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ON AN ADDITIVE PROBLEM OF UNLIKE POWERS IN SHORT INTERVALS

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Abstract. We prove that almost all positive even integers n can be represented as $p_2^2 + p_3^3 + p_4^4 + p_5^5$ with $|p_k^k - \frac{1}{4}N| \leqslant N^{1-1/54+\varepsilon}$ for $2 \leqslant k \leqslant 5$. As a consequence, we show that each sufficiently large odd integer N can be written as $p_1 + p_2^2 + p_3^3 + p_4^4 + p_5^5$ with $|p_k^k - \frac{1}{5}N| \leqslant N^{1-1/54+\varepsilon}$ for $1 \leqslant k \leqslant 5$.

Keywords: Waring-Goldbach problem; exponential sum over prime in short interval; circle method

MSC 2020: 11P32, 11P05, 11P55

1. Introduction

In 1951, Roth in [9] proved that almost all positive integers n can be written as $m_2^2 + m_3^3 + m_4^4 + m_5^5$ with m_i $(2 \leqslant i \leqslant 5)$ being positive integers. Prachar in [6] improved the above result by showing that almost all positive even integers n can be represented in the form

$$n = p_2^2 + p_3^3 + p_4^4 + p_5^5, \tag{1.1}$$

where p_i $(2 \leqslant i \leqslant 5)$ are prime numbers. Denote by $E(N)$ the number of positive even integers n up to N that cannot be written in the form (1.1). Prachar in [6] proved that $E(N) \ll N(\log N)^{-30/47+\varepsilon}$, where $\varepsilon > 0$ is arbitrarily small. Bauer in [1] improved Prachar's result to $E(N) \ll N^{1-\theta+\varepsilon}$ with $\theta = \frac{1}{2742}$. Ren and Tsang improved this result to $\theta = \frac{1}{66}$ in [7], and to $\theta = \frac{1}{48}$ in [8]. Bauer in [2] further proved that $\theta = \frac{47}{1680}$ is acceptable. The best known result is $\theta = \frac{1}{16}$ proved by Zhao, see [11].

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In this note, we investigate this problem in short intervals, i.e.,

$$\begin{cases} n = p_2^2 + p_3^3 + p_4^4 + p_5^5, \\ \left|p_k^k - \dfrac{N}{4}\right| \leqslant U, \quad 2 \leqslant k \leqslant 5, \end{cases} \tag{1.2}$$

where $U = N^{1-\delta+\varepsilon}$ with $\delta > 0$. Let $E(N,U)$ be the number of all positive even integers $n \in [N-4U, N+4U]$ that cannot be expressed by (1.2). We want to show that there exists a δ such that $E(N,U) \ll U^{1-\varepsilon}$. As is known, the quality of this problem is described by δ. In 2012, Li and Tang in [4] considered this problem firstly and proved that $\delta = \frac{1}{264}$ is acceptable. In 2016, Zhang in [10] improved their result to $\delta = \frac{4}{325}$. In this paper, we establish the following theorem.

Theorem 1.1. *For $U = N^{1-\delta+\varepsilon}$ with $\delta = \frac{1}{54}$, we have*

$$E(N,U) \ll U^{1-\varepsilon}.$$

As a consequence, we give the following result.

Corollary 1.1. *For each sufficiently large odd positive integer N, the equation*

$$\begin{cases} N = p_1 + p_2^2 + p_3^3 + p_4^4 + p_5^5, \\ \left|p_k^k - \dfrac{N}{5}\right| \leqslant U, \quad 1 \leqslant k \leqslant 5, \end{cases}$$

is solvable for $U = N^{1-1/54+\varepsilon}$.

We use the circle method to establish Theorem 1.1. For the integrals on the major arc, we cite Proposition 1 in [10] and give a slight modification. For the treatment of the integrals on the minor arc, we prove an estimate for the integral $\int_{\mathfrak{m}} |f_2^2(\alpha) f_3^{1/2}(\alpha) f_4^2(\alpha) f_5^2(\alpha)|\,\mathrm{d}\alpha$ and then apply the estimates for the exponential sum over primes in short intervals in [5].

Throughout this paper, the letter ε denotes an arbitrarily small positive number which may be different at different occurrences. In what follows, we use N to denote a large positive number and set $L = \log N$. As usual, we write $e(x) = \mathrm{e}^{2\pi \mathrm{i} x}$.

