if $A \in J(M_2(R))$, or $I_2 - A \in J(M_2(R))$, or A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R)$, $\mu \in 1 + J(R)$, and the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

In Section 3, we are concerned on strongly J-clean power series over a noncommutative rings. If $f(0) \in R$ is optimally J-clean, we prove that $f(x) \in R[[x]]$ is strongly J-clean. This provides new kind of ring elements which can be written as the sum of an idempotent and a radical element.

Throughout, all rings are associative with identity. The symbol $M_n(R)$ denotes the ring of all $n \times n$ matrices over R and $GL_n(R)$ stands for the n-dimensional general linear group of R. Let M be a right module, end(M) and aut(M) stand for the ring of endomorphism and automorphism of M, respectively. Let $R[[x]]$ denote the ring of power series over R. We always use $[a, b]$ to denote the commutator $ab - ba$ for any $a, b \in R$.

2. Strongly J-clean matrices

In this section, we characterize a strongly J-clean matrix over projective-free rings in terms of the solvability of the quadratic equation.

Theorem 2.1. *Let R be projective-free. Then $A \in M_2(R)$ is strongly J-clean if and only if $A \in J(M_2(R))$ or $I_2 - A \in J(M_2(R))$ or A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in 1 + J(R)$, $\beta \in J(R)$.*

Proof. $\Leftarrow$ If $A \in J(M_2(R))$, then $A = 0 + A$ is strongly J-clean. If $I_2 - A \in J(M_2(R))$, then $A = I_2 + (A - I_2)$ is strongly J-clean. If A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in 1 + J(R)$, $\beta \in J(R)$, then there exists some $U \in GL_2(R)$ such that

$$A = U^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U + U^{-1} \begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix} U \quad \text{is stroongly } J\text{-clean.}$$

$\Rightarrow$ By hypothesis, there exists an idempotent $E \in M_2(R)$ and $W \in J(M_2(R))$ such that $A = E + W$ with $EW = WE$. Suppose that A and $I_2 - A$ are not in $J(M_2(R))$. Since R is projective-free, there exists $U \in GL_2(R)$ such that $UEU^{-1} = \text{diag}(1, 0)$. Hence, $UAU^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + UWU^{-1}$. Set $V = (v_{ij}) := UWU^{-1}$. Then $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V = V \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, whence, $v_{12} = v_{21} = 0$ and $v_{11}, v_{22} \in J(R)$. Therefore, A is similar to $\begin{pmatrix} 1+v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$, which completes the proof. □

Lemma 2.2 ([4], Theorem 2.1). *Let $E = \operatorname{end}({}_RM)$ and let $\alpha \in E$. Then the following statements are equivalent:*

(1) *α is strongly J-clean in E.*

(2) *$M = P \oplus Q$, where P and Q are α-invariant, and $\alpha|_P \in J(\operatorname{end}(P))$ and $(1_M - \alpha)|_Q \in J(\operatorname{end}(Q))$.*

Lemma 2.3. *Let R be projective-free and let $A \in M_2(R)$ be strongly J-clean. Then $A \in J(M_2(R))$ or $I_2 - A \in J(M_2(R))$ or A is similar to a matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R)$, $\mu \in 1 + J(R)$.*

Proof. Suppose that $A, I_2 - A \notin J(M_2(R))$. By virtue of Theorem 2.1, we have $P \in GL_2(R)$ such that $PAP^{-1} = \begin{pmatrix} 1+\alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha, \beta \in J(R)$. Thus, we check that

$$UAU^{-1} = \begin{pmatrix} 0 & -(1+\alpha)(1+\alpha-\beta)^{-1}\beta(1+\alpha-\beta) \\ 1 & (1+\alpha-\beta)^{-1}\beta(1+\alpha-\beta) + (1+\alpha) \end{pmatrix},$$

where

$$U = \begin{pmatrix} 1 & -1-\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1-\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (1+\alpha-\beta)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Set $\lambda = -(1+\alpha)(1+\alpha-\beta)^{-1}\beta(1+\alpha-\beta)$ and $\mu = (1+\alpha-\beta)^{-1}\beta(1+\alpha-\beta)+(1+\alpha)$. Then $\lambda \in J(R)$ and $\mu \in 1 + J(R)$, as desired. $\square$

Many authors studied strongly clean matrices over a ring, see [6], [7], [8]. This inspires us to investigate strongly J-clean matrices over a projective-free ring. We are ready to prove:

Theorem 2.4. *Let R be projective-free. Then $A \in M_2(R)$ is strongly J-clean if and only if*

