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Ying Li-Jun; Qin Feng

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MIGRATIVITY PROPERTIES OF 2-UNINORMS OVER SEMI-T-OPERATORS

Li-Jun Ying and Feng Qin

In this paper, we analyze and characterize all solutions about α-migrativity properties of the five subclasses of 2-uninorms, i. e. C^k, C_k^0, C_k^1, C_1^0, C_0^1, over semi-t-operators. We give the sufficient and necessary conditions that make these α-migrativity equations hold for all possible combinations of 2-uninorms over semi-t-operators. The results obtained show that for $G \in C^k$, the α-migrativity of G over a semi-t-operator $F_{\mu,\nu}$ is closely related to the α-section of $F_{\mu,\nu}$ or the ordinal sum representation of t-norm and t-conorm corresponding to $F_{\mu,\nu}$. But for the other four categories, the α-migrativity over a semi-t-operator $F_{\mu,\nu}$ is fully determined by the α-section of $F_{\mu,\nu}$.

Keywords: 2-uninorms, uninorms, semi-t-operators, triangular norms, triangular conorms

Classification: 03B52,94D05

1. INTRODUCTION

Aggregation functions are usually used in the process of combining and integrating a given number of data into a representative value. This process is indispensable in some step for many fields and this is a reason why aggregation functions are considered as an important tool in many applications, from computer science and mathematics to social sciences and economics. Therefore, in recent decades, there is a significant increase about the interest in aggregation functions, and the publications of some monographs focusing completely on aggregation functions have supported this interest for a long time [4, 6, 11, 15, 33, 37].

Recently, bringing uninorms and nullnorms together, a new class of aggregation functions called 2-uninorms was introduced by Akella [2]. This concept shares the same idea with the ordinal sum of t-norms and generalizes uninorms in such a way that the global neutral element is replaced by two local neutral elements e_1 and e_2. Since then, many scholars have devoted themselves to study of the structural characterization of 2-uninorms. Specifically, some classes of 2-uninorms have been characterized under some appropriate continuity conditions. In 2018, Zong et al. have characterized the structures of all five mutually exclusive classes of 2-uninorms [37]. The most eye-catching is that, in a systematic analysis, the 2-uninorms with continuous underlying operators were obtained by decomposing into the form of ordinal sum [19, 20, 21, 22]. In recent years,

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under the assumption of removing commutativity of t-operators, t-operators and nullnorms wwere generalized and the notion of semi-t-operators was introduced by Drygaś [8]. These operators are interesting and important not only from the theoretical point of view, but also from their applications because they also play an important and essential role in many fields like fuzzy quantifiers, fuzzy logic framework, neural networks or expert system [25]. Meanwhile, based on the important role of semi-t-operator in different fields, a complete characterization concerning it was presented in [8].

In addition to the characterization mentioned above, another research direction of aggregation function is considered from their application aspect. One of the application properties is the migrativity. Investigations of this property are directed towards finding all solutions for different kinds of aggregation operations such as t-norms, t-conorms, uninorms, 2-uninorms, nullnorms or semi-t-operators. As far as we know, the significance of the migrativity property of different aggregation operators stems from its role in different fields, for example, image processing, decision making processes and aggregation of information.

To our best knowledge, the concept of α-migrativity properties of the t-norms was put forward first as an open problem about whether there are strictly monotone t-norm solutions to the migrative equation [16, 17], and then it was formally affirmed by Durante and Sarkoci [7] with the purpose to construct new t-norms by the means of convex combinations about t-norms with this property and the drastic product t-norm T_D. For any $\alpha \in [0,1]$ and a t-norm T, this property is the functional equation

$$T(\alpha x, y) = T(x, \alpha y) \qquad for\ all\ x, y \in [0,1]. \tag{1}$$

The interest of this property not only comes from its applications as mentioned above, but also from the theoretical point of view because it is useful and essential for strcturing a new t-norm by convex combinations between two given ones [7, 28, 29]. Recently, with the significant development of aggregation functions in theory, researchers have also extended this property (Eq.(1)) by replacing the t-norm T and the product αx with more general aggregation operators. In this sense, the most general functional equation about α-migrativity for aggregation functions is described as

$$C(D(\alpha, x), y) = C(x, D(\alpha, y)), \tag{2}$$

where C and D are two aggregation functions, and it is called the (α, D)-migrativity of C or called that C is α-migrative over D. Since then, the α-migrativity of aggregation functions has attracted wider attention of many scholars. One of the most impressive and influential results is probably given by Li et al. [14]. Their results show that the migrative properties of two uninorms with continuous underlying operators is mainly determined by the migrative properties of two representable uninorms or the α-section. Since we are going to deal with 2-uninorm in this work, over here, we only recall some results of migrative property related to 2-uninorms. Indeed, to our best knowledge, only the migrativity involving nullnorms and 2-uninorm has been studied. Specifically speaking, in 2019, Wang et al. [35] analyzed and investigated the migrativity between nullnorms and 2-uninorms. Based on whether the absorbing elements of unllnorms and 2-uninorms are the same, authors gave all solutions of the migrativity equations for every possible combinations between 2-uninorms and nullnorms. Considering the wide application of migrative property and the importance of aggregation operators as already mentioned above, the goal of this paper is to fill the gap mentioned above. In other words, as a supplement of this topic from the theoretical point of view, we will continue to study the migrativity of 2-uninorms. Specifically, we investigate the migrativity property of 2-uninorms over semi-t-operators. We put forward the sufficient and necessary conditions when the equation holds for all possible combinations about 2-uninorms over semi-t-operators. In the process of our research, it is interesting to find that for $G \in C^k$, the α-migrativity of G over a semi-t-operator $F_{\mu,\nu}$ is closely related to the α-section of $F_{\mu,\nu}$ or the ordinal sum representation of t-norm and t-conorm corresponding to $F_{\mu,\nu}$. But for the other four categories, the α-migrativity over a semi-t-operator $F_{\mu,\nu}$ is fully determined by the α-section of $F_{\mu,\nu}$. Since semi-t-operators are generalization of nullnorms, in this sense, the results obtained in this paper generalize the conclusions of Wang [35].

Next, we will specifically add a preliminary section to review the structures of uninorms, 2-uninorms, nullnorms and semi-t-operators after this introduction, which will be used throughout this paper. In section 3, we investigate the migrativity property of 2-uninorms over semi-t-operators for the case $F \in F_{\mu,\nu}$ with $\mu < \nu$ in detail. In section 4, we minutely discuss the migrativity property of 2-uninorms over semi-t-operators for another case, i. e. $F \in F_{\mu,\nu}$ with $\nu < \mu$. Section 5 is the Conclusion.

2. PERLIMINARIES

We will suppose that readers are already familiar with the basic facts and results related to t-norms, t-conorms and uninorms. Meanwhile, we can also find the knowledge about the definitions, notations and results of them in [1, 3, 9, 12, 14]. In this section, we will just give some basic notations and facts of semi-t-operators and 2-uninorms.

Definition 2.1. (Drygaś [8], Klement et al. [12]) A pseudo-t-norm T is a binary operation T: $[0,1]^2 \to [0,1]$ which is increasing, associative and has 1 as the neutral element. A pseudo-t-conorm S is a binary operation S: $[0,1]^2 \to [0,1]$ which is increasing, associative and has 0 as the neutral element.

Obviously, if the commutativity of t-norm (t-conorm) is omitted, we can just get pseudo-t-norm (pseudo-t-conorm) from the above definition.

Definition 2.2. (Drygaś [8]) A *semi-t-operator* is a binary operation F: $[0,1]^2 \to [0,1]$ which is increasing, associative, fulfills F(0,0)=0, F(1,1)=1 and such that the functions F^0, F^1, F_0, F_0 are continuous, where $F^0(x) = F(x,0)$, $F^1(x) = F(x,1)$, $F_0(x) = F(0,x)$, $F_1(x) = (1,x)$. We will use $F_{\mu,\nu}$ to denote the family of all semi-t-operators F satisfing that $F(0,1) = \mu$ and $F(1,0) = \nu$.

Theorem 2.3. (Drygaś [8]) Let F: $[0,1]^2 \to [0,1]$, $F(0,1) = \mu$ and $F(1,0) = \nu$. $F \in F_{\mu,\nu}$ if and only if there are *a pseudo-t-norm T and a pseudo-t-conorm S* such that

$$F(x,y) = \begin{cases} \mu S(\frac{x}{\mu}, \frac{y}{\mu}) & \text{if } x,y \in [0,\mu], \\ \nu + (1-\nu)T(\frac{x-\nu}{1-\nu}, \frac{y-\nu}{1-\nu}) & \text{if } x,y \in [\nu,1], \\ \mu & \text{if } x \le \mu \le y, \\ \nu & \text{if } y \le \nu \le x, \\ x & \text{if } \mu \le x \le \nu, \end{cases} \tag{3}$$

when $\mu \leq \nu$ and

$$F(x,y) = \begin{cases} \nu S(\frac{x}{\nu}, \frac{y}{\nu}) & \text{if } x,y \in [0,\nu], \\ \mu + (1-\mu)T(\frac{x-\mu}{1-\mu}, \frac{y-\mu}{1-\mu}) & \text{if } x,y \in [\mu,1], \\ \mu & \text{if } x \leq \mu \leq y, \\ \nu & \text{if } y \leq \nu \leq x, \\ y & \text{if } \nu \leq y \leq \mu, \end{cases} \tag{4}$$

when $\nu \leq \mu$.

