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A QUADRATIC FORM WITH PRIME VARIABLES ASSOCIATED WITH HECKE EIGENVALUES OF A CUSP FORM

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Abstract. Let f be a normalized primitive holomorphic cusp form of even integral weight k for the full modular group $\mathrm{SL}(2,\mathbb{Z})$, and denote its nth Fourier coefficient by $\lambda_f(n)$. We consider the hybrid problem of quadratic forms with prime variables and Hecke eigenvalues of normalized primitive holomorphic cusp forms, which generalizes the result of D. Y. Zhang, Y. N. Wang (2017).

Keywords: circle method; cusp form; Fourier coefficient

MSC 2020: 11F30, 11F41, 11N37

1. Introduction

Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ be the full modular group. Denote by H_k the set of normalized primitive holomorphic cusp forms of even integral weight $k \geqslant 2$ for Γ, which consists of the common eigenfunctions f of all Hecke operators T_n. If $f \in H_k$ is a Hecke eigenform, then it has the Fourier expansion at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz), \quad \Im(z) > 0,$$

where $e(z) = \mathrm{e}^{2\pi \mathrm{i} z}$ and the normalized Fourier coefficients $\lambda_f(n)$ are the eigenvalues of all Hecke operators T_n. It is known that $\lambda_f(n)$ is a real and multiplicative function. Furthermore, Deligne in [4] proved the Ramanujan-Petersson conjecture which asserts that

$$|\lambda_f(n)| \leqslant d(n),$$

where $d(n)$ is the divisor function.

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The study of properties of quadratic forms has attracted lots of attention in the literature. Let $l \geqslant 3$ be a positive integer, many authors consider the asymptotic formula of the form

$$\sum_{1\leqslant n_1,n_2,\ldots,n_l\leqslant\sqrt{x}} d_k(n_1^2+n_2^2+\ldots+n_l^2), \tag{1.1}$$

where $d_k(n)$ denotes the kth divisor function. In particular, we have $d_2(n)=d(n)$. For $k=2$, $l=3$, there have been a number of results (see [1], [6], [17]) in this direction. And the best records in this direction is given by Zhao (see [17]) who showed that

$$\sum_{1\leqslant n_1,n_2,n_3\leqslant\sqrt{x}} d(n_1^2+n_2^2+n_3^2) = c_1x^{3/2}\log x + c_2x^{3/2} + O(x\log^7 x),$$

where $c_1, c_2 > 0$ are positive constants. Later, Sun and Zhang in [13] considered the triple divisor function $d_3(n)$ for $l=3$, and they proved that

$$\sum_{1\leqslant n_1,n_2,n_3\leqslant\sqrt{x}} d_3(n_1^2+n_2^2+n_3^2) = c_1x^{3/2}(\log x)^2 + c_2x^{3/2}\log x + c_3x^{3/2} + O(x^{11/8+\varepsilon}),$$

where $c_1, c_2, c_3 > 0$ are constants and $\varepsilon > 0$ is an arbitrarily small positive number. By adopting the method of Sun and Zhang in [13], Hu and Yang in [10] derived an asymptotic formula for the case of $l=4$. Very recently, Hu and Lü in [9] generalized the above results and studied the general problem (1.1). For $k \geqslant 4$ and $l \geqslant 3$ integers, they established the following results:

$$\begin{aligned}\sum_{1\leqslant n_1,n_2,\ldots,n_l\leqslant\sqrt{x}} & d_k(n_1^2+n_2^2+\ldots+n_l^2)\\ &= \sum_{j=0}^{k-1} a_{k,l,j}\sum_{i=0}^{j} b_{k,l,i}x^{l/2}(\log x)^{j-i} + O(x^{1/2-\vartheta_{k,l}+\varepsilon}),\end{aligned}$$

where $a_{k,l,j}, b_{k,l,i} > 0$ are some suitable constants and $\vartheta_{k,l}$ are given by

$$\vartheta_{k,l} = \begin{cases} \dfrac{10-k}{12k+24} & \text{if } l=3 \text{ and } 4\leqslant k\leqslant 9,\\[2mm] \dfrac{l-2}{2k+l-2} & \text{if } l=4,5 \text{ and } k\geqslant 4,\\[2mm] \dfrac{3(l-2)}{(k+2)l}, & \text{if } l\geqslant 6 \text{ and } k\geqslant 4.\end{cases}$$

On the one hand, in 1963 Vinogradov (see [15]) and Chen (see [3]) independently studied the sphere problem and showed that

$$\sum_{\substack{m_1^2+m_2^2+m_3^2\leqslant x\\ m_j\in\mathbb{Z}\ (j=1,2,3)}} 1 = \frac{4}{3}\pi x^{3/2} + O(x^{2/3}). \tag{1.2}$$