(1) *$A \in J(M_2(R))$, or*

(2) *$I_2 - A \in J(M_2(R))$, or*

(3) *A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R)$, $\mu \in 1+J(R)$, and the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.*

Proof. Suppose that $A \in M_2(R)$ is strongly J-clean, and that $A, I_2 - A \notin J(M_2(R))$. It follows by Lemma 2.3 that A is similar to the matrix $B = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R)$, $\mu \in 1 + J(R)$. Hence, $B \in M_2(R)$ is strongly J-clean. In view of Lemma 2.2, we have $2R = C \oplus D$, where $(I_2 - B)|_C \in J(\operatorname{end}(C))$ and $B|_D \in J(\operatorname{end}(D))$. Thus, $B|_C \in \operatorname{aut}(C)$ and $(I_2 - B)|_D \in \operatorname{aut}(D)$. Since R is projective-free,

C and D are free. As $B, I_2-B \notin J(M_2(R))$, we see that $C, D \cong R$. Assume that (a,b) and (c,d) are bases of C and D, respectively. Then $C = R(a,b)$, $D = R(c,d)$. Then

$$R(a,b)\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = R(a,b).$$

Set $\overline{R} = R/J(R)$. Then

$$\overline{R}(\bar{a}, \bar{b}) \subseteq \overline{R}(\bar{1}, \bar{1}).$$

Similarly,

$$\overline{R}(\bar{c}, \bar{d}) \subseteq \overline{R}(\bar{1}, \bar{0}).$$

Write $(\bar{a}, \bar{b}) = s(\bar{1}, \bar{1})$ and $(\bar{c}, \bar{d}) = t(\bar{1}, \bar{0})$. Then

$$(\bar{1}, \bar{1}) = z(\bar{a}, \bar{b}) + z'(\bar{c}, \bar{d}) = zs(\bar{1}, \bar{1}) + z't(\bar{1}, \bar{0}).$$

This implies that $1 - zs \in J(R)$, and so $s \in R$ is left invertible. Hence, $s \in U(R)$, as R is directly finite. Clearly, $a - s, b - s \in J(R)$, and so $1 - a^{-1}b \in J(R)$. $C = R(a,b) = R(1,\alpha)$, where $\alpha = a^{-1}b \in 1 + J(R)$. Analogously, $D = R(1,\beta)$, where $\beta = c^{-1}d \in J(R)$. As C is B-invariant, we see that

$$(1,\alpha)\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = r(1,\alpha)$$

for some $r \in R$. It follows that $\alpha = r$ and $\lambda + \alpha\mu = r\alpha$, and therefore $\alpha^2 - \alpha\mu - \lambda = 0$, i.e., $x^2 - x\mu - \lambda = 0$ has a root $\alpha \in 1 + J(R)$. Likewise, this equation has a root $\beta \in J(R)$, as desired.

Conversely, if (1) or (2) holds, then $A \in M_2(R)$ is strongly J-clean, and so we assume (3) holds. As strong J-cleanness is invariant under similarity, we will suffice to check if $B = \left(\begin{smallmatrix} 0 & \lambda \\ 1 & \mu \end{smallmatrix}\right)$ is strongly J-clean. By hypothesis, the equation $x^2 - x\mu - \lambda = 0$ has roots $c \in J(R)$ and $d \in 1 + J(R)$. Then $c^2 - c\mu - \lambda = 0$ and $d^2 - d\mu - \lambda = 0$. Choose $C = R(1,c)$ and $D = R(1,d)$. Since

$$(1,c)\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = c(1,c) \in C,$$

where C is B-invariant. Similarly, D is B-invariant. If $r(1,c) = s(1,d) \in C \cap D$, then $r = s$ and $rc = sd$; hence, $r(c-d) = 0$. Since $c - d \in U(R)$, we get $r = 0$. Thus, $C \cap D = 0$. Let $(a,b) \in 2R$. Choose $s = (b - ac)(d-c)^{-1}$ and $r = a - s$. Then $(a,b) = r(1,c) + s(1,d) \in C \oplus D$. Hence, $2R = C \oplus D$. Let $\gamma \in \mathrm{end}(C)$. Then

$$1_C - B|_C\gamma\colon\ C \to C; \quad r(1,c) \mapsto r(1,c) - rc(1,c)\gamma.$$

