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MINIMIZING AND MAXIMIZING A LINEAR OBJECTIVE FUNCTION UNDER A FUZZY MAX $-*$ RELATIONAL EQUATION AND AN INEQUALITY CONSTRAINT

Zofia Matusiewicz

This paper provides an extension of results connected with the problem of the optimization of a linear objective function subject to max $-*$ fuzzy relational equations and an inequality constraint, where $*$ is an operation. This research is important because the knowledge and the algorithms presented in the paper can be used in various optimization processes.

Previous articles describe an important problem of minimizing a linear objective function under a fuzzy max $-*$ relational equation and an inequality constraint, where $*$ is the t-norm or mean. The authors present results that generalize this outcome, so the linear optimization problem can be used with any continuous increasing operation with a zero element where $*$ includes in particular the previously studied operations. Moreover, operation $*$ does not need to be a t-norm nor a pseudo-t-norm.

Due to the fact that optimal solutions are constructed from the greatest and minimal solutions of a max $-*$ relational equation or inequalities, this article presents a method to compute them.

We note that the linear optimization problem is valid for both minimization and maximization problems. Therefore, for the optimization problem, we present results to find the largest and the smallest value of the objective function.

To illustrate this problem a numerical example is provided.

Keywords: fuzzy optimization, minimizing a linear objective function, maximizing a linear objective function, fuzzy relational equations, system of equations, fuzzy relational inequalities, system of inequalities, max $-*$ composition, solution family, minimal solutions

Classification: 90C05, 03E72, 15A06, 15A39, 46N10

1. INTRODUCTION AND MOTIVATION OF THE WORK

The optimization problem is crucial for different aspects of economic and social life. Note that, for example, during the COVID-19 pandemic there is a need for various optimization methods and processes. In the meantime, governments attempt to concentrate their actions on two social problems. The first one being connected with medical research and the other with a group of problems linked to a wide variety of optimization issues. For instance, optimization problems can be described by the minimization of costs on

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the one hand, and the maximization of profits on the other. Just as fuzzy relational equations and inequalities are useful in studying the relationship between processes and states at hand, so is scientific research connected to fuzzy optimization problems quite essential.

Moreover, over the last two decades there have been many papers presenting applications of linear optimization under a fuzzy $\max - *$ relational equation and an inequality constraint with a variety of methods employed to achieve specific objectives (see e. g. [19, 23, 24] and [26]). So far, optimization has been associated with minimization (such as of costs). We introduce a distinction between the upper and the lower solution in the optimization process. The introduction of the upper solution is intended to indicate the application of this method to profit maximization, for example.

Previous studies describe important results of minimizing a linear objective function under a fuzzy $\max - *$ relational equation and an inequality constraint, where $*$ is one of the following operation: the minimum (see [6]), the product (see [17]), the Lukasiewicz t-norm (see [15]), the Archimedean t-norm (see [8]) or the average (see [11]) or other ones. We present results which generalize that the linear optimization problem can be used with any continuous increasing operation with the zero element, and where $*$ includes in particular the operations previously studied.

Moreover, operation $*$ does not need to be a t-norm or a pseudo-t-norm or the average. To conclude, minimizing and maximizing a linear objective function under a fuzzy $\max - *$ relational equation and an inequality constraint is significant and remains in strict connection with fuzzy relational equations and their optimization problem.

The work is divided as follows. First, we recall the basic results on relational systems of equations. Then, we review the methods for determining minimal solutions. Based on this knowledge, we present results concerning the maximization and minimization problem.

Let us begin by outlining the knowledge needed. Calculus and main results concerning fuzzy relations are presented in many articles. Our focus is on some of the early results in articles [25] or [21], and books (especially monographs) [4, 18]. As a consequence of examining the finite fuzzy relations, most scientific papers present equations in the form of a matrix. Study, for example, articles [2, 10], and monograph [1] which demonstrate such an approach.

To present the outcome of our scientific research, our effort concentrates only on such data which would be useful in presenting our work in a clear and effective way. Firstly, we introduce some specific notation and cite the most important definitions and facts.

Let $m, n \in \mathbb{N}$, $M = \{1, \ldots, m\}$, $N = \{1, \ldots, n\}$. Vectors $x, y \in [0,1]^n$ and matrices $A, B \in [0,1]^{m \times n}$ are ordered as follows:

$$(x \leqslant y) \Leftrightarrow (x_j \leqslant y_j), \quad (A \leqslant B) \Leftrightarrow (a_{ij} \leqslant b_{ij}), \quad i \in M, j \in N. \tag{1}$$

The zero vector is denoted by $\mathbf{0}$ and $\mathbf{1}$ and the vector consisting of ones. Let $* : [0,1]^2 \to [0,1]$. In this paper we use the product of a matrix, introduced by Zadeh (see [25]). By employing the $\max - *$ product of matrix $A = [a_{ij}] \in [0,1]^{m \times n}$ and vector $x = [x_j] \in [0,1]^n$, we call $A \circ x \in [0,1]^m$, where:

$$(A \circ x)_i = \bigvee_{j=1}^{n} (a_{ij} * x_j), \quad i \in M. \tag{2}$$

Following Sanchez ([21]) and Drewniak ([3, 4]), we consider a system of equations $A \circ x = b$ and a system of inequalities $A \circ x \leqslant b$ and $A \circ x \geqslant b$. The sets of the solutions of equations are denoted as $S(A, b, *)$ and inequalities as $S_{\leqslant}(A, b, *)$, $S_{\geqslant}(A, b, *)$ respectively.