If a semi-t-operator $F \in F_{\mu,\nu}$ satisfies commutativity, then $\mu = \nu$. It means that F becomes a nullnorm with the zero element μ by equations (3) and (4). As a result, the class of the nullnorms is a subclass of the semi-t-operators.

Now, let us take a look back to the definition and some results on 2-uninorms.

Definition 2.4. (Akella [2]) Assume that $0 \leq e_1 \leq k \leq e_2 \leq 1$, where $k \in [0,1]$, $e_1 \in [0,k]$ and $e_2 \in [k,1]$. A binary operation G: $[0,1]^2 \to [0,1]$ is called a 2-uninorm if it is increasing, commutative, associative and fulfills that $G(e_1,y) = y$ for all $y \in [0,k]$ and $G(e_2,y) = y$ for all $y \in [k,1]$.

Obviously, e_1 is the neutral element of G in $[0,k]$ and e_2 is the neutral element of G in $[k,1]$. Thus, for any $k \in (0,1)$, we get easily that the operator works as a uninorm in $[0,k]^2$ or $[k,1]^2$. In this paper, we will denote the underlying uninorm of the 2-uninorm in $[0,k]^2$ as U_1 and in a similar way, the underlying uninorm in $[k,1]^2$ as U_2. Meanwhile, we will use U^2 to denote the class of all 2-uninorms.

Lemma 2.5. (Drygaś [2]) Assume that $0 \leq e_1 \leq k \leq e_2 \leq 1$ and $G \in U^2$, then $G(0,1) \in \{0,k,1\}$.

Based on this fact, we obtain three subclasses of operators in U^2 denoted by C^0, C^k, C^1. Next, we will give the structures of every kind of the 2-uninorms.

Theorem 2.6. (Drygaś [2]) Let $G \in U^2$. If $G(x,1)$ is continuous except at the points $x = e_1$ and $x = e_2$, then $G(0,1) = 0$ and $G(1,k) = k$ if and only if $0 < e_1 \leq k < e_2 \leq 1$ as well as there exist two t-norms T^d, T^c and two t-conorms S^d, S^c such that

$$G(x,y) = \begin{cases} T^d & \text{if } (x,y) \in [0,e_1]^2, \\ S^d & \text{if } (x,y) \in [e_1,k]^2, \\ T^c & \text{if } (x,y) \in [k,e_2]^2, \\ S^c & \text{if } (x,y) \in [e_2,1]^2, \\ k & \text{if } (x,y) \in [e_1,k) \times (k,1] \cup (k,1] \times [e_1,k), \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{5}$$

We denote this kind of all 2-uninorms as C^0_k.

Theorem 2.7. (Drygaś [2]) Let $G \in U^2$. If $G(x,1)$ is continuous except at the point e_1 and $G(x,e_1)$ is continuous except at the point e_2, then $G(0,1) = 0$ and $G(1,k) = 1$ if and only if $0 < e_1 \leq k \leq e_2 < 1$ as well as there exist two t-norms T^d, T^c and two t-conorms S^d, S^c such that

$$G(x,y)=\begin{cases} T^d & \text{if } (x,y)\in[0,e_1]^2,\\ S^d & \text{if } (x,y)\in[e_1,k]^2,\\ T^c & \text{if } (x,y)\in[k,e_2]^2,\\ S^c & \text{if } (x,y)\in[e_2,1]^2,\\ \min(x,y) & \text{if } (x,y)\in[0,e_1)\times(e_1,1]\cup(e_1,1]\times[0,e_1),\\ \max(x,y) & \text{if } (x,y)\in[e_1,e_2)\times(e_2,1]\cup(e_2,1]\times[e_1,e_2),\\ k & \text{if } (x,y)\in[e_1,k)\times(k,e_2]\cup(k,e_2]\times[e_1,k).\end{cases} \tag{6}$$

We use C_1^0 to denote this kind of all 2-uninorms.

Theorem 2.8. (Drygaś [2]) Let $G\in U^2$. If $G(x,0)$ is continuous except at the points e_1 and e_2, then $G(0,1)=1$ and $G(0,k)=k$ if and only if $0\le e_1<k\le e_2<1$ as well as there exist two t-norms T^d, T^c and two t-conorms S^d, S^c such that

$$G(x,y)=\begin{cases} T^d & \text{if } (x,y)\in[0,e_1]^2,\\ S^d & \text{if } (x,y)\in[e_1,k]^2,\\ T^c & \text{if } (x,y)\in[k,e_2]^2,\\ S^c & \text{if } (x,y)\in[e_2,1]^2,\\ k & \text{if } (x,y)\in[0,k)\times(k,e_2]\cup(k,e_2]\times[0,k),\\ \max(x,y) & \text{otherwise}.\end{cases} \tag{7}$$

We denote this kind of all 2-uninorms as C_k^1.

Theorem 2.9. (Drygaś [2]) Let $G\in U^2$. If $G(x,0)$ is continuous except at the point e_2 and $G(x,e_2)$ is continuous except at the point e_1, then $G(0,1)=1$ and $G(0,k)=0$ if and only if $0<e_1\le k\le e_2<1$ as well as there exist two t-norms T^d, T^c and two t-conorms S^d, S^c such that

$$G(x,y)=\begin{cases} T^d & \text{if } (x,y)\in[0,e_1]^2,\\ S^d & \text{if } (x,y)\in[e_1,k]^2,\\ T^c & \text{if } (x,y)\in[k,e_2]^2,\\ S^c & \text{if } (x,y)\in[e_2,1]^2,\\ \min(x,y) & \text{if } (x,y)\in[0,e_1)\times(e_1,e_2]\cup(e_1,e_2]\times[0,e_1),\\ \max(x,y) & \text{if } (x,y)\in[0,e_2)\times(e_2,1]\cup(e_2,1]\times[0,e_2),\\ k & \text{if } (x,y)\in[e_1,k)\times(k,e_2]\cup(k,e_2]\times[e_1,k).\end{cases} \tag{8}$$

We use C_0^1 to denote this kind of all 2-uninorms.

Theorem 2.10. (Drygaś [2]) Let $G\in U^2$. If $G(x,0)$ is continuous except at the point e_1 and $G(x,1)$ is continuous except at the point e_2, then $G(0,1)=k$ if and only if $0\le e_1<k<e_2\le 1$ as well as there exist two t-norms T^d, T^c and two t-conorms S^d, S^c such that

$$G(x,y)=\begin{cases} T^d & \text{if } (x,y)\in[0,e_1]^2,\\ S^d & if\ (x,y)\in[e_1,k]^2,\\ T^c & \text{if } (x,y)\in[k,e_2]^2,\\ S^c & \text{if } (x,y)\in[e_2,1]^2,\\ \max(x,y) & \text{if } (x,y)\in[0,e_1)\times(e_1,k]\cup(e_1,k]\times[0,e_1),\\ \min(x,y) & \text{if } (x,y)\in[k,e_2)\times(e_2,1]\cup(e_2,1]\times[k,e_2),\\ k & \text{if } (x,y)\in[0,k)\times(k,1]\cup(k,1]\times[0,k).\end{cases} \tag{9}$$

We use C^k to denote this kind of all 2-uninorms.

In this paper, we denote the underlying t-norms and underlying t-conorms of a 2-uninorm G as T^d, T^c and S^d, S^c, as well as, the underlying t-norms and underlying t-conorms of $F_{\mu,\nu}$ as T_F and S_F, respectively.

Definition 2.11. Let $\alpha \in (0,1)$, $G \in U^2$ and $F \in F_{\mu,\nu}$. A 2-uninorm G is said to be α-migrative with respect to F or (α, F)-migrative if

$$G(F(\alpha,x),y) = G(x,F(\alpha,y)) \qquad \text{for all } x,y \in [0,1]. \tag{10}$$

Next, we investigate the migrativity property of 2-uninorms over semi-t-operators in detail. Since $F_{\mu,\nu}$ is degenerated into a nullnorm when $\mu = \nu$, we presuppose that $F_{\mu,\nu}$ meet the condition of $\mu \neq \nu$ and μ, $\nu \notin \{0,1\}$. Meanwhile, in order to highlight the unique structure of 2-uninorm and distinguish it from the other aggregation operators, we stipulate $G \in U^2$ meet the condition of $0 < e_1 < k < e_2 < 1$. For analysing and investigating the migrativity properties of 2-uninorms over semi-t-operators, depending on Theorem 2.3, there are two cases to consider: a) $F \in F_{\mu,\nu}$ with $\mu < \nu$, b) $F \in F_{\mu,\nu}$ with $\mu > \nu$.