Write $(1,c)\gamma = b(1,c)$ for $a,b \in R$. If $(r(1,c))(1_C - B|_C\gamma) = 0$, then $r(1,c) - rcb(1,c) = 0$, hence, $r(1-cb)(1,c) = 0$. It follows from $c \in J(R)$ that $r = 0$, and so $r(1,c) = 0$. Thus, $1_C - B|_C\gamma$ is monomorphic. For any $r(1,c) \in C$ we see that

$$(r(1-cb)^{-1}(1,c))(r(1,c))(1_C - B|_C\gamma) = r(1,c).$$

This implies that $1_C - B|_C\gamma$ is epimorphic. As a result, $1_C - B|_C\gamma$ is isomorphic. We infer that $B|_C \in J(\mathrm{end}(C))$. Similarly, $(I_2 - B)|_D \in J(\mathrm{end}(D))$. In light of Lemma 2.3, $B \in M_2(R)$ is strongly J-clean. □

A matrix $A \in M_2(R)$ is cyclic if there exists a column α such that $(\alpha, A\alpha) \in GL_2(R)$.

Corollary 2.5. *Let R be a commutative projective-free ring, and let $A \in M_2(R)$. Then A is strongly J-clean if and only if*

(1) *$A \in J(M_2(R))$, or*

(2) *$I_2 - A \in J(M_2(R))$, or*

(3) *A is cyclic and $x^2 - \mathrm{tr}(A)x + \det(A) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.*

Proof. Suppose that A is strongly J-clean. If $A, I_2 - A \notin J(M_2(R))$, then A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R)$, $\mu \in 1+J(R)$, and the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$, by Theorem 2.4. In view of [3], Lemma 7.4.6, A is cyclic. As R is commutative, we see that $\mathrm{tr}(A) = \mu$ and $\det(A) = -\lambda$, and so $x^2 - \mathrm{tr}(A)x + \det(A) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

Conversely, if $A \in J(M_2(R))$ or $I_2 - A \in J(M_2(R))$, then A is strongly J-clean. We now assume that A is cyclic and $x^2 - \mathrm{tr}(A)x + \det(A) = 0$ has a root α in $J(R)$ and a root β in $1+J(R)$. In view of [3], Lemma 7.4.6, A is isomorphic to a companion matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$. This shows that $\mu = \mathrm{tr}(A)$ and $\det(A) = -\lambda$. Since

$$\alpha^2 - \mathrm{tr}(A)\alpha + \det(A) = 0 \quad \text{and} \quad \beta^2 - \mathrm{tr}(A)\beta + \det(A) = 0,$$

we get $\mathrm{tr}(A) = \alpha + \beta$ and $\det(A) = \alpha\beta$. Hence, $\mu = \alpha + \beta \in 1 + J(R)$ and $\lambda = -\alpha\beta \in J(R)$. Therefore, we complete the proof, by Theorem 2.4. □

3. Power series over rings

This section is concerned on strongly J-clean decompositions in power series rings. An element $a \in R$ is optimally J-clean provided that there exists an idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$, and that for any $b \in R$ there exists $c \in R$ such that $[a,c] = [e,b]$. We now derive:

Lemma 3.1. *Let R be a ring and let $a \in R$. Then the following statements are equivalent:*

(1) *$a \in R$ is optimally J-clean.*

(2) *There exists an idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$, and that for any $b \in R$ there exists $c \in eR(1-e) + (1-e)Re$ such that $[a, c] + [e, b] = 0$.*

Proof. (1) $\Rightarrow$ (2) Since $a \in R$ is optimally J-clean, there exists an idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$, and that for any $b \in R$ there exists $c \in R$ such that $[a, c] = [e, b]$. It is easy to check that

$$[a, ec(1-e) + (1-e)ce] = [a, ec(1-e)] + [a, (1-e)ce] = e[a, c](1-e) + (1-e)[a, c]e$$
$$= e[e, b](1-e) + (1-e)[e, b]e = [e, b],$$

and therefore $[a, -ec(1-e) - (1-e)ce] + [e, b] = 0$.

(2) $\Rightarrow$ (1) There exists an idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$, and that for any $b \in R$ there exists $c \in eR(1-e) + (1-e)Re$ such that $[a, c] + [e, b] = 0$. Choose $c' = -c$. Then $[a, c'] = [e, b]$, as required. $\square$

Lemma 3.2 ([10], Lemma 3.2.1)**.** *Let R be a ring and let $n \geqslant 2$. If $e_0 = e_0^2 \in R$ and $e_k(1 - e_0) = \sum\limits_{i=0}^{k-1} e_i e_{k-i}$ $(0 < k < n)$, then*

$$e_0\left(\sum_{i=1}^{n-1} e_i e_{n-i}\right) = \left(\sum_{i=1}^{n-1} e_i e_{n-i}\right) e_0.$$