Reviews of the general results and new scientific research directions connected with fuzzy $\max -*$ equations are presented in many papers, especially in the work of P. Li, S. Ch. Fang ([14]). Let us only focus on the first one where fuzzy systems of equations and their family solutions are addressed with the following products: $\max - \min$ and $\max - product$. Subsequently, fuzzy systems of equations with $\max - t - norm$, $\max - avg$ and other ones are examined. Because of different scientific and commercial utilization of such systems of equations, we search for the most general form. In this paper, we would like to draw on article [16]. Moreover, we would like to extend the most important aspects of the optimization problem, which was presented in [8]. In this paper, we consider:

$$\text{Optimize:} \quad z(x) = \sum_{j=1}^{n} c_j x_j, \tag{3}$$

$$\text{subject to:} \;\; x \in S(A, b, *), \text{ and to: } \; x \in S_{\leqslant}(A, b, *),\; x \in S_{\geqslant}(A, b, *). \tag{4}$$

In the 21st century, there has been interest in a class of minimization problems with fuzzy relational equation constraints. The optimization problem is described in [6] and detailed in [20] and [8]. Various algorithms for their determination have been constructed (see [7]). Applications of this type of a problem are used for finding the minimum cost (for example [13]). Now we would like to use this method to study maximum profits. Following this idea, we present an extended concept of optimization:

Definition 1.1. Let z be a linear objective function where $c \in \mathbb{R}^n$ and $x \in S(A, b, *)$ or $x \in S_{\geqslant}(A, b, *)$ or $x \in S_{\leqslant}(A, b, *)$. Vector x is called the lower solution if the value of $z(x)$ (3) is the lowest (minimization). Vector x is called the upper solution if the value of $z(x)$ (3) is the greatest (maximization).

To make an abridgement, we would like to use the following notation.

Remark 1.2. By D we denote the family of all increasing operations $* : [0, 1]^2 \to [0, 1]$, left-continuous on the second argument.

2. SOLUTION SET OF SYSTEMS OF INEQUALITIES AND SYSTEMS OF EQUATIONS

We consider solution sets of inequality $A \circ x \geqslant b$ and $A \circ x \leqslant b$

$$S_{\geqslant}(A, b, *) = \{x \in [0, 1]^n : A \circ x \geqslant b\}, S_{\leqslant}(A, b, *) = \{x \in [0, 1]^n : A \circ x \leqslant b\}, \tag{5}$$

where $A \in [0, 1]^{m \times n}$ and $b \in [0, 1]^m$.

Therefore, we can consider the set of equations as its intersection:

$$S(A, b, *) = S_{\geqslant}(A, b, *) \cap S_{\leqslant}(A, b, *) = \{x \in [0, 1]^n : A \circ x = b\}. \tag{6}$$

Definition 2.1. (Matusiewicz and Drewniak [16], Definition 5) By minimal solutions of system $A \circ x \geqslant b$ ($A \circ x = b$) with the max $-*$ product we call minimal elements in $S_{\geqslant}(A, b, *)$ (in $S(A, b, *)$) with respect to the partial order (1) (if any). The set of all minimal solutions is denoted by $S^0_{\geqslant}(A, b, *)$ ($S^0(A, b, *)$).

Lemma 2.2. (see Matusiewicz and Drewniak [16], Lemma 8, Corollary 2) If operation $*$ has a right-site zero element, then $\mathbf{0} \in S_{\leqslant}(*, A, b)$.

Definition 2.3. By the greatest solution of system $A \circ x = b$ with the max $-*$ product we call the greatest element in $S(A, b, *)$ with respect to the partial order (1) (if any).

Lemma 2.4. (see Matusiewicz and Drewniak [16], Corollary 4) If operation $*$ is increasing, then

$$\mathbf{1} \in S_{\geqslant}(A, b, *) \Leftrightarrow S_{\geqslant}(A, b, *) \neq \emptyset. \tag{7}$$

To compute the greatest solution of equation $A \circ x = b$ the following operations are needed:

Definition 2.5. (Drewniak [3], Definition 1) The induced implication $\overset{*}{\to}$ and dual induced implication $\overset{*}{\leftarrow}$ in $[0, 1]$ are the following operations:

$$a \overset{*}{\to} b = \max\{t \in [0, 1] : a * t \leqslant b\}, \tag{8}$$

$$a \overset{*}{\leftarrow} b = \min\{t \in [0, 1] : a * t \geqslant b\}, \tag{9}$$

if they exist for $a, b \in [0, 1]$.