3. MIGRATIVITY PROPERTIES OF 2-UNINORMS OVER SEMI-T-OPERATORS FOR THE CASE $F \in F_{\mu,\nu}$ WITH $\mu < \nu$

In this section, we mainly investigate the migrativity properties of a 2-uninorm G over a semi-t-operator F for the case $F \in F_{\mu,\nu}$ with $\mu < \nu$, where $G \in C^k \cup C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$. Further, according to the following proof, $C^k \cup C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ are divided into two subclasses: 1) C^k, 2) $C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$. Now, let us discuss the first case.

3.1. $G \in C^k$

Over here, by the order relation of α and μ, ν, we divide the discussion into the following cases: (a) $\alpha \in (0,\mu)$; (b) $\alpha \in [\mu,\nu]$; (c) $\alpha \in (\nu,1)$. Now, we start by discussing the case (a) $\alpha \in (0,\mu)$.

Theorem 3.1. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\mu < \nu$. If $\alpha \in (0,\mu)$, then the following conclusions are obtained.

(i) If $\mu \le e_1 < k$, then G is not (α, F)-migrative.

(ii) If $e_1 < \mu \le k$, then G is (α, F)-migrative if and only if $\mu = k$ and U_1 is $(\frac{\alpha}{k}, S_F)$-migrative.

(iii) If $e_1 < k < \mu$, then G is (α, F)-migrative if and only if

$$F(\alpha,y) = \begin{cases} G(F(\alpha,e_1),y) & \text{if } y \in [0,k], \\ G(\mu,y) & \text{if } y \in [k,1]. \end{cases} \tag{11}$$

P r o o f. Our discussions are divided into the following situations.

- If $\mu \le e_1 < k$, then $\alpha = G(\alpha, e_1) = G(F(\alpha, 0), e_1) = G(0, F(\alpha, e_1)) = G(0, \mu) = 0$, which contradicts with $\alpha \neq 0$.

- If $e_1 < \mu \le k$, then we get $\alpha \in (0, k)$ from $\mu \le k$. Based on this fact, we have $k = G(\alpha, 1) = G(F(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, \mu) = \max(0, \mu) = \mu$. It means that when $\alpha \in (0, \mu)$ and $e_1 < \mu \le k$, it must be that $\mu = k$ since G is (α, F)-migrative. In addition, for $(x, y) \in [0, k]^2$, we have $kU_1(S_F(\frac{\alpha}{k}, \frac{x}{k}), \frac{y}{k}) = kU_1(\frac{kS_F(\frac{\alpha}{k}, \frac{x}{k})}{k}, \frac{y}{k}) = G(F(\alpha, x), y) = G(x, F(\alpha, y)) = kU_1(\frac{x}{k}, \frac{kS_F(\frac{\alpha}{k}, \frac{y}{k})}{k}) = kU_1(\frac{x}{k}, S_F(\frac{\alpha}{k}, \frac{y}{k}))$ since G is (α, F)-migrative. Next, let $x^1 = \frac{x}{k}$ and $y^1 = \frac{y}{k}$, it is obvious that $(x^1, y^1) \in [0, 1]^2$. Thus, we have $U_1(S_F(\frac{\alpha}{k}, x^1), y^1) = U_1(x^1, S_F(\frac{\alpha}{k}, y^1))$ for $(x^1, y^1) \in [0, 1]^2$. So U_1 is $(\frac{\alpha}{k}, S_F)$-migrative. On the contrary, when $\mu = k$ and U_1 is $(\frac{\alpha}{k}, S_F)$-migrative, we easily get G is (α, F)-migrative by a simple calculation.

- If $e_1 < k < \mu$. First of all, we prove that $e_2 > \mu$. If not, i. e. $e_2 \le \mu$, then we obtain that $1 = G(F(\alpha, e_2), 1) = G(e_2, F(\alpha, 1)) = G(e_2, \mu) = \mu$ by $F(\alpha, e_2) \ge e_2$. This is impossible for $\mu \neq 1$. So we get $e_2 > \mu$. In this case, for any $y \in [k, 1]$, we have $G(\mu, y) = G(F(\alpha, e_2), y) = G(e_2, F(\alpha, y)) = F(\alpha, y)$ by $F(\alpha, y) \ge F(\alpha, k) \ge k$. Assume $y = k$, it is straightforward that $F(\alpha, k) = G(\mu, k) = k$. Therefore, when $y \in [0, k]$, we get clearly that $G(F(\alpha, e_1), y) = G(e_1, F(\alpha, y)) = F(\alpha, y)$ by $F(\alpha, y) \le F(\alpha, k) = k$. On the contrary, note that

$$F(\alpha, y) = \begin{cases} G(F(\alpha, e_1), y) & \text{if } y \in [0, k], \\ G(\mu, y) & \text{if } y \in [k, 1]. \end{cases} \tag{12}$$

  Next, our discussions are divided into the following situations.

  – Assume $(x, y) \in [0, k]^2$, then we get $G(F(\alpha, x), y) = G(G(F(\alpha, e_1), x), y) = G(G(x, F(\alpha, e_1)), y) = G(x, G(F(\alpha, e_1), y)) = G(x, F(\alpha, y))$.

  – Assume $(x, y) \in [k, 1]^2$, then it is obvious that $G(F(\alpha, x), y) = G(G(\mu, x), y) = G(G(x, \mu), y) = G(x, G(\mu, y)) = G(x, F(\alpha, y))$.

  – Assume $(x, y) \in [0, k] \times [k, 1]$, we get $G(F(\alpha, x), y) = G(G(F(\alpha, e_1), x), y) = G(x, G(F(\alpha, e_1), y))$ and $G(x, F(\alpha, y)) = G(x, G(\mu, y)) = G(y, G(\mu, x))$. Note that $G(F(\alpha, e_1), y) \ge G(F(\alpha, e_1), k) = k$, by the structure of $G \in C^k$, we obtain that $G(x, G(F(\alpha, e_1), y)) = k = G(y, G(\mu, x)) = G(y, k) = k$.

  – Assume $(y, x) \in [0, k] \times [k, 1]$, then it holds $G(F(\alpha, x), y) = G(G(\mu, x), y) = G(x, G(\mu, y))$ and $G(x, F(\alpha, y)) = G(x, G(F(\alpha, e_1), y)) = G(y, G(F(\alpha, e_1), x))$. Note that $G(F(\alpha, e_1), x) \ge G(F(\alpha, e_1), k) = k$, from the structure of $G \in C^k$, we get $G(x, G(\mu, y)) = G(x, k) = k = G(y, G(F(\alpha, e_1), x))$.

□

Remark 3.2. For the above Theorem 3.1 (ii), the migrativity for uninorms over t-conorms has already been investigated by Mas et al. in [26, 27].

Similar to Theorem 3.1, we get the following theorem for the case (c), i. e. $\alpha \in (\nu, 1)$.

Theorem 3.3. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\mu < \nu$. If $\alpha \in (\nu, 1)$, then the following conclusions are obtained.

(i) If $k < e_2 \leq \nu$, then G is not (α, F)-migrative.

(ii) If $k \leq \nu < e_2$, then G is (α, F)-migrative if and only if $\nu = k$ and U_2 is $(\frac{\alpha-k}{1-k}, T_F)$-migrative.

(iii) If $\nu < k < e_2$, then G is (α, F)-migrative if and only if

$$F(\alpha, y) = \begin{cases} G(\nu, y) & \text{if } y \in [0, k], \\ G(F(\alpha, e_2), y) & \text{if } y \in [k, 1]. \end{cases} \tag{13}$$

Remark 3.4. For the above Theorem 3.3 (ii), the migrativity for uninorms over t-norms has already been investigated by Mas et al. in [26, 27].

In the end, we discuss the case $\alpha \in [\mu, \nu]$.

Theorem 3.5. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\mu < \nu$. If $\alpha \in [\mu, \nu]$, then G is (α, F)-migrative if and only if $\alpha = k$.

Proof. First, let us assert that $\alpha > k$ is impossible. Otherwise, by the fact that G is (α, F)-migrative, we have $\alpha = G(\alpha, e_2) = G(F(\alpha, k), e_2) = G(k, F(\alpha, e_2)) = G(k, \alpha) = k$. But this contradicts with the assumption that $\alpha > k$. Further, we claim that $\alpha < k$ is also impossible. Indeed, since G is (α, F)-migrative, we have $\alpha = G(\alpha, e_1) = G(F(\alpha, k), e_1) = G(k, F(\alpha, e_1)) = G(k, \alpha) = k$, which also contradicts with the assumption that $\alpha < k$. Therefore, it must be that $\alpha = k$.

On the contrary, when $\alpha = k$, we know G is (α, F)-migrative by a simple calculation. □

For the following discussions, we stipulate that G belongs to one of the other four classes, i. e. $G \in C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$.

3.2. $G \in C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$

Over here, by the order relation of α and μ, ν, we divide the discussion into the following cases: (a) $\alpha \in (0, \mu)$; (b) $\alpha \in [\mu, \nu]$; (c) $\alpha \in (\nu, 1)$. Now, we start by discussing the case (a) $\alpha \in (0, \mu)$.

Theorem 3.6. Let $G \in C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ and $F \in F_{\mu,\nu}$ with $\mu < \nu$. If $\alpha \in (0, \mu)$, then the following conclusions are obtained.

