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CURVATURE TENSORS AND RICCI SOLITONS WITH RESPECT TO ZAMKOVOY CONNECTION IN ANTI-INVARIANT SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLD

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Abstract. The present paper deals with the study of some properties of anti-invariant submanifolds of trans-Sasakian manifold with respect to a new non-metric affine connection called Zamkovoy connection. The nature of Ricci flat, concircularly flat, ξ-projectively flat, M-projectively flat, ξ-M-projectively flat, pseudo projectively flat and ξ-pseudo projectively flat anti-invariant submanifolds of trans-Sasakian manifold admitting Zamkovoy connection are discussed. Moreover, Ricci solitons on Ricci flat, concircularly flat, M-projectively flat and pseudo projectively flat anti-invariant submanifolds of trans-Sasakian manifold admitting the aforesaid connection are studied. At last, some conclusions are made after observing all the results and an example of an anti-invariant submanifold of a trans-Sasakian manifold is given in which all the results can be verified easily.

Keywords: anti-invariant submanifold; trans-Sasakian manifold; Zamkovoy connection; η-Einstein manifold; Ricci curvature tensor; concircular curvature tensor; projective curvature tensor; M-projective curvature tensor; pseudo projective curvature tensor; Ricci soliton

MSC 2020: 53C05, 53C15, 53C20, 53C25, 53C40

1. Introduction

The notion of Zamkovoy connection was introduced by Zamkovoy in 2009, see [32]. Later Biswas and Baishya applied this connection on generalized pseudo Ricci symmetric Sasakian manifolds (see [1]) and on almost pseudo symmetric Sasakian manifolds (see [2]). This connection was further studied by Blaga in 2015, see [3]. In 2020, Mandal and Das worked in detail on various curvature tensors of Sasakian and Lorentzian para-Sasakian manifolds admitting this new connection (see [11], [12],

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[13], [6]), and recently in 2021, they discussed LP-Sasakian manifolds equipped with this new connection and conharmonic curvature tensor, see [14].

For an n-dimensional almost contact metric manifold $M(\varphi, \xi, \eta, g)$ consisting of a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g with the Riemannian connection ∇, *Zamkovoy connection* ∇^* is defined as (see [32])

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\varphi Y. \tag{1.1}$$

Ricci flow was introduced by Hamilton in 1982 (see [8]). He observed that it can be used well in simplifying the structure of a manifold. He developed the concept to answer Thurston's geometric conjecture stating that each closed 3-manifold admits a geometric decomposition. The *Ricci flow equation* (see [8]) is given by

$$\frac{\partial g}{\partial t} = -2S,$$

where g, S, t are, respectively, the Riemannian metric, Ricci curvature tensor and time. *Ricci soliton*, which is a self similar solution of the above equation, was also introduced by Hamilton in [9]. It is represented by the triplet (g, V, λ) (where V, λ are, respectively, a vector field and a constant) satisfying the equation

$$L_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $L_V g$ is the Lie derivative of g along V (see [9]). Ricci soliton is called *shrinking*, *steady* or *expanding* according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively.

Curvature is the central subject in Riemannian geometry. It measures the distance between an manifold and a Euclidean space.

Yano introduced the notion of *concircular curvature tensor* C of type (1,3) on Riemannian manifold for an n-dimensional manifold M as

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]$$

for all smooth vector fields $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor of type (1,3) and r is the scalar curvature (see [29]).

Hence, if we consider C^* as the concircular curvature tensor with respect to Zamkovoy connection, then for a $(2n+1)$-dimensional manifold we have

$$C^*(X,Y)Z = R^*(X,Y)Z - \frac{r^*}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{1.3}$$

for all smooth vector fields $X, Y, Z \in \chi(M)$, where R^* is the curvature tensor and r^* is the scalar curvature with respect to Zamkovoy connection.

Definition 1.1. A $(2n+1)$-dimensional manifold M is called *Ricci flat* with respect to Zamkovoy connection if $S^*(X,Y) = 0$ for all $X, Y \in \chi(M)$.

Definition 1.2 ([6]). A $(2n+1)$-dimensional manifold M is called *concircularly flat* with respect to Zamkovoy connection if $C^*(X,Y)Z = 0$ for all $X, Y, Z \in \chi(M)$.