Theorem 2.6. (Drewniak and Matusiewicz [5], Theorem 8) If operation $*$ is increasing and right-continuous on the second argument, then

$$S_{\geqslant}(A, b, *) = \bigcup_{v \in S^0_{\geqslant}(A, b, *)} [v, \mathbf{1}]. \tag{10}$$

Example 2.7. Examples of operations from family D include triangular norms T_M, T_P, T_L, T_{FD}, left-continuous on the second argument pseudo triangular norms and geometric mean.

Based on the definition of induced implication by operation $*$ (8) we can construct the implication introduced by the matrix product: max $-*$ (2)

$$A \overset{\circ}{\to} b = \max\{x \in [0, 1]^n : A \circ x \leqslant b\}. \tag{11}$$

Let $u \in [0, 1]^n$ and $u = u(A, b, *)$.

Theorem 2.8. (Drewniak and Matusiewicz [5], Theorem 2) If $* \in D$ and $1 * 0 = 0$, then the greatest element exists there (11) and

$$(A \overset{\circ}{\to} b)_j = \bigwedge_{i=1}^{m} (a_{ij} \overset{*}{\to} b_i) \quad \text{for} \quad j \in N. \tag{12}$$

Corollary 2.9. Let $* \in D$ and $1 * 0 = 0$. We have

$$S(A,b,*) \neq \emptyset \Leftrightarrow A \overset{\circ}{\to} b \in S(A,b,*) \Leftrightarrow \max S(A,b,*) = A \overset{\circ}{\to} b. \tag{13}$$

Corollary 2.10. If $* \in D$ and $1*0 = 0$, then vector $u(A,b,*) = A \overset{\circ}{\to} b$ has the following form

$$u_j = \bigwedge_{i=1}^{m} (a_{ij} \overset{*}{\to} b_i) \quad \text{for } j \in N. \tag{14}$$

Theorem 2.11. (Drewniak and Matusiewicz [5], Theorem 11) Let $* \in D$ and $1*0 = 0$. If $S(A,b,*) \neq \emptyset$, then

$$S(A,b,*) = \bigcup_{v \in S^0(A,b,*)} [v, A \overset{\circ}{\to} b]. \tag{15}$$

Corollary 2.12. Let $* \in D$ and $1 * 0 = 0$. If $S_{\leqslant}(A,b,*) \neq \emptyset$, then

$$S_{\leqslant}(A,b,*) = [\mathbf{0}, A \overset{\circ}{\to} b]. \tag{16}$$

3. METHOD OF DETERMINATION OF MINIMAL SOLUTIONS

In this study, the results obtained are based on the properties of the family of solutions, in particular, the greatest solutions and the minimum solutions. In this section, we present a method for determining the minimum solutions. In this way, the results obtained are of a usable nature. Algorithm I and Algorithm I are described and presented in detail in paper [16].

Let $* \in D$ and $1 * 0 = 0$, $A \in [0,1]^{m \times n}$, $b \in [0,1]^m$.

Definition 3.1. (Matusiewicz and Drewniak [16], Definition 7) By reduced matrix of equation $A \circ x = b$ with respect to solution $x \in S(A,b,*)$ we call matrix $A'_b(x)$, where

$$a'_{ij}(x) = \begin{cases} a_{ij}, & \text{if } a_{ij} * x_j = b_i \\ 0, & \text{in other case} \end{cases}, i \in M, j \in N. \tag{17}$$

By reduced matrix of inequality $A \circ x \geqslant b$ with respect to solution $x \in S_{\geqslant}(A,b,*)$ we call matrix $A'_{\geqslant b}(x)$, where

$$a^{\geqslant}_{ij}(x) = \begin{cases} a_{ij}, & \text{if } a_{ij} * x_j \geqslant b_i \\ 0, & \text{in other case} \end{cases}, i \in M, j \in N. \tag{18}$$

Now, we present algorithms from [16], which can be useful to compute minimal solutions of $A \circ x \geqslant b$ and $A \circ x = b$.

Firstly, we change the order of rows as follows

$$b_m \leqslant \ldots \leqslant b_2 \leqslant b_1. \tag{19}$$

ALGORITHM I

Let $S_{\geqslant}(A,b,*) \neq \emptyset$, $x \in S_{\geqslant}(A,b,*)$, $b_i > 0$ for $i \in M$ and $* \in RC$. We assume that $b_m > 0$ because rows with $b_i = 0$ can be omitted (true inequalities).

Step 1. Determine reduced matrix $A'_{\geqslant b}(x)$ from (17). Let $i := 1$, $K := \emptyset$, $V := M$.