Lemma 3.3 ([10], Theorem 3.2.2)**.** *Let R be a ring and let $n \geqslant 2$. If $e_0 = e_0^2 \in R$, $e_k(1 - e_0) = \sum\limits_{i=0}^{k-1} e_i e_{k-i}$ and $[r_0, e_k] + [r_1, e_{k-1}] + \ldots + [r_k, e_0] = 0$ for all $0 < k < n$, then*

$$\left[r_0, \sum_{i=1}^{n-1} e_i e_{n-i}\right] = (1 - e_0)\left(\sum_{i=1}^{n-1} [e_i, r_{n-i}]\right) - \left(\sum_{i=1}^{n-1} [e_i, r_{n-i}]\right) e_0.$$

In [10], Shifflet studied strongly clean power series by means of the optimally clean condition. We now extend Theorem 3.2.2 of [10] to strongly J-clean power series and come now to the main result of this section.

Theorem 3.4. *Let R be a ring and let $f(x) \in R[[x]]$. If $f(0) \in R$ is optimally J-clean, then $f(x) \in R[[x]]$ is strongly J-clean.*

P r o o f. Write $f(x) = \sum\limits_{i=0}^{\infty} r_i x^i$. Then we can find an idempotent e_0 such that $r_0 = e_0 + (r_0 - e_0)$ is an optimally J-clean decomposition of r_0. In view of Lemma 3.1, there exists some $e_1 \in (1-e_0)Re_0 + e_0R(1-e_0)$ such that $[r_0, e_1] + [e_0, r_1] = 0$. Clearly, $e_1 = e_0e_1 + e_1e_0$. We shall prove that there exist $e_2, \ldots, e_k, \ldots \in R$ such that

$$e_k = e_0e_k + e_1e_{k-1} + \ldots + e_ke_0 \quad \text{and} \quad [r_0, e_k] + [r_1, e_{k-1}] + \ldots + [r_k, e_0] = 0.$$

Assume that this is true for all $1 \leqslant k \leqslant n-1$. Set $f_n = (1-2e_0)(e_1e_{n-1} + e_2e_{n-2} + \ldots + e_{n-1}e_1)$ and $s_n = r_n + [e_0, [e_1, r_{n-1}] + [e_2, r_{n-2}] + \ldots + [e_{n-1}, r_1]]$. By virtue of Lemma 3.1, we have some $g_n \in (1-e_0)Re_0 + e_0R(1-e_0)$ such that $[r_0, g_n] = [e_0, s_n]$. Let $e_n = f_n + g_n$. In light of Lemma 3.2, analogously to Theorem 3.2.2 of [10], we obtain

$$\sum_{i=1}^{n-1} e_ie_{n-i} = (1-e_0)e_n - e_ne_0.$$

Thus, $e_n = \sum\limits_{i=1}^{n} e_ie_{n-i}$. Furthermore,

$$\begin{aligned}[r_0, f_n] &= \left[r_0, (1-e_0)\left(\sum_{i=1}^{n-1} e_ie_{n-i}\right)\right] - \left[r_0, \left(\sum_{i=1}^{n-1} e_ie_{n-i}\right)e_0\right] \\ &= (1-e_0)\left[r_0, \left(\sum_{i=1}^{n-1} e_ie_{n-i}\right)\right](1-e_0) - e_0\left[r_0, \left(\sum_{i=1}^{n-1} e_ie_{n-i}\right)\right]e_0.\end{aligned}$$

By using Lemma 3.3, we have

$$\left[r_0, \sum_{i=1}^{n-1} e_ie_{n-i}\right] = (1-e_0)\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right) - \left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right)e_0,$$

and then

$$[r_0, f_n] = (1-e_0)\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right)(1-e_0) + e_0\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right)e_0.$$

Moreover,

$$\begin{aligned}[r_0, g_n] = [e_0, s_n] &= [e_0, r_n] + \left[e_0, \left[e_0, \sum_{i=1}^{n-1}[e_i, r_{n-i}]\right]\right] \\ &= [e_0, r_n] + e_0\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right)(1-e_0) + (1-e_0)\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right)e_0.\end{aligned}$$