Yano and Bochner introduced the notion of *projective curvature tensor* P of type (1,3) for an n-dimensional manifold M as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]$$

for all smooth vector fields $X, Y, Z \in \chi(M)$, where S is the Ricci tensor of type (0,2) (see [30]). Thus, for dimension $(2n+1)$ we have

$$P^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{2n}[S^*(Y,Z)X - S^*(X,Z)Y], \tag{1.4}$$

where we consider P^* and S^*, respectively, as the projective curvature tensor and the Ricci curvature tensor with respect to Zamkovoy connection. Both of the above curvature tensors represent the deviation of a manifold from being a manifold of constant curvature (see [30], [29]).

Definition 1.3 ([12]). A $(2n+1)$-dimensional manifold M is called *projectively flat* with respect to Zamkovoy connection if $P^*(X,Y)Z = 0$ for all $X, Y, Z \in \chi(M)$.

Definition 1.4 ([12]). A $(2n+1)$-dimensional manifold M is called ξ*-projectively flat* with respect to Zamkovoy connection if $P^*(X,Y)\xi = 0$ for all $X, Y \in \chi(M)$.

Pokhariyal and Mishra introduced the notion of M-projective curvature tensor on a Riemannian manifold in 1971 (see [21]). Later Ojha studied its properties in [17], [18], [19]. This curvature tensor was further discussed by many researchers, see [4], [5], [11], [22], [26]. The *M-projective curvature tensor* $\overline{M}$ of rank 3 on an n-dimensional manifold M is given by

$$\begin{aligned}\overline{M}(X,Y)Z = R(X,Y)Z &- \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y]\\ &- \frac{1}{2(n-1)}[g(Y,Z)QX - g(X,Z)QY]\end{aligned}$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where Q is the Ricci operator (see [21]).

Thus, for a $(2n+1)$-dimensional manifold, considering $\overline{M}^*$ as the M-projective curvature tensor with respect to Zamkovoy connection we get

$$\begin{aligned}\overline{M}^*(X,Y)Z = R^*(X,Y)Z &- \frac{1}{4n}[S^*(Y,Z)X - S^*(X,Z)Y]\\ &- \frac{1}{4n}[g(Y,Z)Q^*X - g(X,Z)Q^*Y],\end{aligned} \tag{1.5}$$

where Q^* is the Ricci operator with respect to Zamkovoy connection.

Definition 1.5 ([11])**.** A $(2n+1)$-dimensional manifold M is called *M-projectively flat* with respect to Zamkovoy connection if $\overline{M}^*(X,Y)Z = 0$ for all $X, Y, Z \in \chi(M)$.

Definition 1.6 ([11])**.** A $(2n+1)$-dimensional manifold M is called *ξ-M-projectively flat* with respect to Zamkovoy connection if $\overline{M}^*(X,Y)\xi = 0$ for all $X, Y \in \chi(M)$.

Prasad introduced the notion of pseudo projective curvature tensor in a Riemannian manifold of dimension $n > 2$ in 2002, see [23]. Its properties were further studied by many researchers on various manifolds (see [13], [15], [16], [24], [28]). The *pseudo projective curvature tensor* $\overline{P}$ of rank 3 on an n-dimensional manifold M is given by

$$\overline{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] + cr[g(Y,Z)X - g(X,Z)Y]$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where a, b, c are nonzero constants related as $c = -n^{-1}(a(n-1)^{-1} + b)$, see [23].

Thus, for a $(2n+1)$-dimensional manifold, considering $\overline{P}^*$ as the pseudo projective curvature tensor with respect to Zamkovoy connection, we get

$$\begin{aligned}\overline{P}^*(X,Y)Z = aR^*(X,Y)Z &+ b[S^*(Y,Z)X - S^*(X,Z)Y] \\ &+ cr^*[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.6}$$

where a, b, c are nozero constants related as

$$c = -\frac{1}{2n+1}\Big(\frac{a}{2n} + b\Big). \tag{1.7}$$

Definition 1.7 ([13])**.** A $(2n+1)$-dimensional manifold M is called *pseudo projectively flat* with respect to Zamkovoy connection if $\overline{P}^*(X,Y)Z = 0$ for all $X, Y, Z \in \chi(M)$.