Step 2. Choose k_i such that $a'_{ik_i} > 0$, compute K $:= K \cup \{k_i\}$ and insert (cf. (9))

$$v_{k_i} = a'_{ik_i} \overset{*}{\leftarrow} b_i. \tag{20}$$

Step 3. Determine the set

$$V := V \cap \{s \in M : s > i \text{ and } a'_{sk_i} * v_{k_i} < b_s\}. \tag{21}$$

Step 4. If $V \neq \emptyset$, then $i := \min V$ and return to Step 2. Otherwise, go to Step 5.

Step 5. If $k \in N \backslash K$, then $v_k := 0$.

Remark 3.2. By $Alg(x)$ we denote all effects of Algorithm I.

Theorem 3.3. (Matusiewicz and Drewniak [16], Theorem 6.) If operation $*$ is increasing and left-continuous on the second argument, then

$$S^0_{\geqslant}(A,b,*) \subset Alg(\mathbf{1}). \tag{22}$$

Example 3.4. Let $x * y = \sqrt{x \cdot y}$ for $x, y \in [0,1]$ and

$$A = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.6 & 0.25 & 0.15 \\ 1 & 0.2 & 0.5 \end{bmatrix}, \quad b = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.4 \end{bmatrix}.$$

The geometric mean is the increasing and continuous operation. From Definition 2.5 we obtain:

$$a \overset{*}{\leftarrow} b = \begin{cases} \frac{b^2}{a} & \text{for } a \neq 0 \\ 0 & \text{for } a = 0 \end{cases}, \quad a, b \in [0,1]. \tag{23}$$

Using (17) we get

$$A'_{\geqslant b}(\mathbf{1}) = \begin{bmatrix} 0.4 & 0.5 & 0 \\ 0.6 & 0.25 & 0 \\ 1 & 0 & 0.5 \end{bmatrix}.$$

We determine all effects of Algorithm I:

1. For $i = 1$ we choose $k_1 = 1$. We determine $v_1 = a_{11} \overset{*}{\leftarrow} b_1 = 0.4 \overset{*}{\leftarrow} 0.6 = 0.9$. Because $V = \emptyset$, $K = \{1\}$, so $v^1_2 = 0$, $v^1_3 = 0$ and $v^1 = [0.9, 0, 0]^T$.

2. For $i = 1$ we choose $k_1 = 2$. We determine $v_2 = a_{12} \overset{*}{\leftarrow} b_1 = 0.5 \overset{*}{\leftarrow} 0.6 = 0.072$. We obtain $V = \{3\}$, $K = \{2\}$. For $i = 3$ we choose $k_3 = 1$ and we determine $v_1 = a_{31} \overset{*}{\leftarrow} b_3 = 1 \overset{*}{\leftarrow} 0.4 = 0.16$. Because $V = \emptyset$ and $K = \{1, 2\}$, so $v_3 = 0$ and $v^2 = [0.16, 0.72, 0]^T$.

3. Return to $k_1 = 2$, $V = \{3\}$, $K = \{2\}$, $i = 3$.

In the next step we have $k_3 = 3$ and we determine $v_3 = a_{33} \overset{*}{\leftarrow} b_3 = 0.5 \overset{*}{\leftarrow} 0.4 = 0.32$. Because $V = \emptyset$, $K = \{2, 3\}$, so $v_1 = 0$ and $v^3 = [0, 0.72, 0.32]^T$.

So we have $Alg(\mathbf{1}) = \{v^1, v^2, v^3\}$, where

$$v^1 = \begin{bmatrix} 0.9 \\ 0 \\ 0 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 0.16 \\ 0.72 \\ 0 \end{bmatrix}, \quad v^3 = \begin{bmatrix} 0 \\ 0.72 \\ 0.32 \end{bmatrix}.$$

From Theorem 3.3, we have $S^0_{\geqslant}(A, b, *) \subset Alg(\mathbf{1}) = \{v^1, v^2, v^3\}$. Because v^1, v^2 and v^3 are incomparable, it means that all of them are minimal solutions.

ALGORITHM II

We assume that operation $*$ is increasing, continuous on the second argument, and satisfies $1 * 0 = 0$ and $b_i > 0$, $i \in M$.

Step 0. Compute $u = A \overset{\circ}{\to} b$ from (14).

Step 1. Determine $Alg(u)$ from Algorithm I.

Step 2. Determine $S^0(A, b, *)$ as the set of minimal elements in $Alg(u)$.

Theorem 3.5. (see Matusiewicz and Drewniak [16], Theorem 12) Let $b \neq \mathbf{0}$. If operation $*$ is increasing, continuous on the second argument, and satisfies $1 * 0 = 0$, then

$$S^0(A, b, *) \subset Alg(u). \tag{24}$$

Corollary 3.6. Algorithm II has a stronger assumption than Algorithm I (as the additional assumption of greatest solution $A \circ x = b$ is required to exist).