(i) If $\mu \leq e_1 < k$, then G is not (α, F)-migrative.

(ii) If $e_1 < \mu \leq k$, then G is not (α, F)-migrative.

(iii) If $e_1 < k < \mu$, then G is (α, F)-migrative if and only if $e_2 > \mu$ and

$$F(\alpha, y) = \begin{cases} G(F(\alpha, e_1), y) & \text{if } y \in [0, k], \\ G(\mu, y) & \text{if } y \in [k, 1]. \end{cases} \tag{14}$$

Proof. Let us prove these results one by one.

- Suppose that $\mu \le e_1 < k$. Similar to proof of the case $G \in C^k$ in Theorem 3.1 (i), we have $G \in C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ is not (α, F)-migrative.

- Suppose that $e_1 < \mu \le k$ and G is (α, F)-migrative. We discuss this subcase by two steps.

  – If $G \in C_k^0$. Considering $\alpha \in (0, \mu)$ and $e_1 < \mu \le k$, then we have $G(F(\alpha, 0), 1) = G(\alpha, 1) \in \{\alpha, k\}$. However, from structure of C_k^0, it holds that $G(0, F(\alpha, 1)) = G(0, \mu) = 0$, which contradicts with the fact that $\alpha \neq 0$ and $k \neq 0$.

  – For the other three cases, i. e. $G \in C_1^0 \cup C_k^1 \cup C_0^1$. From the basic facts that $G(x, y) = \max(x, y)$ for any $(x, y) \in [e_1, k] \times (e_2, 1]$ and $F(\alpha, e_1) \in [e_1, k]$, we get easily $G(F(\alpha, e_1), 1) = \max(F(\alpha, e_1), 1) = 1$. On the other hand, we have $G(e_1, F(\alpha, 1)) = G(e_1, \mu) = \mu$, which contradicts with the fact that $\mu \neq 1$.

  So, we get the conclusion by summarizing the discussions above.

- Suppose that $e_1 < k < \mu$. Similar to the case $G \in C^k$ in Theorem 3.1 (iii), we obtain the necessity is true for all the four categories.

  To see the sufficiency, we get it by a simple calculation.

□

Similarly to the results above, we can obtain the following theorem.

Theorem 3.7. Let $G \in C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ and $F \in F_{\mu,\nu}$ with $\mu < \nu$. If $\alpha \in (\nu, 1)$, then the following conclusions are obtained.

(i) If $k < e_2 \le \nu$, then G is not (α, F)-migrative.

(ii) If $k \le \nu < e_2$, then G is not (α, F)-migrative.

(iii) If $\nu < k < e_2$, then G is (α, F)-migrative if and only if $e_1 < \nu$ and

$$F(\alpha, y) = \begin{cases} G(\nu, y) & \text{if } y \in [0, k], \\ G(F(\alpha, e_2), y) & \text{if } y \in [k, 1]. \end{cases} \tag{15}$$

In the end, we investigate the case (b), i. e. $\alpha \in [\mu, \nu]$.

Theorem 3.8. Let $G \in C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ and $F \in F_{\mu,\nu}$ with $\mu < \nu$. If $\alpha \in [\mu, \nu]$, then G is not (α, F)-migrative.

Proof. Assume that $G \in C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ and G is (α, F)-migrative. Then we have $\alpha = G(\alpha, e_1) = G(F(\alpha, y), e_1) = G(y, F(\alpha, e_1)) = G(y, \alpha) = G(\alpha, y)$ for any fixed $\alpha \in [\mu, \nu]$ and $\alpha \le k$. Meanwhile, it also holds that $\alpha = G(\alpha, e_2) = G(F(\alpha, y), e_2) = G(y, F(\alpha, e_2)) = G(y, \alpha) = G(\alpha, y)$ for any fixed $\alpha \in [\mu, \nu]$ and $\alpha > k$. So, these two facts imply that $\alpha = G(\alpha, y)$ for any $y \in [0, 1]$ when $\alpha \in [\mu, \nu]$ and G is (α, F)-migrative. Next, to complete our proof, we need consider the following cases.

- If $G \in C_k^0 \cup C_1^0$. From the structure of $G \in C_k^0 \cup C_1^0$, we have $G(x,y) = \min(x,y)$ for any $(x,y) \in [0,1] \times [0,e_1)$, then we have $\alpha = G(\alpha,0) = 0$ for all $\alpha \in [0,1]$, and it is clearly true for all $\alpha \in [\mu,\nu]$, which contradicts with $\alpha \neq 0$.
- If $G \in C_k^1 \cup C_0^1$. From the structure of $G \in C_k^1 \cup C_0^1$, we have $G(x,y) = \max(x,y)$ for any $(x,y) \in [0,1] \times (e_2,1]$, and then we have $\alpha = G(\alpha,1) = 1$ for all $\alpha \in [0,1]$, and it is obviously true for all $\alpha \in [\mu,\nu]$, which contradicts with $\alpha \neq 1$.

Summing up, we get the desired conclusion. □

Next, let us analyse and investigate the migrativity properties of 2-uninorms over semi-t-operators for the case $F \in F_{\mu,\nu}$ with $\nu < \mu$.

4. MIGRATIVITY PROPERTIES OF 2-UNINORMS OVER SEMI-T-OPERATORS FOR THE CASE $F \in F_{\mu,\nu}$ WITH $\nu < \mu$

Over here, we mainly analyse and investigate migrativity properties of 2-uninorms G over semi-t-operators F for the case $F \in F_{\mu,\nu}$ with $\nu < \mu$, where $G \in C^k \cup C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$. Firstly, we introduce the following lemma.

Lemma 4.1. Let $G \in C^k \cup C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If G is (α,F)-migrative, then it must be $e_1 < \nu$ and $e_2 > \mu$.

Proof. Firstly, we prove $e_1 < \nu$. Otherwise, we have $e_1 \geq \nu$. Then since G is (α,F)-migrative and the structure of F and G, it holds that $F(\alpha,e_1) \leq e_1$ for any $\alpha \in (0,1)$ and $0 = G(F(\alpha,e_1),0) = G(e_1,F(\alpha,0))$. Depending on the value of $G(e_1,F(\alpha,0))$, we need consider the following situations. If $\alpha \in (0,\nu]$, then we obtain $G(e_1,F(\alpha,0)) = G(e_1,\alpha) = \alpha$ by $\alpha \leq \nu \leq e_1$, which is a contradition. If $\alpha \in [\nu,1)$, then we obtain $G(e_1,F(\alpha,0)) = G(e_1,\nu) = \nu$, which is also a contradiction. So, it must be $e_1 < \nu$ if G is (α,F)-migrative for any $\alpha \in (0,1)$.

Next, we prove that $e_2 > \mu$. On the contrary, suppose that $e_2 \leq \mu$. Then since G is (α,F)-migrative and the structure of F and G, it holds that $F(\alpha,e_2) \geq e_2$ for any $\alpha \in (0,1)$ and $1 = G(F(\alpha,e_2),1) = G(e_2,F(\alpha,1))$. According the value of $G(e_2,F(\alpha,1))$, we divide our discussion into the following situations. If $\alpha \in (0,\mu]$, then we obtain $G(e_2,F(\alpha,1)) = G(e_2,\mu) = \mu$, which is a contradition. If $\alpha \in [\mu,1)$, then we obtain $G(e_2,F(\alpha,1)) = G(e_2,\alpha) = \alpha$ by $\alpha \geq \mu \geq e_2$, which is also a contradiction. So, if G is (α,F)-migrative for any $\alpha \in (0,1)$, then it must be $e_2 > \mu$. □

According to the following proof, $C^k \cup C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ are divided into two subclasses: 1) C^k, 2) $C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$. Now, let us discuss the first case.

4.1. $G \in C^k$

Note that $G \in C^k \cup C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If G is (α,F)-migrative, then from Lemma 4.1, we know that it must be $e_1 < \nu$ and $e_2 > \mu$. Therefore, in the following, we always start our discussions with the assumptions of $e_1 < \nu$ and $e_2 > \mu$, i.e. $0 < e_1 < \nu < \mu < e_2 < 1$. On the other hand, in virtue of the possible position of k, there are three cases to discuss, i. e. (1) $k \in (e_1, \nu]$; (2) $k \in (\nu, \mu)$; (3) $k \in [\mu, e_2)$. Hence, we know that there only exists three corresponding order relations among e_1, k, e_2, ν and μ as following: (a) $0 < e_1 < k \le \nu < \mu < e_2 < 1$; (b) $0 < e_1 < \nu < k < \mu < e_2 < 1$; (c) $0 < e_1 < \nu < \mu \le k < e_2 < 1$.

Now, we start with the case (a) $0 < e_1 < k \le \nu < \mu < e_2 < 1$.

Proposition 4.2. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < k \le \nu < \mu < e_2 < 1$ and $\alpha \in (0, e_1] \cup [e_2, 1)$, then G is not (α, F)-migrative.