2. Proof of Theorem 1.1

Let $1 < P < \frac{1}{2}Q$. By Dirichlet's rational approximation theorem, each $\alpha \in [1/Q, 1+1/Q]$ may be written in the form

$$\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \leqslant \frac{1}{qQ} \tag{2.1}$$

for some integers a, q with $1 \leqslant a \leqslant q \leqslant Q$ and $(a,q)=1$. Let $I(q,a)$ be the set of α satisfying (2.1). Define the *major arc* $\mathfrak{M}$ and *minor arc* $\mathfrak{m}$ as

$$\mathfrak{M} = \bigcup_{q \leqslant P} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} I(q,a), \quad \mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus \mathfrak{M}. \tag{2.2}$$

For $1 < U < N$, write

$$Y = \frac{N}{4} - U, \quad X = \frac{N}{4} + U.$$

Put

$$\mathcal{R}(n,U) = \sum_{\substack{n = p_2^2 + p_3^3 + p_4^4 + p_5^5 \\ Y \leqslant p_k^k \leqslant X}} (\log p_2) \ldots (\log p_5)$$

and

$$f_k(\alpha) = \sum_{Y \leqslant p^k \leqslant X} (\log p) e(p^k \alpha). \tag{2.3}$$

Then

$$\mathcal{R}(n,U) = \int_0^1 \left(\prod_{k=2}^{5} f_k(\alpha) \right) e(-n\alpha)\, \mathrm{d}\alpha.$$

By orthogonality of the exponential functions, we have

$$\begin{aligned} \mathcal{R}(n,U) &= \int_{1/Q}^{1+1/Q} \left(\prod_{k=2}^{5} f_k(\alpha) \right) e(-n\alpha)\, \mathrm{d}\alpha \\ &= \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}} \right) \left(\prod_{k=2}^{5} f_k(\alpha) \right) e(-n\alpha)\, \mathrm{d}\alpha. \end{aligned} \tag{2.4}$$

To describe the contribution from the major arc, we introduce some notations. Let

$$C_k(q,a) = \sum_{\substack{h=1 \\ (h,q)=1}}^{q} e\Big(\frac{ah^k}{q}\Big) \quad \text{and} \quad B(n,q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\Big(-\frac{an}{q}\Big) \prod_{k=2}^{5} C_k(q,a).$$

Define the singular series

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \frac{B(n,q)}{\varphi^4(q)}. \tag{2.5}$$

In the following argument, we set

$$P = U^2 N^{-23/12 - 3\varepsilon}, \quad Q = N^{11/12 + 2\varepsilon}, \quad U = N^{1 - 1/54 + \varepsilon}. \tag{2.6}$$

Proposition 2.1. *Let the major arc $\mathfrak{M}$ be defined in* (2.2) *with P, Q, U defined in* (2.6)*. Then for $N-4U \leqslant n \leqslant N+4U$, one has*

$$\int_{\mathfrak{M}} \Big(\prod_{k=2}^{5} f_k(\alpha) \Big) e(-n\alpha) \, \mathrm{d}\alpha = \frac{1}{120} \mathfrak{S}(n) \mathfrak{J}(n,U) + O(U^3 N^{-163/60} L^{-A}),$$

where $\mathfrak{J}(n,U)$ is defined as

$$\mathfrak{J}(n,U) = \sum_{\substack{n=m_2+m_3+m_4+m_5 \\ Y \leqslant m_2,\dots,m_5 \leqslant X}} \Big\{ \prod_{k=2}^{5} m_k^{1/k-1} \Big\}$$

and satisfies $\mathfrak{J}(n) \asymp U^3 N^{-163/60}$. Moreover, the singular series $\mathfrak{S}(n)$ defined in (2.5) *is absolutely convergent and satisfies $\mathfrak{S}(n) \gg (\log\log n)^{-c}$ for some absolute positive constant c.*

Proof. This is a slight modification of Proposition 1 in [10]. Actually, the requirements for the parameters in Proposition 1 of [10] are the following:

$$P^{1+\varepsilon} \ll U^2 N^{-23/12}, \quad Q \gg N^{11/12}, \quad PQ \ll U^{1-\varepsilon}. \tag{2.7}$$

One can easily check that for P, Q, U defined in (2.6), the conditions (2.7) are satisfied. Hence, the argument of Proposition 1 in [10] is true for our choice of parameters. $\square$