Example 3.7. Algorithm II cannot be used for the arithmetic mean because it does not satisfy $1 * 0 = 0$.

4. OPTIMIZATION OF A LINEAR OBJECTIVE FUNCTION UNDER $A \circ X = B$ CONSTRAIN

Let $A \in [0, 1]^{m \times n}$, $b \in [0, 1]^m$ and $S(A, b, *) \neq \emptyset$. As a generalization of Lemmas 5 and 6 from [8] we obtain Theorem 4.3.

Lemma 4.1. Let $* \in D$, $1 * 0 = 0$. If $c \leqslant \mathbf{0}$, then vector $u(A, b, *)$ is the lower solution.

Proof. Let $u = A \overset{\circ}{\to} b$. Because (13) $\max S(A, b, *) = u$, then for any $x \in S(A, b, *)$ we have $x \leqslant u$. It means that $x_j \leqslant u_j$, so $c_j x_j \geqslant c_j u_j$ for $j \in N$. Thus we obtain:

$$z(x) = \sum_{j=1}^{n} c_j x_j \geqslant \sum_{j=1}^{n} c_j u_j = z(u),$$

so $u(A, b, *)$ is the lower solution. $\square$

Lemma 4.2. Let $* \in D$ and $1 * 0 = 0$. If $c \geqslant \mathbf{0}$, then one or more solutions from $S^0(A, b, *)$ are the lower solutions.

P r o o f. Let $S(A, b, *) \neq \emptyset$. Based on Theorem 2.11 for any $x \in S(A, b, *)$ there exists such $v \in S^0(A, b, *)$, that $v \leqslant x$, so $v_j \leqslant x_j$ for all $j \in N$. It means that $c_j v_j \leqslant c_j x_j$ for $j \in N$, so we obtain:

$$z(v) = \sum_{j=1}^{n} c_j v_j \leqslant \sum_{j=1}^{n} c_j x_j = z(x).$$

Because $z(v^i) \in \mathbb{R}$ for $v^i \in S^0(A, b, *)$ are comparable, so there exists such vector $v \in S^0(A, b, *)$ for which value $z(v)$ is the lowest. Therefore the lower solutions are in finite set $S^0(A, b, *)$. □

Let us denote:

$$c_j^1 = \begin{cases} c_j & \text{for } c_j < 0 \\ 0 & \text{for } c_j \geqslant 0 \end{cases}, \quad c_j^2 = \begin{cases} 0 & \text{for } c_j \leqslant 0 \\ c_j & \text{for } c_j > 0 \end{cases}, \quad j \in N, \tag{25}$$

so $c = c^1 + c^2$. Therefore the objective function for $x \in S(A, b, *)$ we can present as:

$$z(x) = \sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n} (c_j^1 + c_j^2) x_j = \sum_{j=1}^{n} c_j^1 x_j + \sum_{j=1}^{n} c_j^1 x_j = z^1(x) + z^2(x). \tag{26}$$

Now, we introduce the following vector $q \in [0, 1]^n$:

$$q_j = \begin{cases} u_j & \text{for } c_j^1 \leqslant 0 \\ v_j & \text{for } c_j^2 > 0, \end{cases} \tag{27}$$

where $v \in S^0(A, b, *)$ is the optimum solution for c^2.

Theorem 4.3. If $* \in D$ and $1 * 0 = 0$, then vector q of the following form (27) is the lower solution.

P r o o f. Let $x \in S(A, b, *)$, $u = A \overset{\circ}{\to} b$ and $v \in S^0(A, b, *)$. Because $v \leqslant q \leqslant u$, then $q \in S(A, b, *)$, based on Theorem 2.11. Using Lemmas 4.1, 4.2, and (26) we obtain:

$$z(x) = z^1(x) + z^2(x) = \sum_{j=1}^{n} c_j^1 x_j + \sum_{j=1}^{n} c_j^2 x_j \geqslant \sum_{j=1}^{n} c_j^1 u_j + \sum_{j=1}^{n} c_j^2 v_j = z^1(q) + z^2(q) = z(q).$$

□

In a similar way, we can determine the upper solutions.

Lemma 4.4. Let $* \in D$ and $1 * 0 = 0$. If $c \geqslant \mathbf{0}$, then vector $u(A, b, *)$ is the upper solution.

Lemma 4.5. Let $* \in D$ and $1 * 0 = 0$. If $c \leqslant \mathbf{0}$, then one or more solution from $S^0(A, b, *)$ are the upper solutions.

Let us introduce the following vector $r \in [0,1]^n$:

$$r_j = \begin{cases} v_j & \text{for } c_j^1 \leqslant 0 \\ u_j & \text{for } c_j^2 > 0, \end{cases} \tag{28}$$

where $v \in S^0(A,b,*)$.