And the exponent in the remainder term of (1.2) has been refined by a number of authors (see e.g. [2], [7]). Let $\pi_3(x)$ denote the number of integer points by

$$\pi_3(x) := \sharp\{(m_1,m_2,m_3)\in\mathbb{Z}^3:\ m_1^2+m_2^2+m_3^2 = p\leqslant x\}.$$

Friedlander and Iwaniec in [5] proved that

$$\pi_3(x) \sim \frac{4\pi}{3}\frac{x^{3/2}}{\log x},\quad x\to\infty,$$

which can be regarded as a generalization of the prime number theorem. In 2012, Guo and Zhai in [6] considered the quadratic form

$$\pi_\Lambda(x) := \sum_{m_1^2+m_2^2+m_3^2\leqslant x} \Lambda(m_1^2+m_2^2+m_3^2),$$

where $\Lambda(n)$ is the von Mangoldt function, and they obtained

$$\pi_\Lambda(x) = cx^{3/2} + O(x^{3/2}\log^{-A} x), \tag{1.3}$$

where $c>0$ is a suitable constant and $A>0$ is any fixed number. As a corollary of (1.3), they also proved that

$$\pi_3(x) = c'\int_2^x \frac{t^{1/2}}{\log t}\,\mathrm{d}t + O(x^{3/2}\log^{-A} x),$$

where $c'>0$ is a positive constant and $A>0$ is any fixed number. In 2017, Zhang and Wang in [16] studied the sum

$$S_\lambda(x) := \sum_{\substack{n\leqslant x\\ n=p_1^2+p_2^2+p_3^2}} \lambda_f(n),$$

and they obtained

$$S_\lambda(x) \ll x^{3/2-2/67+\varepsilon} \tag{1.4}$$

for $\varepsilon>0$, where the implied constant depends on f and ε. Later, Hu et al. in [8] improved and generalized the result (1.4) which asserts that

$$\sum_{\substack{1\leqslant n\leqslant x\\ n=p_1^2+p_2^2+\ldots+p_l^2}} \lambda_f(n+b) \ll_{f,\varepsilon} x^{l/2-\delta_l+\varepsilon}, \tag{1.5}$$

where $\lambda_f(n)$ denotes the nth normalized Fourier coefficients of holomorphic cusp form or Hecke-Maass cusp form for $\mathrm{SL}(2,\mathbb{Z})$, and $b \ll x$ is an integer, here $\delta_3 = \frac{1}{4}$ and $\delta_l = \frac{1}{2}$ $(l \geqslant 4)$.

In this paper, we consider the average behaviour of normalized Fourier coefficients related to quadratic form with prime variables of the type

$$S_{\lambda,r,l}(x) := \sum_{\substack{n \leqslant x \\ n = p_1^r + p_2^r + \ldots + p_l^r}} \lambda_f(n), \tag{1.6}$$

where $r \geqslant 2$ and $l \geqslant 3$ are positive integers. More precisely, we will be able to establish the following result.

Theorem 1.1. *Let $f \in H_k$ be a normalized primitive holomorphic cusp form of even integral weight k for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, and denote its nth normalized Fourier coefficients by $\lambda_f(n)$. For $S_{\lambda,r,l}$ defined by (1.6), we have*

$$S_{\lambda,r,l}(x) \ll_{f,\varepsilon,r,l} x^{l/r - 2(l-2^{r-1})/(16+3(l-2^{r-1}))+\varepsilon},$$

where $r \geqslant 2$ and $l \geqslant 2^{r-1}+1$ are positive integers and the implied constant depends on f, ε, r, l.

Throughout the paper, we will work with the vector space H_k. Here, $\varepsilon > 0$ denotes an arbitrarily small number which may take different values in different occurrences. The letter p, with or without subscript, always denotes a prime number.

2. Preliminaries

In this section, we first give an outline of the circle method following the line of [16], then we collect some lemmas which play an important role in the proof of the main result.

Let x be a large number and $L = \log x$. For any $\alpha \in \mathbb{R}$ and $y > 1$ we define

$$S_1(\alpha; y) = \sum_{1 \leqslant p \leqslant y} e(p^r \alpha), \quad S_2(\alpha; y) = \sum_{1 \leqslant n \leqslant y} \lambda_f(n) e(n\alpha). \tag{2.1}$$

By the orthogonality relation, we have

$$S_{\lambda,r,l}(x) = \int_0^1 S_1^l(\alpha; x^{1/r}) S_2(-\alpha; x)\,\mathrm{d}\alpha. \tag{2.2}$$

Let $2P \leqslant Q$ be large positive numbers such that

$$P = x^{4/(16+3(l-2^{r-1}))}, \quad Q = \frac{x}{PL^B}, \tag{2.3}$$

where $B > 0$ is a large parameter.