Proof. Since proof of the case $\alpha \in [e_2, 1)$ is similar to that of the case $\alpha \in (0, e_1]$, we just prove the case $\alpha \in (0, e_1]$ over here. At present, we start with the assumptions of G is (α, F)-migrative and $\alpha \in (0, e_1]$. Then we have $\alpha = G(\alpha, e_1) = G(F(\alpha, 0), e_1) = G(0, F(\alpha, e_1)) = \{0, k, F(\alpha, e_1)\}$ from $e_1 \le F(\alpha, e_1) \le \nu < e_2$. Note that $\alpha = k$ or $\alpha = 0$ is impossible because it contradicts with the assumption of $\alpha \in (0, e_1]$, it means that $\alpha = F(\alpha, e_1)$. In this case, we get $\alpha = G(0, F(\alpha, e_1)) = G(0, \alpha) = 0$, which contradicts with $\alpha \neq 0$. So, G is not (α, F)-migrative. □

Proposition 4.3. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < k \le \nu < \mu < e_2 < 1$ and $\alpha \in (e_1, k]$, then G is (α, F)-migrative if and only if

$$F(\alpha, y) = \begin{cases} G(\alpha, y) & \text{if } y \in [0, k], \\ G(\mu, y) & \text{if } y \in [k, 1]. \end{cases} \tag{16}$$

Proof. Firstly, assume that G is (α, F)-migrative, then, for $\alpha \in (e_1, k]$ and $y \in [k, 1]$, it must be $F(\alpha, y) = G(F(\alpha, y), e_2) = G(y, F(\alpha, e_2)) = G(y, \mu) = G(\mu, y)$ from $F(\alpha, y) \ge F(\alpha, k) \ge k$. Based on this fact, we have $F(\alpha, k) = G(\mu, k) = k$ by taking $y = k$. It means that $e_1 \le F(\alpha, e_1) \le F(\alpha, k) = k$. Therefore, it holds $\alpha = G(\alpha, e_1) = G(F(\alpha, 0), e_1) = G(0, F(\alpha, e_1)) = \{0, F(\alpha, e_1)\}$ because of $F(\alpha, e_1) \in [e_1, k]$, and then we get $\alpha = F(\alpha, e_1)$ because $\alpha \neq 0$. Further, for any $y \in [0, k]$, we have $F(\alpha, y) = G(F(\alpha, y), e_1) = G(y, F(\alpha, e_1)) = G(y, \alpha) = G(\alpha, y)$ from $F(\alpha, y) \le F(\alpha, k) = k$.

Conversely, it is a simple calculation. □

Proposition 4.4. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < k \le \nu < \mu < e_2 < 1$ and $\alpha \in (k, \nu]$, then G is not (α, F)-migrative.

Proof. Assume G is (α, F)-migrative, then we have $\alpha = G(\alpha, e_2) = G(F(\alpha, 0), e_2) = G(0, F(\alpha, e_2)) = G(0, \mu) = k$ by $\mu \in (k, e_2)$. This contradicts with the assumption of $\alpha \in (k, \nu]$. □

Proposition 4.5. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < k \le \nu < \mu < e_2 < 1$ and $\alpha \in (\nu, e_2)$, then G is (α, F)-migrative if and only if $F(\alpha, y) = G(F(\alpha, e_2), y)$ for all $y \in [0, 1]$.

Proof. If G is (α, F)-migrative. Then from $F(\alpha, e_2) \in (k, e_2]$, we get $k = G(F(\alpha, e_2), y) = G(e_2, F(\alpha, y)) = G(e_2, \nu) = \nu$ for any $y \in [0, k]$. It means that $\nu = k$ from G is (α, F)-migrative. Therefore, we have $F(\alpha, y) = \nu = k = G(F(\alpha, e_2), y)$ for any $y \in [0, k]$. For any $y \in [k, 1]$, we have $F(\alpha, y) = G(F(\alpha, y), e_2) = G(y, F(\alpha, e_2)) = G(F(\alpha, e_2), y)$ from $F(\alpha, y) \geq F(\alpha, k) = \nu = k$.

Conversely, we have $G(F(\alpha, x), y) = G(G(F(\alpha, e_2), x), y) = G(G(x, F(\alpha, e_2)), y) = G(x, G(F(\alpha, e_2), y)) = G(x, F(\alpha, y))$ for any $(x, y) \in [0, 1]^2$. □

Proposition 4.6. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. Suppose that T^c and T_F are continuous. If $0 < e_1 < k \leq \nu < \mu < e_2 < 1$ and $\alpha \in [\mu, e_2)$, then the following conclusions are obtained.

(i) If α is an idempotent element of F, then G is (α, F)-migrative if and only if $\nu = k$ and $G(\alpha, \alpha) = \alpha$.

(ii) If α is not an idempotent element of F, then G is (α, F)-migrative if and only if $\nu = k$ and T^c and T_F have ordinal sums of the form $T^c = (...\langle \frac{a-k}{e_2-k}, \frac{b-k}{e_2-k}, T^c_A\rangle ...)$ and $T_F = (...\langle \frac{a-\mu}{1-\mu}, \frac{b-\mu}{1-\mu}, T_{FA}\rangle ...)$, where $a, b \in [\mu, e_2]$, T^c_A and T_{FA} are continuous Archimedean t-norms such that T^c_A is $(\frac{\alpha-a}{b-a}, T_{FA})$-migrative.

Proof. Let us prove these conclusions one by one.

- If α is an idempotent element of F, then we have $F(\alpha, e_2) \in (k, e_2]$ for any fixed $\alpha \in [\mu, e_2)$. Further, from the structure of G and F as well as proof of Proposition 4.5, we get $\nu = k$ and $\alpha = G(\alpha, e_2) = G(F(\alpha, 1), e_2) = G(1, F(\alpha, e_2)) = \{1, F(\alpha, e_2)\}$ for $\alpha \in [\mu, e_2)$. Furthermore, we get $\alpha = F(\alpha, e_2)$ because of $\alpha \neq 1$. Then from Proposition 4.5, we have $F(\alpha, y) = G(F(\alpha, e_2), y) = G(\alpha, y)$ for any $y \in [0, 1]$. In particular, let $y = \alpha$, we have $G(\alpha, \alpha) = \alpha$. Conversely, since α is the idempotent element of G and F together with continuity of T^c and T_F, we know T^c and T_F have ordinal sums of the form

$$T^c = \Big(\langle 0, \frac{\alpha - k}{e_2 - k}, T^{c1}\rangle, \langle \frac{\alpha - k}{e_2 - k}, 1, T^{c2}\rangle\Big)$$

and

$$T_F = \Big(\langle 0, \frac{\alpha - \mu}{1 - \mu}, T^1_F\rangle, \langle \frac{\alpha - \mu}{1 - \mu}, 1, T^2_F\rangle\Big),$$

where T^{c1}, T^{c2} and T^1_F, T^2_F are continuous t-norms. From $\nu = k$ as well as the structure of G and F, we get easily that $F(\alpha, y) = G(\alpha, y)$ for any $y \in [0, 1]$. Therefore, it is obvious that G is (α, F)-migrative.

- If α is not an idempotent element of F. By proof the above item (i), we know for any fixed $\alpha \in [\mu, e_2)$, if G is (α, F)-migrative, then it holds $F(\alpha, y) = G(\alpha, y)$ for any $y \in [0, 1]$ and $\nu = k$. Let $y = \alpha$, then we have $G(\alpha, \alpha) = F(\alpha, \alpha) < \alpha$. Since T^c and T_F are continuous, T^c and T_F have ordinal sums of the following form

$$T^c = (...\langle \frac{a - k}{e_2 - k}, \frac{b - k}{e_2 - k}, T^c_A\rangle ...)$$

and

$$T_F = (...\langle \frac{a^1 - \mu}{1 - \mu}, \frac{b^1 - \mu}{1 - \mu}, T_{FA}\rangle ...),$$

where $a, b \in [k, e_2]$, $a^1, b^1 \in [\mu, 1]$, T_A^c and T_{FA} are continuous Archimedean t-norms. Next, let us prove $a = a^1$ and $b = b^1$. Over here, we just prove $a = a^1$ because the case of $b = b^1$ is similar. Firstly, we prove $a = a^1$. Otherwise, suppose that $a \neq a^1$, then there two cases to consider. If $a < a^1$, then, taking $y \in (a, \min\{a^1, b\})$, y meets the condition of $y \in (a, b)$ and $\nu = k \leq a < y < a^1$, and then we have $G(\alpha, y) < y$ since T_A^c is continuously Archimedean. On the other hand, we also have $F(\alpha, y) = y$, which is a contradiction. If $a > a^1$, then taking $y \in (a^1, \min\{a, b^1\})$, y meets the condition of $y \in (a^1, b^1)$ and $k < \mu \leq a^1 < y < a$, and then we have $F(\alpha, y) < y$ because T_{FA} is continuously Archimedean. On the other hand, it also holds that $G(\alpha, y) = y$, which is a contradiction. So we get $a = a^1$. Similarly, we have $b = b^1$ and $a, b \in [\mu, e_2]$. Finally, from $F(\alpha, y) = G(\alpha, y)$ for any $y \in [0, 1]$, we have $T_A^c(\frac{\alpha-a}{b-a}, \frac{x-a}{b-a}) = T_{FA}(\frac{\alpha-a}{b-a}, \frac{x-a}{b-a})$ for any $x \in [a, b]$, which leads to $T_A^c(\frac{\alpha-a}{b-a}, x) = T_{FA}(\frac{\alpha-a}{b-a}, x)$ for any $x \in [0, 1]$. Therefore, we obtain T_A^c is $(\frac{\alpha-a}{b-a}, T_{FA})$-migrative. Conversely, from $\nu = k$ as well as the structure of F and G, we obtain that $F(\alpha, y) = G(\alpha, y)$ for any $y \in [0, 1]$. Based on this fact, it is obvious that G is (α, F)-migrative.