Theorem 4.6. If $* \in D$ and $1 * 0 = 0$, then vector r (28) is the upper solution.

Proof. Let $x \in S(A,b,*)$, $u = A \overset{\circ}{\to} b$ and $v \in S^0(A,b,*)$. Because $v \leqslant r \leqslant u$, then $r \in S(A,b,*)$, based on Theorem 2.11. Using Lemmas 4.4, 4.5 and (28) we obtain:

$$z(x) = z^1(x) + z^2(x) = \sum_{j=1}^{n} c_j^1 x_j + \sum_{j=1}^{n} c_j^2 x_j \leqslant \sum_{j=1}^{n} c_j^1 v_j + \sum_{j=1}^{n} c_j^2 u_j = z^1(r) + z^2(r) = z(r).$$

□

Because of Theorem 4.3 and using Theorem 2.11 we obtain:

Corollary 4.7. Let $S(A,b,*) \neq \emptyset$, $* \in D$ and $1 * 0 = 0$. In set $S(A,b,*)$ at least one lower solution and at least one upper solution exist.

Corollary 4.8. Based on (14) and an algorithm to compute all minimal solutions of $A \circ x = b$ we can obtain the lower and upper solutions.

5. MINIMIZING A LINEAR OBJECTIVE FUNCTION UNDER A FUZZY $\max - *$ RELATIONAL INEQUALITY CONSTRAINT

Let $A \in [0,1]^{m \times n}$, $b \in [0,1]^m$ and $S_{\geqslant}(A,b,*) \neq \emptyset$.

Lemma 5.1. Let $*$ be an increasing operation. If $c \geqslant \mathbf{0}$, then vector $\mathbf{1}$ is the upper solution.

Proof. Let $S_{\geqslant}(A,b,*) \neq \emptyset$. Because we have $x \leqslant \mathbf{1}$, it means that $x_j \leqslant 1$, so $c_j \cdot x_j \leqslant c_j \cdot 1$ for $j = 1,\ldots,n$. Thus we obtain:

$$z(x) = \sum_{j=1}^{n} c_j \cdot x_j \leqslant \sum_{j=1}^{n} c_j \cdot 1 = z(\mathbf{1}),$$

so $\mathbf{1}$ is the upper solution. □

In a similar way we obtain:

Lemma 5.2. Let $*$ be an increasing operation. If $c \leqslant \mathbf{0}$, then vector $\mathbf{1}$ is the lower solution.

Lemma 5.3. Let $*$ be an increasing, right-continuous on the second argument operation. If $c \geqslant \mathbf{0}$, then one or more solutions from $S^0_{\geqslant}(A,b,*)$ are the lower solutions.

P r o o f. Let $S_{\geqslant}(A,b,*) \neq \emptyset$. Based on Theorem 2.6 for any $x \in S_{\geqslant}(A,b,*)$ there exists such $v \in S^0_{\geqslant}(A,b,*)$, that $v \leqslant x$, so $v_j \leqslant x_j$. It means $c_j v_j \leqslant c_j x_j$ for $j = 1,\ldots,n$, so we have:

$$z(v) = \sum_{j=1}^{n} c_j v_j \leqslant \sum_{j=1}^{n} c_j x_j = z(x).$$

Because $z(v^i) \in \mathbb{R}$ for $v^i \in S^0_{\geqslant}(A,b,*)$ are comparable, so exists such vector $v \in S^0_{\geqslant}(A,b,*)$ for which value $z(v)$ is the lowest. Therefore the optimal solution is found in finite $S^0_{\geqslant}(A,b,*)$. □

In a similar way, we can determine the upper solutions.

Lemma 5.4. Let $*$ be an increasing, right-continuous on the second argument-operation. If $c \leqslant \mathbf{0}$, then one or more solutions from $S^0_{\geqslant}(A,b,*)$ are the upper solutions.

So, for the given vectors:

$$q_j^{\geqslant} = \begin{cases} 1 & \text{for } c_j^1 \geqslant 0 \\ v_j & \text{for } c_j^2 < 0 \end{cases}, \quad r_j^{\geqslant} = \begin{cases} v_j & \text{for } c_j^1 \geqslant 0 \\ 1 & \text{for } c_j^2 < 0 \end{cases} \tag{29}$$

where $v \in S^0(A,b,*)$.

Theorem 5.5. If $*$ is an increasing, right-continuous on the second argument operation, then vector $q^{\geqslant}$ is the upper solution and $r^{\geqslant}$ is the lower solution (see (29)).