Definition 1.8 ([13])**.** A $(2n+1)$-dimensional manifold M is called *ξ-pseudo projectively flat* with respect to Zamkovoy connection if $\overline{P}^*(X,Y)\xi = 0$ for all $X, Y \in \chi(M)$.

Motivated by the works mentioned above, in this paper the study was done on Ricci flat, concircularly flat, ξ-projectively flat, M-projectively flat, ξ-M-projectively flat, pseudo projectively flat and ξ-pseudo projectively flat anti-invariant submanifolds of a trans-Sasakian manifold with respect to Zamkovoy connection. This paper consists of seven sections. After introduction, the second section consists of a short description of trans-Sasakian manifold and anti-invariant submanifold. In the third, fourth, sixth and seventh section, Ricci flat, concircularly flat, M-projectively flat and pseudo projectively flat anti-invariant submanifolds of a trans-Sasakian manifold admitting Zamkovoy connection are discussed, respectively. Ricci solitons on those submanifolds are also studied. Also we have found out the conditions under which an anti-invariant submanifold of a trans-Sasakian manifold is ξ-projectively, ξ-M-projectively and ξ-pseudo projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection in the fifth, sixth and seventh section, respectively. At last, three conclusions are made after observing all the results and an example of an anti-invariant submanifold of a trans-Sasakian manifold is given in which all the results can be verified easily.

2. Preliminaries

Let M be an odd dimensional differentiable manifold equipped with a metric structure (φ, ξ, η, g) consisting of a (1,1) tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying the relations

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \quad \eta(X) = g(X, \xi) \quad \forall\, X, Y \in \chi(M). \tag{2.3}$$

Then M is called *almost contact metric manifold* (see [7]). An almost contact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is called *trans-Sasakian manifold of type* (α, β) (α, β are smooth functions on M) if for all $X, Y \in \chi(M)$ (see [7])

$$(\nabla_X \varphi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X], \tag{2.4}$$

$$\nabla_X \xi = -\alpha\varphi X + \beta[X - \eta(X)\xi]. \tag{2.5}$$

In a trans-Sasakian manifold of type (α, β), we have the following relations (see [7])

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta[g(X, Y) - \eta(X)\eta(Y)], \tag{2.6}$$

$$\begin{aligned} R(X, Y)\xi = {} & (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] \\ & + [(Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y], \end{aligned} \tag{2.7}$$

$$\begin{aligned} R(\xi, Y)X = {} & (\alpha^2 - \beta^2)[g(X, Y)\xi - \eta(X)Y] + 2\alpha\beta[g(\varphi X, Y)\xi + \eta(X)\varphi Y] \\ & + g(\varphi X, Y)(\operatorname{grad}\alpha) - g(\varphi X, \varphi Y)(\operatorname{grad}\beta) \\ & + (X\alpha)\varphi Y + (X\beta)[Y - \eta(Y)\xi], \end{aligned} \tag{2.8}$$

$$S(X, \xi) = [2n(\alpha^2 - \beta^2) - \xi\beta]\eta(X) - (\varphi X)\alpha - (2n-1)(X\beta), \tag{2.9}$$

$$Q\xi = [2n(\alpha^2 - \beta^2) - \xi\beta]\xi + \varphi(\operatorname{grad}\alpha) - (2n-1)(\operatorname{grad}\beta). \tag{2.10}$$

Now we state the following lemma.

Lemma 2.1 ([25]). *In a $(2n+1)$-dimensional trans-Sasakian manifold of type (α, β), if $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$, then $\xi\beta = 0$.*

In 1977, anti-invariant submanifolds of Sasakian space forms were discussed by Yano and Kon (see [31]). In 1985, Pandey and Kumar investigated anti-invariant submanifolds of almost para-contact manifolds (see [20]). Recently, Karmakar and Bhattacharyya studied anti-invariant submanifolds of some indefinite almost contact and paracontact manifolds (see [10]).

Let φ be a differentiable map from a manifold M into a manifold $\widetilde{M}$ and let the dimensions of M, $\widetilde{M}$ be n, m, respectively. If at each point p of M, $(\varphi_*)_p$ is a 1-1 map, i.e., if rank φ=n, then φ is called an *immersion* of M into $\widetilde{M}$.