Proposition 2.2. *Let the minor arc $\mathfrak{m}$ be defined as in* (2.2) *with P and Q defined in* (2.6)*. Then we have*

$$\int_{\mathfrak{m}} |f_2^2(\alpha) f_3^2(\alpha) f_4^2(\alpha) f_5^2(\alpha)| \, \mathrm{d}\alpha \ll U^{19/4+\varepsilon} N^{-129/40}.$$

Remark 2.1. The above bound is $O(U^{589/112+\varepsilon} N^{-6239/1680})$ by [10], Proposition 2. It is easy to see that the bound in Proposition 2.2 is a slight improvement.

We will prove Proposition 2.2 in the next section. With Propositions 2.1 and 2.2 ready, we can establish Theorem 1.1 and Corollary 1.1 in the remaining part of this section.

Proof of Theorem 1.1. Let $\mathcal{E}(n,U)$ be the set of positive integers $n \in [N-4U, N+4U]$ which cannot be represented as (1.2) and $E(n,U) = |\mathcal{E}(n,U)|$. Then, $\mathcal{R}(n,U) = 0$ for $n \in \mathcal{E}(n,U)$. By (2.4) we have

$$\sum_{n \in \mathcal{E}(n,U)} \Big(\int_{\mathfrak{M}} + \int_{\mathfrak{m}} \Big) \Big(\prod_{k=2}^{5} f_k(\alpha) \Big) e(-n\alpha) \, \mathrm{d}\alpha = 0. \tag{2.8}$$

Thus, by Proposition 2.1

$$\left|\sum_{n\in\mathcal{E}(n,U)}\int_{\mathfrak{M}}\left(\prod_{k=2}^{5} f_k(\alpha)\right)e(-n\alpha)\,\mathrm{d}\alpha\right| \gg U^3N^{-163/60}(\log\log N)^{-c}E(n,U). \tag{2.9}$$

On the other hand, by Cauchy's inequality, we have

$$\begin{aligned}
&\left|\sum_{n\in\mathcal{E}(n,U)}\int_{\mathfrak{m}}\left(\prod_{k=2}^{5} f_k(\alpha)\right)e(-n\alpha)\,\mathrm{d}\alpha\right| \\
&\quad\leqslant \left(\int_{\mathfrak{m}}\left|\prod_{k=2}^{5} f_k^2(\alpha)\right|\mathrm{d}\alpha\right)^{1/2}\left(\int_0^1\left|\sum_{n\in\mathcal{E}(n,U)} e(-n\alpha)\right|^2\mathrm{d}\alpha\right)^{1/2} \\
&\quad= \mathcal{I}^{1/2}(2)E^{1/2}(n,U),
\end{aligned}$$

where

$$\mathcal{I}(2) = \int_{\mathfrak{m}} |f_2^2(\alpha)f_3^2(\alpha)f_4^2(\alpha)f_5^2(\alpha)|\,\mathrm{d}\alpha.$$

This together with (2.8) and (2.9) gives

$$E(n,U) \ll U^{-6+\varepsilon}N^{163/30}\mathcal{I}(2).$$

It follows from Proposition 2.2 and the definition of U in (2.6) that

$$E(n,U) \ll U^{-5/4+\varepsilon}N^{53/24} \ll U^{1-\varepsilon}.$$

This completes the proof of Theorem 1.1. $\square$

Proof of Corollary 1.1. Let $\overline{N} = \frac{4}{5}N$, $\overline{U} = \overline{N}^{1-\delta+\varepsilon}$. By Theorem 1.1, all but $O(\overline{U}^{1-\varepsilon})$ exceptions of n in $[\overline{N}-4\overline{U}, \overline{N}+4\overline{U}]$ can be written as $p_2^2+p_3^3+p_4^4+p_5^5$, where $\frac{1}{4}\overline{N}-\overline{U} \leqslant p_i^i \leqslant \frac{1}{4}\overline{N}+\overline{U}$, i.e., $\frac{1}{5}N-\overline{U} \leqslant p_i^i \leqslant \frac{1}{5}N+\overline{U}$. Consider the subset of primes

$$\mathcal{P} = \left\{p\colon \left|p-\frac{N}{5}\right| \leqslant \overline{U}\right\}.$$