□

The following theorem is a summary of Propositions 4.2, 4.3, 4.4, 4.5, 4.6.

Theorem 4.7. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $k \in (e_1, \nu]$, i.e. $0 < e_1 < k \leq \nu < \mu < e_2 < 1$, then the following conclusions are obtained.

(i) If $\alpha \in (0, e_1] \cup (k, \nu] \cup [e_2, 1)$, then G is not (α, F)-migrative.

(ii) If $\alpha \in (\nu, e_2)$, then G is (α, F)-migrative if and only if $F(\alpha, y) = G(F(\alpha, e_2), y)$ for any $y \in [0, 1]$. In particular, if $\alpha \in [\mu, e_2)$ as well as T^c and T_F are continuous, then we have the following statements are ture.

  (a) If α is an idempotent element of F, then G is (α, F)-migrative if and only if $\nu = k$ and $G(\alpha, \alpha) = \alpha$.

  (b) If α is not an idempotent element of F, then G is (α, F)-migrative if and only if $\nu = k$, T^c and T_F have ordinal sums of the form $T^c = (...\langle \frac{a-k}{e_2-k}, \frac{b-k}{e_2-k}, T_A^c\rangle...)$ and $T_F = (...\langle \frac{a-\mu}{1-\mu}, \frac{b-\mu}{1-\mu}, T_{FA}\rangle...)$, where $a, b \in [\mu, e_2]$, T_A^c and T_{FA} are continuous Archimedean t-norms such that T_A^c is $(\frac{\alpha-a}{b-a}, T_{FA})$-migrative.

(iii) If $\alpha \in (e_1, k]$, then G is (α, F)-migrative if and only if

$$F(\alpha, y) = \begin{cases} G(\alpha, y) & \text{if } y \in [0, k], \\ G(\mu, y) & \text{if } y \in [k, 1]. \end{cases} \tag{17}$$

Analogously to the previous case, we have the following results for the case (c) $0 < e_1 < \nu < \mu \leq k < e_2 < 1$.

Theorem 4.8. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $k \in [\mu, e_2)$, i.e. $0 < e_1 < \nu < \mu \leq k < e_2 < 1$, then the following conclusions are obtained.

(i) If $\alpha \in (0, e_1] \cup [\mu, k) \cup [e_2, 1)$, then G is not (α, F)-migrative.

(ii) If $\alpha \in (e_1, \mu)$, then G is (α, F)-migrative if and only if $F(\alpha, y) = G(F(\alpha, e_1), y)$ for any $y \in [0, 1]$. In particular, if $\alpha \in (e_1, \nu]$ as well as S^d and S_F are continuous, then the following statements are true.

   (a) If α is an idempotent element of F, then G is (α, F)-migrative if and only if $\mu = k$, $G(\alpha, \alpha) = \alpha$.

   (b) If α is not an idempotent element of F, then G is (α, F)-migrative if and only if $\mu = k$ and S^d and S_F have ordinal sums of the form $S^d = (...\langle \frac{a-e_1}{k-e_1}, \frac{b-e_1}{k-e_1}, s^d_A\rangle ...)$ and $S_F = (...\langle \frac{a}{\nu}, \frac{b}{\nu}, S_{FA}\rangle ...)$, where $a, b \in [e_1, \nu]$, S^d_A and S_{FA} are continuous Archimedean t-conorms such that S^d_A is $(\frac{\alpha-a}{b-a}, S_{FA})$-migrative.

(iii) If $\alpha \in [k, e_2)$, then G is (α, F)-migrative if and only if

$$F(\alpha, y) = \begin{cases} G(\nu, y) & \text{if } y \in [0, k], \\ G(\alpha, y) & \text{if } y \in [k, 1]. \end{cases} \tag{18}$$

Finally, let us discuss the case (b) $0 < e_1 < \nu < k < \mu < e_2 < 1$.

Proposition 4.9. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < \nu < k < \mu < e_2 < 1$ and $\alpha \in (0, e_1] \cup [e_2, 1)$, then G is not (α, F)-migrative.

Proof. Over here, we just prove the case $\alpha \in (0, e_1]$ because proof of the case $\alpha \in [e_2, 1)$ is similar. Now, let us start with the assumptions that G is (α, F)-migrative and $\alpha \in (0, e_1]$. In this case, we have $\alpha = G(\alpha, e_1) = G(F(\alpha, 0), e_1) = G(0, F(\alpha, e_1)) = \{0, F(\alpha, e_1)\}$ by $e_1 \leq F(\alpha, e_1) \leq \nu < k$. On the other hand, we get $\alpha = F(\alpha, e_1)$ because of $\alpha \neq 0$. Based on this fact, we have $\alpha = G(0, F(\alpha, e_1)) = G(0, \alpha) = 0$, which contradicts with the fact that $\alpha \neq 0$. □

Proposition 4.10. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. Suppose that S^d and S_F are continuous. If $0 < e_1 < \nu < k < \mu < e_2 < 1$ and $\alpha \in (e_1, \nu]$, then the following statements are true.

(i) If α is an idempotent element of F, then G is (α, F)-migrative if and only if $G(\alpha, \alpha) = \alpha$ and $G(\mu, y) = \mu$ for any $y \in (\mu, e_2]$.

(ii) If α is not an idempotent element of F, then G is (α, F)-migrative if and only if $G(\mu, y) = \mu$ for any $y \in (\mu, e_2]$, as well as, S^d and S_F have ordinal sums of the form $S^d = (...\langle \frac{a-e_1}{k-e_1}, \frac{b-e_1}{k-e_1}, S^d_A\rangle ...)$ and $S_F = (...\langle \frac{a}{\nu}, \frac{b}{\nu}, S_{FA}\rangle ...)$, where $a, b \in [e_1, \nu]$, S^d_A and S_{FA} are continuous Archimedean t-conorms such that S^d_A is $(\frac{\alpha-a}{b-a}, S_{FA})$-migrative.

Proof. We prove the results one by one.

- If α is an idempotent element of F. We start our discussion with the assumptions that G is (α, F)-migrative and $\alpha \in (e_1, \nu]$. Then we have $F(\alpha, k) = k$ by the structure of F and $k \in (\nu, \mu)$. So, for any $y \in [0, k]$, we obtain $F(\alpha, y) = G(F(\alpha, y), e_1) = G(y, F(\alpha, e_1))$ by $F(\alpha, y) \leq F(\alpha, k) = k$. In particular, let $y = 0$, then it holds that $\alpha = F(\alpha, 0) = G(0, F(\alpha, e_1)) = \{0, F(\alpha, e_1)\}$ by $e_1 \leq F(\alpha, e_1) \leq \nu < k$. On the other hand, we get $\alpha = F(\alpha, e_1)$ because of $\alpha \neq 0$. Therefore, we have $F(\alpha, y) = G(y, F(\alpha, e_1)) = G(y, \alpha) = G(\alpha, y)$ for any $y \in [0, k]$. Further, let $y = \alpha$, we get $G(\alpha, \alpha) = F(\alpha, \alpha) = \alpha$ because of $\alpha \leq \nu < k$. Thus, for any $y \in (\mu, e_2]$, it follows that $G(\mu, y) = G(F(\alpha, 1), y) = G(1, F(\alpha, y)) = G(1, \mu) = \mu$ by $k < \mu < e_2$. Conversely, since α is an idempotent element of F and G, as well as, S^d and S_F are continuous, we get easily that S^d and S_F have ordinal sums of the form
$$S^d = (\langle 0, \frac{\alpha - e_1}{k - e_1}, S^{d1}\rangle, \langle \frac{\alpha - e_1}{k - e_1}, 1, S^{d2}\rangle)$$
and
$$S_F = (\langle 0, \frac{\alpha}{\nu}, S_F^1\rangle, \langle \frac{\alpha}{\nu}, 1, S_F^2\rangle),$$
where S^{d1}, S^{d2} and S_F^1, S_F^2 are continuous t-conorms. On the other hand, we get that $F(\alpha, y) = G(\alpha, y)$ for any $y \in [0, k]$ from the structure of G and F. Based on this fact, from $G(\mu, y) = \mu$ for any $y \in (\mu, e_2]$, we have that G is (α, F)-migrative by a simple verification.
- If α is not an idempotent element of F. Similarly to proof of item (i) above, we have $G(\mu, y) = \mu$ for any $y \in (\mu, e_2]$ and $F(\alpha, y) = G(\alpha, y)$ for any $y \in [0, k]$. So, $G(\alpha, \alpha) = F(\alpha, \alpha) > \alpha$ is ture. It means that α is not an idempotent element of F and G. similarly to proof of Proposition 4.6 (ii), from the conditions that $F(\alpha, y) = G(\alpha, y)$ for any $y \in [0, k]$ as well as S^d and S_F are continuous, we obtain the desired conclusion. Conversely, from the structure of G and F, we also get that $F(\alpha, y) = G(\alpha, y)$ for any $y \in [0, k]$. Further, from $G(\mu, y) = \mu$ for any $y \in (\mu, e_2]$, we obtain easily that G is (α, F)-migrative by a simple calculation.