Thus, we have

$$\begin{aligned}[r_0, e_n] &= [r_0, f_n] + [r_0, g_n] \\ &= [e_0, r_n] + e_0\Big(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\Big)(1-e_0) + (1-e_0)\Big(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\Big)e_0 \\ &\quad + (1-e_0)\Big(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\Big)(1-e_0) + e_0\Big(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\Big)(1-e_0) \\ &= \sum_{i=0}^{n-1}[e_i, r_{n-i}],\end{aligned}$$

hence, $\sum\limits_{i=0}^{n}[r_i, e_{n-i}] = 0$. By induction, the claim is true. Thus, $\sum\limits_{i=0}^{\infty} e_i x^i = \Big(\sum\limits_{i=0}^{\infty} e_i x^i\Big)^2 \in R[[x]]$ and $f(x)\Big(\sum\limits_{i=0}^{\infty} e_i x^i\Big) = \Big(\sum\limits_{i=0}^{\infty} e_i x^i\Big)f(x)$. Since $f(0) - e(0) \in J(R)$, we see that $f(x) - \sum\limits_{i=0}^{\infty} e_i x^i \in J(R[[x]])$. Therefore, $f(x) \in R[[x]]$ is strongly J-clean, as asserted. □

Corollary 3.5. *Let R be an abelian ring and let $f(x) \in R[[x]]$. If $f(0) \in R$ is strongly J-clean, then $f(x) \in R[[x]]$ is strongly J-clean.*

Proof. Suppose $f(0) \in R$ is strongly J-clean. Then there exists an idempotent $e \in R$ such that $f(0) - e \in J(R)$ and $f(0)e = ef(0)$. For any $b \in R$, we choose $c = 0 \in eR(1-e) + (1-e)Re$. Then $[f(0), c] + [e, b] = 0$, hence, $f(0)$ is J-optimally clean. Therefore, $f(x) \in R[[x]]$ is strongly J-clean in terms of Theorem 3.4. □

Corollary 3.6. *Let R be a ring and let $f(x) \in R[[x]]$. Then the following statements are equivalent:*
(1) *$f(0) \in R$ is optimally J-clean.*
(2) *$f(x) \in R[[x]]$ is optimally J-clean.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 3.4, $f(x) \in R[[x]]$ is strongly J-clean. Hence, there exists an idempotent $e(x) \in R[[x]]$ such that $w(x) := f(x) - e(x) \in J(R[[x]])$ and $f(x)e(x) = e(x)f(x)$. Thus, $f(x) = (1 - e(x)) + (2e(x) - 1 + w(x))$. As $(2e(x)-1)^2 = 1$, we see that $(2e(x) - 1 + w(x)) = (2e(x) - 1)(1 + (2e(x) - 1)w(x)) \in U(R[[x]])$. By virtue of [10], Theorem 3.3.2, $f(x) \in R[[x]]$ is optimally clean. For any $b(x) \in R[[x]]$ there exists $c(x) \in R[[x]]$ such that $[f(x), -c(x)] = [1 - e(x), b(x)]$. This implies that $[f(x), c(x)] = [e(x), b(x)]$. Therefore, $f(x) \in R[[x]]$ is optimally J-clean, as desired.

(2) $\Rightarrow$ (1) This is obvious. □

Example 3.7. Let $\mathbb{Z}_{(2)} = \{m/n\colon m,n \in \mathbb{Z},\, n \neq 0,\, (m,n) = 1, 2 \nmid n\}$ and

$$A(x) = \begin{pmatrix} \sum\limits_{n=0}^{\infty} x^n & \sum\limits_{n=0}^{\infty} x^{n+1} \\ -\sum\limits_{n=0}^{\infty} \dfrac{1}{3^{n+1}} x^n & \sum\limits_{n=0}^{\infty} \dfrac{2}{3^{n+1}} x^n \end{pmatrix} \in M_2(\mathbb{Z}_{(2)}[[x]]).$$

Then $A(x) \in M_2(\mathbb{Z}_{(2)}[[x]])$ is strongly J-clean.

P r o o f. Clearly, $A(0) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$. Since the characteristic equation $\chi_{A(0)} = x^2 - \frac{5}{3}x + \frac{2}{3}$ has roots 1 and $\frac{2}{3}$, we see that $A(0)$ is similar to $C = \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$. Let $E = \mathrm{diag}(1,0)$. Then $E^2 = E$, $EC = CE$ and $C - E = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \in J(M_2(\mathbb{Z}_{(2)}))$.

Let $B = (b_{ij}) \in M_2(\mathbb{Z}_{(2)})$. Choose $x_1 = 3b_{12}$ and $x_2 = 3b_{21}$. Set $X = \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix}$. Then

$$[C,X] = \begin{pmatrix} 0 & b_{12} \\ -b_{21} & 0 \end{pmatrix} = [E,B].$$

Accordingly, C is optimally J-clean. Hence, $A(0) \in M_2(\mathbb{Z}_{(2)})$ is J-optimally clean. Therefore, $A(x) \in M_2(\mathbb{Z}_{(2)}[[x]])$ is strongly J-clean by Theorem 3.4. $\square$

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