By the prime number theorem in short intervals [3], Chapter 7, Theorem 2, one has $|\mathcal{P}| \gg \overline{U}/L$. Thus, $\#\,\{N-p\colon p\in\mathcal{P}\} \gg \overline{U}/L$. There are $\gg \overline{U}/L$ even integers n such that $n = N-p$ and $\overline{N}-\overline{U} \leqslant n \leqslant \overline{N}+\overline{U}$. Note that $\overline{U}/L \gg \overline{U}^{1-\varepsilon}$, so Theorem 1.1 implies that there exists $p\in\mathcal{P}$ such that the equation

$$N-p = p_2^2+p_3^3+p_4^4+p_5^5, \quad \left|p_k^k-\frac{N}{5}\right| \leqslant \overline{U}, \quad k=2,\ldots,5,$$

has solutions. Then Corollary 1.1 follows by noticing that $\overline{U} \leqslant U$. $\square$

3. Proof of Proposition 2.2

Lemma 3.1. *Let $\mathfrak{m}$ be the minor arc defined as in* (2.2) *with P, Q in* (2.6). *Then, for $\frac{8}{9} < \theta \leqslant 1$ and $0 < \varrho \leqslant \min\left\{\frac{1}{12}(3\theta-2), \frac{1}{6}(9\theta-8)\right\}$, one has*

$$\sup_{\alpha\in\mathfrak{m}}\left|\sum_{x<n\leqslant x+x^\theta}\Lambda(n)e(n^3\alpha)\right| \ll x^{\theta-\varrho+\varepsilon} + x^{\theta+\varepsilon}P^{-1/2}.$$

Proof. See [5], Theorem 1. □

Lemma 3.2. *Let $f_k(\alpha)$ be defined as in* (2.3). *Then for $U \gg N^{1-1/30}$, we have*

$$\int_0^1 |f_2^2(\alpha)f_3^2(\alpha)f_5^2(\alpha)|\,\mathrm{d}\alpha \ll U^{4+\varepsilon}N^{-44/15}.$$

Proof. See [4], Lemma 8. □

Lemma 3.3. *Let $f_k(\alpha)$ be defined as in* (2.3). *Then for $U = o(N)$ we have*

$$\int_0^1 |f_2^2(\alpha)f_4^4(\alpha)|\,\mathrm{d}\alpha \ll U^{3+\varepsilon}N^{-2} \tag{3.1}$$

and

$$\int_0^1 |f_2^2(\alpha)f_5^6(\alpha)|\,\mathrm{d}\alpha \ll U^{9/2+\varepsilon}N^{-33/10}. \tag{3.2}$$

Proof. The integral in (3.1) counts the number of integer solutions of the equation

$$x_1^2 - x_2^2 = y_1^4 + y_2^4 - y_3^4 - y_4^4 \tag{3.3}$$

with $Y^{1/2} \leqslant x_1,\ x_2 \leqslant X^{1/2}$ and $Y^{1/4} \leqslant y_i \leqslant X^{1/4}$ for $i = 1, \ldots, 4$. For any fixed (y_1, y_2, y_3, y_4) if $y_1^4 + y_2^4 \neq y_3^4 + y_4^4$, then the number of (x_1, x_2) is at most $d(y_1^4 + y_2^4 - y_3^4 - y_4^4)$. So in this case, the number of solutions of (3.3) is

$$\ll (X^{1/4} - Y^{1/4})^{4+\varepsilon} \ll (UN^{-3/4})^{4+\varepsilon} \ll U^{4+\varepsilon}N^{-3}.$$

If $y_1^4 + y_2^4 = y_3^4 + y_4^4$, then $x_1 = x_2$. Hence, the number of solutions of (3.3) is bounded by

$$\begin{aligned}&\ll (X^{1/2} - Y^{1/2})\int_0^1 |f_4^4(\alpha)|\,\mathrm{d}\alpha \ll (X^{1/2} - Y^{1/2})(X^{1/4} - Y^{1/4})^{2+\varepsilon}\\&\ll UN^{-1/2}(UN^{-3/4})^{2+\varepsilon} \ll U^{3+\varepsilon}N^{-2}.\end{aligned}$$

Thus, we get

$$\int_0^1 |f_2^2(\alpha) f_4^4(\alpha)| \,\mathrm{d}\alpha \ll U^{4+\varepsilon} N^{-3} + U^{3+\varepsilon} N^{-2} \ll U^{3+\varepsilon} N^{-2}.$$