By Dirichlet's lemma on rational approximations, we know that for any $\alpha \in [1/Q, 1+1/Q]$ can be written as

$$\alpha = \frac{a}{q} + \beta, \quad |\beta| \leqslant \frac{1}{qQ}$$

for some integers a, q with $1 \leqslant a \leqslant q \leqslant Q$ and $(a,q) = 1$. We define the major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{M} := \bigcup_{1\leqslant q\leqslant P} \bigcup_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} \mathfrak{M}(a,q), \quad \mathfrak{m} := \Big[\frac{1}{Q}, 1+\frac{1}{Q}\Big] \setminus \mathfrak{M}, \tag{2.4}$$

where

$$\mathfrak{M}(a,q) = \Big[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\Big].$$

It follows from $2P \leqslant Q$ that the arcs $\mathfrak{M}(a,q)$ are mutually disjoint. Then we can rewrite the integral (2.2) as

$$S_{\lambda,r,l}(x) = \int_{\mathfrak{M}} S_1^l(\alpha; x^{1/r}) S_2(-\alpha; x)\,\mathrm{d}\alpha + \int_{\mathfrak{m}} S_1^l(\alpha; x^{1/r}) S_2(-\alpha; x)\,\mathrm{d}\alpha.$$

Lemma 2.1. *For $\alpha \in \mathfrak{M}$ defined by* (2.1) *we have*

$$S_2(-\alpha; x) \ll (xq)^{\varepsilon}(q^{3/2} + q^{2/3}x^{1/3} + q^{1/2}PL^B + q^{-1/2}P^2L^{2B} + x^{-1/4}P^2L^{2B}),$$

where $B > 0$ is a large parameter.

Proof. This is the main result in [16], Section 4. □

Lemma 2.2. *For $\alpha \in \mathbb{R}$ and fixed positive integer $k \geqslant 1$, let $\beta_k = \frac{1}{2} + 1/\log k$. We have*

$$\sum_{1\leqslant m\leqslant x} \Lambda(m)e(m^k\alpha) \ll (d(q))^{\beta_k}(\log x)^c\Big(x^{1/2}\sqrt{q(1+|\lambda|x^k)} + x^{4/5} + \frac{x}{\sqrt{q(1+|\lambda|x^k)}}\Big),$$

where $c > 0$ is a positive absolute constant.

Proof. This follows the main result of Ren, see [12]. □

Lemma 2.3 (Hua's Lemma). *Let $k \geqslant 1$ be a positive integer and $1 \leqslant j \leqslant k$. Then*
$$\int_0^1 \biggl|\sum_{n=1}^{N} e(n^k\alpha)\biggr|^{2^j} \,\mathrm{d}\alpha \ll N^{2^j-j+\varepsilon}.$$

P r o o f. This is a classical result given by Hua, see [11]. The reader can also refer to [14], Lemma 2.5. $\square$

3. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. By the trivial estimates of $S_1(\alpha; x^{1/r})$ and Lemma 2.1, we have

$$\begin{aligned}
(3.1)\quad \int_{\mathfrak{M}} S_1^l(\alpha; x^{1/r}) S_2(-\alpha; x)\,\mathrm{d}\alpha &= \sum_{1\leqslant q\leqslant P} \sum_{\substack{1\leqslant a\leqslant q\\ (a,q)=1}} \int_{a/q-1/qQ}^{a/q+1/qQ} S_1^l(\alpha; x^{1/r}) S_2(-\alpha; x)\,\mathrm{d}\alpha \\
&\ll \sum_{1\leqslant q\leqslant P} \sum_{\substack{1\leqslant a\leqslant q\\ (a,q)=1}} \frac{x^{l/r}}{qQ}(xq)^{\varepsilon}(q^{3/2}+q^{2/3}x^{1/3}+q^{1/2}P+q^{-1/2}P^2+x^{-1/4}P^2) \\
&\ll \sum_{1\leqslant q\leqslant P} (x^{l/r-1+\varepsilon}q^{3/2+\varepsilon}P + x^{l/r-2/3+\varepsilon}q^{2/3+\varepsilon}P + x^{l/r-1+\varepsilon}q^{1/2+\varepsilon}P^2 \\
&\quad + x^{l/r-1+\varepsilon}q^{-1/2+\varepsilon}P^3 + x^{l/r-5/4+\varepsilon}P^3) \\
&\ll x^{l/r-1+\varepsilon}P^{7/2} + x^{l/r-2/3+\varepsilon}P^{8/3}
\end{aligned}$$

by substituting $Q = x/(PL^B)$ and noting the definition of P in (2.3) along with the restriction that $l \geqslant 2^{r-1}+1$ for $r \geqslant 2$.