□

Proposition 4.11. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. Suppose that T^c and T_F are continuous. If $0 < e_1 < \nu < k < \mu < e_2 < 1$ and $\alpha \in [\mu, e_2)$, then the following statements are true.

(i) If α is an idempotent element of F, then G is (α, F)-migrative if and only if $G(\alpha, \alpha) = \alpha$ and $G(\nu, y) = \nu$ for any $y \in [e_1, \nu)$.

(ii) If α is not an idempotent element of F, then G is (α, F)-migrative if and only if $G(\nu, y) = \nu$ for any $y \in [e_1, \nu)$, as well as, T^c and T_F have ordinal sums of the form $T^c = (...\langle \frac{a-k}{e_2-k}, \frac{b-k}{e_2-k}, T_A^c\rangle...)$ and $T_F = (...\langle \frac{a-\mu}{1-\mu}, \frac{b-\mu}{1-\mu}, T_{FA}\rangle...)$, where $a, b \in [\mu, e_2]$, T_A^c and T_{FA} are continuous Archimedean t-norms such that T_A^c is $(\frac{\alpha-a}{b-a}, T_{FA})$-migrative.

P r o o f . We omit the proof since it is similar to that of Proposition 4.10. □

In the end, we deal with the case $\alpha \in (\nu, \mu)$.

Proposition 4.12. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < \nu < k < \mu < e_2 < 1$ and $\alpha \in (\nu, \mu)$, then G is (α, F)-migrative if and only if

$$F(\alpha, y) = \begin{cases} G(\nu, y) & \text{if } y \in [0, k], \\ G(\mu, y) & \text{if } y \in [k, 1]. \end{cases} \tag{19}$$

Proof. We start with the assumptions that G is (α, F)-migrative and $\alpha \in (\nu, \mu)$. For any $\alpha \in (\nu, \mu)$, we get $F(\alpha, k) = k$ by the structure of F and $\nu < k < \mu$. If $y \in [0, k]$, then by $F(\alpha, y) \leq F(\alpha, k) = k$, we have $F(\alpha, y) = G(F(\alpha, y), e_1) = G(y, F(\alpha, e_1)) = G(y, \nu) = G(\nu, y)$. If $y \in [k, 1]$, then by $F(\alpha, y) \geq F(\alpha, k) = k$, we have $F(\alpha, y) = G(F(\alpha, y), e_2) = G(y, F(\alpha, e_2)) = G(y, \mu) = G(\mu, y)$.

Conversely, we get it by a simple calculation. □

The following theorem is a summary of Propositions 4.9, 4.10, 4.11, 4.12.

Theorem 4.13. Let $G \in C^k$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $k \in (\nu, \mu)$, i. e. $0 < e_1 < \nu < k < \mu < e_2 < 1$, then the following statements are true.

(i) If $\alpha \in (0, e_1] \cup [e_2, 1)$, then G is not (α, F)-migrative.

(ii) If $\alpha \in (e_1, \nu]$, S^d and S_F are continuous. then we have the following statements.

(a) If α is an idempotent element of F, then G is (α, F)-migrative if and only if $G(\alpha, \alpha) = \alpha$ and $G(\mu, y) = \mu$ for any $y \in (\mu, e_2]$.

(b) If α is not an idempotent element of F, then G is (α, F)-migrative if and only if $G(\mu, y) = \mu$ for any $y \in (\mu, e_2]$, as well as, S^d and S_F have ordinal sums of the form $S^d = (...\langle \frac{a-e_1}{k-e_1}, \frac{b-e_1}{k-e_1}, S^d_A \rangle ...)$ and $S_F = (...\langle \frac{a}{\nu}, \frac{b}{\nu}, S_{FA} \rangle ...)$, where $a, b \in [e_1, \nu]$, S^d_A and S_{FA} are continuous Archimedean t-conorms such that S^d_A is $(\frac{\alpha-a}{b-a}, S_{FA})$-migrative.

(iii) If $\alpha \in [\mu, e_2)$, T^c and T_F are continuous, then we have the following statements.

(a) If α is an idempotent element of F, then G is (α, F)-migrative if and only if $G(\alpha, \alpha) = \alpha$ and $G(\nu, y) = \nu$ for any $y \in [e_1, \nu)$.

(b) If α is not an idempotent element of F, then G is (α, F)-migrative if and only if $G(\nu, y) = \nu$ for any $y \in [e_1, \nu)$, as well as, T^c and T_F have ordinal sums of the form $T^c = (...\langle \frac{a-k}{e_2-k}, \frac{b-k}{e_2-k}, T^c_A \rangle ...)$ and $T_F = (...\langle \frac{a-\mu}{1-\mu}, \frac{b-\mu}{1-\mu}, T_{FA} \rangle ...)$, where $a, b \in [\mu, e_2]$, T^c_A and T_{FA} are continuous Archimedean t-norms such that T^c_A is $(\frac{\alpha-a}{b-a}, T_{FA})$-migrative.

(iv) If $\alpha \in (\nu, \mu)$, then G is (α, F)-migrative if and only if

$$F(\alpha, y) = \begin{cases} G(\nu, y) & \text{if } y \in [0, k], \\ G(\mu, y) & \text{if } y \in [k, 1]. \end{cases} \tag{20}$$

Next, we mainly discuss the other four classes, i. e. $G \in C^0_k \cup C^0_1 \cup C^1_k \cup C^1_0$.

4.2. $G \in C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$

From Lemma 4.1, we know that it must be both $e_1 < \nu$ and $e_2 > \mu$. Therefore, in the following, we always start our discussions with the assumptions of $e_1 < \nu$ and $e_2 > \mu$, i. e. $0 < e_1 < \nu < \mu < e_2 < 1$. On the other hand, in virtue of the possible position of k, there are three cases to discuss, i. e. (1) $k \in (e_1, \nu]$; (2) $k \in (\nu, \mu)$; (3) $k \in [\mu, e_2)$. Hence, we know that there only exists three corresponding order relations among e_1, k, e_2, ν and μ as following: (a) $0 < e_1 < k \leq \nu < \mu < e_2 < 1$; (b) $0 < e_1 < \nu < k < \mu < e_2 < 1$; (c) $0 < e_1 < \nu < \mu \leq k < e_2 < 1$.

Now, we start with the case (a) $0 < e_1 < k \leq \nu < \mu < e_2 < 1$.

Proposition 4.14. Let $G \in C_k^0 \cup C_1^0 \cup C_0^1$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < k \leq \nu < \mu < e_2 < 1$ and $\alpha \in (0, 1)$, then G is not (α, F)-migrative.

Proof. On the contrary, suppose that G is (α, F)-migrative. We divide our discussions into the following cases.

- If $\alpha \in (0, k]$. From the structure of $G \in C_k^0 \cup C_1^0 \cup C_0^1$, we can know $G(x, y) = \min(x, y)$ for any $(x, y) \in [0, e_1) \times [e_1, e_2]$. As a consequence, we get $\alpha = G(\alpha, e_1) = G(F(\alpha, 0), e_1) = G(0, F(\alpha, e_1)) = \min(0, F(\alpha, e_1)) = 0$ by $e_1 \leq F(\alpha, e_1) \leq \nu < e_2$, which is a contradiction.

- If $\alpha \in (k, e_2)$. From the structure of $G \in C_k^0 \cup C_1^0 \cup C_0^1$, we can have $G(x, y) = \min(x, y)$ for any $(x, y) \in [0, e_1) \times [k, e_2]$. Meanwhile, by $F(\alpha, 0) \in \{\alpha, \nu\}$, $k < \alpha$ and $k \leq \nu$, we yet have $k \leq F(\alpha, 0)$. Based on these facts, it follows $0 = \min(F(\alpha, e_2), 0) = G(F(\alpha, e_2), 0) = G(e_2, F(\alpha, 0)) = F(\alpha, 0) \in \{\alpha, \nu\}$ by $k < \mu \leq F(\alpha, e_2) \leq e_2$, which is a contradition because $\alpha \neq 0$ and $\nu \neq 0$.

- If $\alpha \in [e_2, 1)$, then we have $\alpha = G(\alpha, e_2) = G(F(\alpha, 1), e_2) = G(1, F(\alpha, e_2))$ since $k < \mu \leq F(\alpha, e_2) \leq e_2$. In the sequence, there are two cases to consider.
  - For the case $G \in C_1^0 \cup C_0^1$. We have $G(x, y) = \max(x, y)$ for any $(x, y) \in [k, e_2] \times (e_2, 1]$ from the structure of G. As a consequence, we get $\alpha = G(1, F(\alpha, e_2)) = \max(1, F(\alpha, e_2)) = 1$, which is a contradiction.
  - For the case $G \in C_k^0$. We have $G(x, y) = \min(x, y)$ for any $(x, y) \in [k, e_2] \times (e_2, 1]$ from the structure of $G \in C_k^0$. And then we have $\alpha = G(1, F(\alpha, e_2)) = F(\alpha, e_2)$, and it means that $\alpha = G(1, F(\alpha, e_2)) = G(1, \alpha) = 1$, but this is impossible.