Treating the integral in (3.2) similarly we get

$$\begin{aligned}
\int_0^1 |f_2^2(\alpha) f_5^6(\alpha)| \,\mathrm{d}\alpha &\ll (X^{\frac{1}{5}} - Y^{\frac{1}{5}})^{6+\varepsilon} + (X^{1/2} - Y^{1/2}) \int_0^1 |f_5^6(\alpha)| \,\mathrm{d}\alpha \\
&\ll (X^{1/5} - Y^{1/5})^{6+\varepsilon} + (X^{1/2} - Y^{1/2}) \\
&\quad \times \left(\int_0^1 |f_5^4(\alpha)| \,\mathrm{d}\alpha \right)^{1/2} \left(\int_0^1 |f_5^8(\alpha)| \,\mathrm{d}\alpha \right)^{1/2} \\
&\ll (X^{1/5} - Y^{1/5})^{6+\varepsilon} + (X^{1/2} - Y^{1/2}) (U N^{-4/5})^{7/2+\varepsilon} \\
&\ll U^{6+\varepsilon} N^{-24/5} + U^{9/2+\varepsilon} N^{-33/10} \ll U^{9/2+\varepsilon} N^{-33/10}.
\end{aligned}$$

This completes the proof of Lemma 3.3. □

P r o o f of Proposition 2.2. To estimate $\mathcal{I}(2)$, we firstly have

$$\mathcal{I}(2) \ll \sup_{\alpha \in \mathfrak{m}} |f_3(\alpha)|^{3/2} \mathcal{I}(1/2).$$

where $\mathcal{I}(\frac{1}{2}) = \int_{\mathfrak{m}} |f_2^2(\alpha) f_3^{1/2}(\alpha) f_4^2(\alpha) f_5^2(\alpha)| \,\mathrm{d}\alpha$. By Lemma 3.1 with $x = (\frac{1}{4}N - U)^{1/3}$ and $x^\theta = (\frac{1}{4}N + U)^{1/3} - (\frac{1}{4}N - U)^{1/3} \asymp U N^{-2/3}$ with U defined in (2.6), we have $\varrho = \frac{1}{12}(3\theta - 2)$. Therefore, we obtain

$$\begin{aligned}
\sup_{\alpha \in \mathfrak{m}} |f_3(\alpha)| &\ll x^{\theta - \varrho + \varepsilon} + x^{\theta + \varepsilon} P^{-1/2} = x^{3\theta/4 + 1/6 + \varepsilon} + x^{\theta + \varepsilon} P^{-1/2} \\
&\ll (U N^{-2/3})^{3/4} \left(\left(\frac{N}{4} - U \right)^{1/3} \right)^{1/6+\varepsilon} + (U N^{-2/3})^{1+\varepsilon} (U^2 N^{-23/12 - 3\varepsilon})^{-1/2} \\
&\ll U^{3/4+\varepsilon} N^{-4/9}
\end{aligned}$$

for $U > N^{53/54}$. By Hölder's inequality, we have

$$\begin{aligned}
\mathcal{I}(1/2) &\leqslant \left(\int_0^1 |f_2^2(\alpha) f_4^4(\alpha)| \,\mathrm{d}\alpha \right)^{1/2} \left(\int_0^1 |f_2^2(\alpha) f_3^2(\alpha) f_5^2(\alpha)| \,\mathrm{d}\alpha \right)^{1/4} \\
&\quad \times \left(\int_0^1 |f_2^2(\alpha) f_5^6(\alpha)| \,\mathrm{d}\alpha \right)^{1/4} \\
&\ll \left(U^{3+\varepsilon} N^{-2} \right)^{1/2} \left(U^{4+\varepsilon} N^{-44/15} \right)^{1/4} \left(U^{9/2+\varepsilon} N^{-33/10} \right)^{1/4} \\
&\ll U^{29/8+\varepsilon} N^{-307/120}.
\end{aligned}$$

Thus, we get

$$\mathcal{I}(2) \ll (U^{3/4+\varepsilon} N^{-4/9})^{3/2} (U^{29/8+\varepsilon} N^{-307/120}) \ll U^{19/4+\varepsilon} N^{-129/40}.$$

This completes the proof of Proposition 2.2. □

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