P r o o f. Let $x \in S_{\geqslant}(A,b,*)$, $u = A \overset{\circ}{\to} b$ and $v^{\geqslant} \in S^0_{\geqslant}(A,b,*)$. Based on Theorem 2.6, $q^{\geqslant} \in S_{\geqslant}(A,b,*)$. Using Lemmas 5.2, 5.3 and (29) we obtain:

$$z(x) = z^1(x) + z^2(x) = \sum_{j=1}^{n} c_j^1 x_j + \sum_{j=1}^{n} c_j^2 x_j \geqslant \sum_{j=1}^{n} c_j^1 \cdot 1 + \sum_{j=1}^{n} c_j^2 v_j = z^1(q^{\geqslant}) + z^2(q^{\geqslant}) = z(q^{\geqslant}).$$

Moreover, using Lemmas 5.1, 5.4 and (29) we obtain:

$$z(x) = z^1(x) + z^2(x) = \sum_{j=1}^{n} c_j^1 x_j + \sum_{j=1}^{n} c_j^2 x_j \leqslant \sum_{j=1}^{n} c_j^1 v_j + \sum_{j=1}^{n} c_j^2 \cdot 1 = z^1(r^{\geqslant}) + z^2(r^{\geqslant}) = z(r^{\geqslant}).$$

□

Corollary 5.6. With the same assumption as in Theorem 5.5 we can find the optimal solutions using Lemma 2.4 and Algorithm II (to compute all minimal solutions of $A \circ x \geqslant b$).

Let us define the following vectors:

$$q_j^{\leqslant} = \begin{cases} u_j & \text{for } c_j^1 \geqslant 0 \\ 0 & \text{for } c_j^2 < 0 \end{cases}, \quad r_j^{\leqslant} = \begin{cases} 0 & \text{for } c_j^1 \geqslant 0 \\ u_j & \text{for } c_j^2 < 0 \end{cases} \tag{30}$$

As a corollary of the previous observations and from Lemma 2.2 and Theorem 2.12 we have:

Theorem 5.7. If $*$ is an increasing, left-continuous on the second argument operation, then vector $q^{\leqslant}$ is the upper solution and $r^{\leqslant}$ is the lower solution.

P r o o f. Let $x \in S_{\leqslant}(A, b, *)$, $u = A \overset{\circ}{\to} b$. Based on Corollary 2.12, we have:

$$z(x) = z^1(x) + z^2(x) = \sum_{j=1}^{n} c_j^1 x_j + \sum_{j=1}^{n} c_j^2 x_j \geqslant \sum_{j=1}^{n} c_j^1 u_j + 0 = z^1(q^{\leqslant}) + 0 = z^1(q^{\leqslant}) + z^2(q^{\leqslant}) = z(q^{\leqslant}).$$

Next, using Lemmas 5.1, 5.4 and (29) we obtain:

$$z(x) = z^1(x) + z^2(x) = \sum_{j=1}^{n} c_j^1 x_j + \sum_{j=1}^{n} c_j^2 x_j \leqslant 0 + \sum_{j=1}^{n} c_j^2 \cdot u_j = z^1(r^{\leqslant}) + 0 = z^1(r^{\leqslant}) + z^2(r^{\leqslant}) = z(r^{\leqslant}).$$

□

6. EXAMPLE ILLUSTRATING THE USE OF RESULTS

The problem of the Covid-19 pandemic indicates that we should develop models to guide social, economic or business activities, based on the incidence of the disease, that take into account various social needs. Based on the current data, we can examine, to some extent, the relationship between the restrictions imposed on various sectors of the economy and the incidence of the disease in a given country or region. We would like to note that one method of examining the relationship is the relational system of equations method ([21]). Thus, following the results described in this article, we can present this approach to verify the effectiveness and social impact of the restrictions imposed. Let set P denote the four-element set of countries participating in the study, Z - introduced restrictions in a given branch of economy or social life (here: closure of open spaces, occupancy of big-box stores, occupancy of swimming pools, hybrid school functioning), L - the increase in illnesses compared to the previous week.

Now, we write the relationship between sets P and L in matrix form:

$$A = \begin{bmatrix} 1 & 0.8 & 0.5 & 0.75 \\ 0.75 & 0.8 & 0.1 & 1 \\ 0.2 & 0.3 & 0.6 & 0.4 \\ 0.4 & 0.5 & 0.6 & 0.5 \end{bmatrix}.$$

So, we study the situation in four selected regions (region corresponds to the row of matrix A). We consider four different factors affecting the transmission of the virus:

- the degree of closure of open spaces such as parks, forests, zoos (column 1);
- the allowed occupancy rate of big-box stores (column 2);
- the maximum occupancy rate of swimming pools (column 3);
- the degree of hybrid operation of schools (column 4).

Vector $b = [0.8, 0.6, 0.3, 0.3]^T$ denotes the degree of changes in virus transmission. The relationship we are looking for (between factors affecting transmission and the degree of incremental virus transmission) is denoted by vector x. By solving the system of equations $A \circ x = b$, we determine the relationships between the considered factors and virus transmission.

Moreover, objective function z illustrates the degree of influence on social sentiment. By determining its minimum and maximum values we will be able to determine the possible social attitudes. This approach allows us to simultaneously study the relationship between the introduced restrictions and the degree of transmission (determining vector x) and public sentiment (studying objective function z).