  Therefore, we get G is not (α, F)-migrative for case $\alpha \in [e_2, 1)$.

□

In the following process, we consider the case $G \in C_k^1$.

Proposition 4.15. Let $G \in C_k^1$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < k \leq \nu < \mu < e_2 < 1$ and $\alpha \in (0, 1)$, then the following statements are true.

(i) If $\alpha \in (0, e_1] \cup (k, \nu] \cup [e_2, 1)$, then G is not (α, F)-migrative.

(ii) If $\alpha \in (e_1, k]$, then G is (α, F)-migrative if and only if

$$F(\alpha, y) = \begin{cases} G(\alpha, y) & \text{if } y \in [0, k], \\ G(\mu, y) & \text{if } y \in [k, 1]. \end{cases} \tag{21}$$

(iii) If $\alpha \in (\nu, e_2)$, then G is (α, F)-migrative if and only if $F(\alpha, y) = G(F(\alpha, e_2), y)$ for any $y \in [0, 1]$.

Proof. Let us prove these results one by one.

- If $\alpha \in (0, e_1] \cup [e_2, 1)$, proof is similar to that of the case of $G \in C^k$ in Proposition 4.2.
- If $\alpha \in (e_1, k]$, it is omitted because proof is completely similar to that of the case of $G \in C^k$ in Proposition 4.3.
- If $\alpha \in (k, \nu]$, proof is completely similar to that of the case of $G \in C^k$ in Proposition 4.4.
- If $\alpha \in (\nu, e_2)$, proof is completely similar to that of the case of $G \in C^k$ in Proposition 4.5.

Therefore, we get the results by summarizing the above discussions. □

Summarizing the above conclusions, we get the following theorem.

Theorem 4.16. Let $G \in C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < k \leq \nu < \mu < e_2 < 1$, then the following statements are true.

(i) If $G \in C_k^0 \cup C_1^0 \cup C_0^1$, then G is not (α, F)-migrative.

(ii) If $G \in C_k^1$ and $\alpha \in (0, e_1] \cup (k, \nu] \cup [e_2, 1)$, then G is not (α, F)-migrative.

(iii) If $G \in C_k^1$ and $\alpha \in (e_1, k]$, then G is (α, F)-migrative if and only if

$$F(\alpha, y) = \begin{cases} G(\alpha, y) & \text{if } y \in [0, k], \\ G(\mu, y) & \text{if } y \in [k, 1]. \end{cases} \tag{22}$$

(iv) If $G \in C_k^1$ and $\alpha \in (\nu, e_2)$, then G is (α, F)-migrative if and only if $F(\alpha, y) = G(F(\alpha, e_2), y)$ for any $y \in [0, 1]$.

Similarly to the above case, we can get the following conclusions for case (c) $0 < e_1 < \nu < \mu \leq k < e_2 < 1$.

Theorem 4.17. Let $G \in C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < \nu < \mu \leq k < e_2 < 1$, then the following statements are true.

(i) If $G \in C_k^0 \cup C_1^0 \cup C_0^1$, then G is not (α, F)-migrative .

(ii) If $G \in C_k^1$ and $\alpha \in (0, e_1] \cup [\mu, k) \cup [e_2, 1)$, then G is not (α, F)-migrative.

(iii) If $G \in C_k^1$ and $\alpha \in [k, e_2)$, then G is (α, F)-migrative if and only if

$$F(\alpha, y) = \begin{cases} G(\nu, y) & \text{if } y \in [0, k], \\ G(\alpha, y) & \text{if } y \in [k, 1]. \end{cases} \quad (23)$$

(iv) If $G \in C_k^1$ and $\alpha \in (e_1, \mu)$, then G is (α, F)-migrative if and only if $F(\alpha, y) = G(F(\alpha, e_1), y)$ for any $y \in [0, 1]$.

In the end, we deal with the case (b) $0 < e_1 < \nu < k < \mu < e_2 < 1$.

Proposition 4.18. Let $G \in C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < \nu < k < \mu < e_2 < 1$ and $\alpha \in (0, \nu] \cup [\mu, 1)$, then G is not (α, F)-migrative.

Proof. Since proof of the case $\alpha \in [\mu, 1)$ is similar to that of the case $\alpha \in (0, \nu]$, we only discuss the case $\alpha \in (0, \nu]$ over here. Let us start our discussions with the assumptions that G is (α, F)-migrative and $\alpha \in (0, \nu]$. Now, we divide discussions into the following cases.

- In the case $G \in C_k^0 \cup C_1^0 \cup C_0^1$, for any $\alpha \in (0, \nu]$, we can have $\alpha = G(\alpha, e_1) = G(F(\alpha, 0), e_1) = G(0, F(\alpha, e_1)) = \min(0, F(\alpha, e_1)) = 0$ by $e_1 \leq F(\alpha, e_1) \leq \nu < k$, which is a contradiction.
- In the case $G \in C_k^1$, for any $\alpha \in (0, \nu]$, we can have $1 = \max(\mu, 1) = G(\mu, 1) = G(F(\alpha, e_2), 1) = G(e_2, F(\alpha, 1)) = G(e_2, \mu) = \mu$ by $k < \mu < e_2$, which contradicts with the assumption of $\mu \neq 1$.

□

Proposition 4.19. Let $G \in C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < \nu < k < \mu < e_2 < 1$ and $\alpha \in (\nu, \mu)$, then G is (α, F)-migrative if and only if

$$F(\alpha, y) = \begin{cases} G(\nu, y) & \text{if } y \in [0, k], \\ G(\mu, y) & \text{if } y \in [k, 1]. \end{cases} \quad (24)$$

Proof. Similarly to proof of Proposition 4.12, we obtain the necessity is also true for these four cases. Conversely, we get it by a simple calculation. □

We get the following theorem by summarizing the results of Propositions 4.18, 4.19.

Theorem 4.20. Let $G \in \{C_k^0 \cup C_1^0 \cup C_k^1 \cup C_0^1\}$ and $F \in F_{\mu,\nu}$ with $\nu < \mu$. If $0 < e_1 < \nu < k < \mu < e_2 < 1$, then the following statements are true.

(i) If $\alpha \in (0, \nu] \cup [\mu, 1)$, then G is not (α, F)-migrative.

(ii) If $\alpha \in (\nu, \mu)$, then G is (α, F)-migrative if and only if

$$F(\alpha, y) = \begin{cases} G(\nu, y) & \text{if } y \in [0, k], \\ G(\mu, y) & \text{if } y \in [k, 1]. \end{cases} \quad (25)$$

5. CONCLUSION

2-uninorms generalize nullnorms and uninorms while semi-t-operators generalize nullnorms and t-operators. Both of them are the relatively special kind of aggregation functions. In addition, migrativity is one of the most significant property for aggregation functions which has been investigated for a long time.

In this work, we have introduced and investigated the concept of α-migrative 2-uninorms over semi-t-operators. Meanwhile, based on the order of μ and ν of $F_{\mu,\nu}$, we have analyzed and characterized all solutions of the α-migrativity equations for all possible combinations for 2-uninorms over semi-t-operators. In the following Table 1, we summarize all theorems in this paper about the α-migrative equations of 2-uninorms over semi-t-operators. For the cases that G is not (α, F)-migrative, we attach '$\times$' in the table.

$F_{\mu,\nu}$ \ Theorem		G: $\mathcal{C}^k$	$\mathcal{C}_k^0$	$\mathcal{C}_1^0$	$\mathcal{C}_k^1$	$\mathcal{C}_0^1$
$\mu < \nu$	$\alpha \in (0, \mu)$	3.1	3.6	3.6	3.6	3.6
	$\alpha \in [\mu, \nu]$	3.5	3.8 $\times$	3.8 $\times$	3.8 $\times$	3.8 $\times$
	$\alpha \in (\nu, 1)$	3.3	3.7	3.7	3.7	3.7
$\mu > \nu$	$k \in (e_1, \nu]$	4.7	4.16 $\times$	4.16$\times$	4.16	4.16 $\times$
	$k \in (\nu, \mu)$	4.13	4.20	4.20	4.20	4.20
	$k \in [\mu, e_2)$	4.8	4.17 $\times$	4.17$\times$	4.17	4.17 $\times$

Tab. 1. All theorems for solutions of the migrative equations of G over $F_{\mu,\nu}$.

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Ying Li-Jun, School of Mathematics and Statistics, Jiangxi Normal University, 330022 Nanchang. P. R. China.
e-mail: 13870305837@163.com

Qin Feng, Corresponding author. School of Mathematics and Statistics, Jiangxi Normal University, 330022 Nanchang. P. R. China.
e-mail: qinfeng923@163.com