If an immersion φ is one-one, i.e., if $\varphi(p) \neq \varphi(q)$ for $p \neq q$, then φ is called an *imbedding* of M into $\widetilde{M}$.

If the manifolds M, $\widetilde{M}$ satisfy the following two conditions, then M is called a *submanifold* of $\widetilde{M}$:

(i) $M \subset \widetilde{M}$,

(ii) the identity map i from M into $\widetilde{M}$ is an imbedding of M into $\widetilde{M}$.

A submanifold M is called *anti-invariant* if $X \in T_x(M) \Rightarrow \varphi X \in T_x^{\perp}(M)$ for all $x \in M$, where $T_x(M)$, $T_x^{\perp}(M)$ are, respectively, the tangent space and the normal space at $x \in M$. Thus, in an anti-invariant submanifold M, we have for all $X, Y \in \chi(M)$,

$$g(X, \varphi Y) = 0. \tag{2.11}$$

3. Ricci flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

This section consists of the study of the nature of a $(2n+1)$-dimensional Ricci flat anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α, β) with respect to Zamkovoy connection and further a Ricci soliton on it.

Using (2.5), (2.6) on (1.1) we get the expression of Zamkovoy connection on $\widetilde{M}$ as

$$\nabla^*_X Y = \nabla_X Y + \eta(X)\varphi Y + \alpha\eta(Y)\varphi X - \beta\eta(Y)X + \beta g(X, Y)\xi - \alpha g(\varphi X, Y)\xi \tag{3.1}$$

with torsion tensor

$$T^*(X, Y) = (1-\alpha)[\eta(X)\varphi Y - \eta(Y)\varphi X] + \beta[\eta(X)Y - \eta(Y)X] + 2\alpha g(X, \varphi Y)\xi. \tag{3.2}$$

Again, we have

$$(\nabla^*_X g)(Y,Z) = \nabla^*_X g(Y,Z) - g(\nabla^*_X Y, Z) - g(Y, \nabla^*_X Z).$$

Then, using (3.1) in the above equation we obtain $\nabla^* g = 0$, i.e., Zamkovoy connection is a metric compatible connection on $\widetilde{M}$.

Now applying (2.11) in (3.1) and (3.2), respectively, we get the expression of Zamkovoy connection on M as

$$\nabla^*_X Y = \nabla_X Y + \eta(X)\varphi Y + \alpha\eta(Y)\varphi X - \beta\eta(Y)X + \beta g(X,Y)\xi \tag{3.3}$$

with torsion tensor

$$T^*(X,Y) = (1-\alpha)[\eta(X)\varphi Y - \eta(Y)\varphi X] + \beta[\eta(X)Y - \eta(Y)X].$$

Applying (2.4), (2.5) and (3.3) on the equation

$$R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z$$

we get for all $X, Y, Z \in \chi(M)$,

$$\begin{aligned} R^*(X,Y)Z = R(X,Y)Z &+ \alpha^2[\eta(X)Y - \eta(Y)X]\eta(Z) \\ &+ \beta[\eta(X)\varphi Y - \eta(Y)\varphi X]\eta(Z) + \beta^2[g(Y,Z)X - g(X,Z)Y] \\ &+ \alpha\beta[g(X,Z)\varphi Y - g(Y,Z)\varphi X] + \beta[\nabla_Y g(X,Z) - \nabla_X g(Y,Z)]\xi. \end{aligned} \tag{3.4}$$

Consequently, if $\xi\beta = 0$, then we have

$$S^*(Y,Z) = S(Y,Z) - 2n\alpha^2\eta(Y)\eta(Z) + 2n\beta^2 g(Y,Z), \tag{3.5}$$

which implies that

$$Q^*Y = QY - 2n\alpha^2\eta(Y)\xi + 2n\beta^2 Y. \tag{3.6}$$

Now if M is Ricci flat with respect to Zamkovoy connection, then $S^*(Y,Z) = 0$, hence (3.5) implies

$$S(Y,Z) = 2n\alpha^2\eta(Y)\eta(Z) - 2n\beta^2 g(Y,Z). \tag{3.7}$$

Thus, using Lemma 2.1 and (3.7) we can state the following theorem.