For the estimate of integrals over minor arcs, the main difference between ours and the estimate in paper [16] is that we apply Ren's estimate and Hua's Lemma from Lemmas 2.2 and 2.3, respectively, and in paper [16] they evaluate the estimates from integrals over minor arcs just by elementary method, which cannot be generalized to dealing with the cases for $r > 2$.

Let
$$\overline{S}(\alpha; x^{1/r}) = \sum_{n\leqslant x^{1/r}} \Lambda(n) e(n^r\alpha), \quad \widetilde{S}(\alpha; x^{1/r}) = \sum_{p\leqslant x^{1/r}} (\log p) e(p^r\alpha).$$

Then we have
$$\overline{S}(\alpha; x^{1/r}) = \widetilde{S}(\alpha; x^{1/r}) + O(x^{1/2r+\varepsilon}).$$

By Lemma 2.2 and for $\alpha \in \mathfrak{m}$, we know that
$$P^{1/2-\varepsilon} \leqslant \sqrt{\min\Bigl(P, \frac{x}{Q}\Bigr)} \leqslant \sqrt{q(1+|\lambda|x)} \leqslant \sqrt{Q+\frac{x}{Q}} \leqslant (x/Q)^{1/2+\varepsilon},$$

then we have

$$S_1(\alpha; x^{1/r}) \ll (x^{1/2r+\varepsilon}(x/Q)^{1/2} + x^{4/5r} + x^{1/r}P^{-1/2+\varepsilon}) \ll x^{1/r}P^{-1/2+\varepsilon}. \tag{3.2}$$

Now we estimate the contributions from the integral on the minor arcs. By the Cauchy-Schwarz inequality, Lemma 2.3 and (3.2), we obtain

$$\begin{aligned}
&\int_{\mathfrak{m}} S_1^l(\alpha; x^{1/r}) S_2(-\alpha; x)\,\mathrm{d}\alpha \\
&\quad\ll \max_{\alpha\in\mathfrak{m}} |S_1(\alpha; x^{1/r})|^{l-2^{r-1}} \int_{\mathfrak{m}} |S_1^l(\alpha; x^{1/r})|^{2^{r-1}} |S_2(-\alpha; x)|\,\mathrm{d}\alpha \\
&\quad\ll \max_{\alpha\in\mathfrak{m}} |S_1(\alpha; x^{1/r})|^{l-2^{r-1}} \left(\int_0^1 |S_1(\alpha; x^{1/r})|^{2^r}\,\mathrm{d}\alpha\right)^{1/2} \\
&\qquad\times \left(\int_0^1 |S_2(-\alpha; x)|^2\,\mathrm{d}\alpha\right)^{1/2} \\
&\quad\ll x^{(l-2^{r-1})/r+\varepsilon} P^{-(l-2^{r-1})/2} x^{(2^r-r+\varepsilon)/2r} x^{1/2},
\end{aligned} \tag{3.3}$$

where we have used the simple observation that

$$\int_0^1 \Bigl|\sum_{p\leqslant x} e(p^j\alpha)\Bigr|^{2^j}\,\mathrm{d}\alpha \leqslant \int_0^1 \Bigl|\sum_{n\leqslant x} e(n^j\alpha)\Bigr|^{2^j}\,\mathrm{d}\alpha$$

since the right-hand side counts the number of solutions

$$\sharp\{(n_1,\ldots,n_{2^j})\colon\ n_1^j+\ldots+n_{2^{j-1}}^j = n_{2^{j-1}+1}^j+\ldots+n_{2^j}^j,\ 1\leqslant n_l\leqslant x,\ 1\leqslant l\leqslant 2^j\}, \tag{3.4}$$

whereas the left-hand side counts the number of solutions of (3.4) in prime variables, and the Rankin-Selberg's well-known estimate

$$\int_0^1 |S_2(-\alpha; x)|^2\,\mathrm{d}\alpha = \sum_{1\leqslant n\leqslant x} \lambda_f^2(n) \ll x.$$

Combining (3.1) and (3.3), we have

$$S_{\lambda,r,l}(x) \ll x^{l/r-1+\varepsilon}P^{7/2} + x^{l/r-2/3}P^{8/3} + x^{(l-2^{r-1})/r+\varepsilon}P^{-(l-2^{r-1})/2}x^{(2^r-r+\varepsilon)/2r}x^{1/2}.$$

Let

$$x^{l/r-2/3}P^{8/3} = x^{(l-2^{r-1})/r}P^{-(l-2^{r-1})/2}x^{(2^r-r)/2r}x^{1/2},$$

which is equivalent to

$$P = x^{4/(16+3(l-2^{r-1}))}.$$

Then

$$S_{\lambda,r,l}(x) \ll x^{l/r-2(l-2^{r-1})/(16+3(l-2^{r-1}))+\varepsilon}.$$

Hence, we complete the proof of Theorem 1.1. □

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