In this example, we consider the optimization problem subject to the max $-*$ fuzzy relational equation, where $*$ is the following operation $x*y = x \cdot \min(x, y)$ for $x, y \in [0, 1]$:
optimize: $z(x) = -4x_1 + 3x_2 + 2x_3 - 7x_4$
subject to: $A \circ x = b$.

First of all, we compute $u = A \overset{\circ}{\rightarrow} b$ (the greatest solution of $A \circ x = b$), and using Algorithm I, we determine minimal solutions of $A \circ x = b$. So, we obtain $S^0(A, b, *) = \{v^1, v^2\}$ (see [16], Example 11):

$$u = \begin{bmatrix} 0.8 \\ 0.75 \\ 0.5 \\ 0.6 \end{bmatrix}, v^1 = \begin{bmatrix} 0.8 \\ 0.75 \\ 0.5 \\ 0 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 0.8 \\ 0 \\ 0.5 \\ 0.6 \end{bmatrix}.$$

Based on (25) we have the following vectors c^1 and c^2:

$$c^1 = \begin{bmatrix} -4 \\ 0 \\ 0 \\ -7 \end{bmatrix}, \quad c^2 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

From Lemma 4.1, we determine solutions (with negative coefficients in objective function) $z^1(x) = -4x_1 - 7x_4$, so $z^1(u) = -4 \cdot 0.8 - 7 \cdot 0.6 = -7.4$.

We compute (with nonnegative coefficients in the objective function):

$z^2(v^1) = 3 \cdot 0.75 + 4 \cdot 0.5 = 4.25$ and $z^2(v^2) = 3 \cdot 0 + 4 \cdot 0.5 = 2$. From Lemma 4.2, v^2 is the lower solution for the objective function $z^2(x) = 3x_2 + 4x_3$. Based on Theorem 4.3, the lower solution for given objective function $z(x)$ is vector $q = [0.8, 0, 0.5, 0.6]$ and the minimal value is $z(q) = -5.4$. In a similar way, we determine the upper solutions $r = [0.8, 0.75, 0.5, 0]^T$ and $z(r) = 0.05$.

Thus, by solving the system of equations: we can examine the degree of dependence between the morbidity rate and the degree of restrictions introduced. Based on Definition 3.1, we can determine reduced matrix $A_b'(x)$, which will show us those areas of the economy in the country which did not appear to cause an increase in the number of patients. Thus, we will increase the comfort of social life without increasing the level of pandemic threat in a given country. At the same time, knowing the set of solutions of such a system of equations, we can determine the predicted largest and smallest value of various economic or social coefficients described by the objective function. Of course, this is one of a very large number of possible systems of equations that can contribute to the understanding of the relationship between socioeconomic activity and the development of an airborne disease epidemic, assuming that the disease develops differently, depending on the region or group studied.

In summary, we would like to emphasize that we can determine the lower solution for other, most well-known operations. As operation $*$ we can choose triangular norms: right-continuous with respect to the second variable, for example T_M, T_L, T_P (see [12]) or pseudo triangular norms: right-continuous with respect to the second variable (by Wang, Yu) e. g. (see [9], Examples 2.2 and 2.3):

$$T_{WY}(x,y) = \begin{cases} y, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases} \qquad \text{(Wang and Yu)}\ , \tag{31}$$

$$T_{Y}(x,y) = \begin{cases} y^{\frac{1}{x}}, & \text{if } x \cdot y > 0 \\ 0, & \text{otherwise} \end{cases} \qquad \text{(Yager)}\ , \tag{32}$$

or right-continuous mean.

7. CONCLUSIONS

This work provides new results and extensions of results connected with the problem of minimizing and maximizing a linear objective function subject to max $-*$ fuzzy relational equations and an inequality constraint.

In their papers, S.-M. Guu and Y.-K. Wu (2010) ([8]) and B.-S. Shieh (2011) ([22]) described the very important problems of minimizing a linear objective function under a fuzzy max $-T$ relation equation constraint, where T is the Archimedean t-norm.

In this paper, their scientific research has been extended and solved for the max $-*$ system of equations, where $*$ is an increasing operation, continuous on the second argument. At the same time, we have hoped to reach a more complete understanding of the problem. We have also presented ideas on the lower and upper solutions for the max $-*$ system of equations and inequalities.

Based on the work of E. Khorram and H. Zarei (2009) ([11]), discussing and obtaining results in this case, seems to be very important. However, we would like to suggest the new problem as equally important as the previous one - how to determine the family of the best solutions for minimizing and maximizing functions. This means that we would like to ask how to efficiently determine the lower solutions where functions are minimized and the upper solutions where others are maximized at the same time (if possible). Moreover, if such family solutions are empty, there is yet another problem to be dealt with. We would like to address approximate optimal solutions.

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