Theorem 3.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is Ricci flat with respect to Zamkovoy connection, then M is η-Einstein if $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

Next, let us consider a Ricci soliton (g, ξ, λ) on M, then from (1.2) we get

$$\begin{aligned} &(L_\xi g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0 \\ &\qquad \Rightarrow g(\nabla_Y \xi, Z) + g(\nabla_Z \xi, Y) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0. \end{aligned}$$

Using (2.5) and (2.11) on the above equation we obtain

$$2S(Y,Z)+2(\lambda+\beta)g(Y,Z)-2\beta\eta(Y)\eta(Z)=0.$$

Setting $Z=\xi$ we get

$$S(Y,\xi)=-\lambda\eta(Y). \tag{3.8}$$

Putting $Z=\xi$ in (3.7) we obtain

$$S(Y,\xi)=2n(\alpha^2-\beta^2)\eta(Y). \tag{3.9}$$

Now, equating (3.8) and (3.9) we get $\lambda=2n(\beta^2-\alpha^2)$, which is <0, $=0$ or >0 according to $|\beta|<|\alpha|$, $|\beta|=|\alpha|$ or $|\beta|>|\alpha|$. Thus, using Lemma 2.1 we can state the following theorem.

Theorem 3.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is Ricci flat with respect to Zamkovoy connection, then a Ricci soliton (g,ξ,λ) on M is shrinking, steady or expanding according to $|\beta|<|\alpha|$, $|\beta|=|\alpha|$ or $|\beta|>|\alpha|$, provided $\varphi(\operatorname{grad}\alpha)=(2n-1)(\operatorname{grad}\beta)$.*

4. Concircularly flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

This section deals with the study of the nature of a $(2n+1)$-dimensional concircularly flat anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) with respect to Zamkovoy connection given by (3.3) and also a Ricci soliton on it. From (3.5) we get

$$r^*=r-2n\alpha^2+2n(2n+1)\beta^2. \tag{4.1}$$

As M is concircularly flat with respect to Zamkovoy connection, from (1.3) we have

$$R^*(X,Y)Z=\frac{r^*}{2n(2n+1)}[g(Y,Z)X-g(X,Z)Y]\Rightarrow S^*(Y,Z)=\frac{r^*}{2n+1}g(Y,Z).$$

Using (3.5) and (4.1) on the above equation we obtain

$$S(Y,Z)=\frac{r-2n\alpha^2}{2n+1}g(Y,Z)+2n\alpha^2\eta(Y)\eta(Z), \tag{4.2}$$

which, using Lemma 2.1, shows the following theorem.

Theorem 4.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is concircularly flat with respect to Zamkovoy connection, then M is η-Einstein if $\varphi(\operatorname{grad}\alpha)=(2n-1)(\operatorname{grad}\beta)$.*

Setting $Z = \xi$ in (4.2) we get

$$S(Y,\xi) = \frac{r + 4n^2\alpha^2}{2n+1}\eta(Y). \tag{4.3}$$

Next, let us consider a Ricci soliton (g, ξ, λ) on M, then equating (3.8) and (4.3) we get

$$\lambda = -\frac{r + 4n^2\alpha^2}{2n+1},$$

which is $< 0, = 0$ or > 0 according to $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$.

Hence, using Lemma 2.1 we have the following theorem.

Theorem 4.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α, β) is concircularly flat with respect to Zamkovoy connection, then a Ricci soliton (g, ξ, λ) on M is shrinking, steady or expanding according to $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

5. ξ-projectively flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

In this section, it will be proved that a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ is ξ-projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Levi-Civita connection under certain conditions.

If $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$, then using Lemma 2.1 from (1.4), (3.4) and (3.5) we have

$$\begin{aligned}
P^*(X,Y)Z &= R^*(X,Y)Z - \frac{1}{2n}[S^*(Y,Z)X - S^*(X,Z)Y] \\
&= R(X,Y)Z + \alpha^2[\eta(X)Y - \eta(Y)X]\eta(Z) \\
&\quad + \beta[\eta(X)\varphi Y - \eta(Y)\varphi X]\eta(Z) + \beta^2[g(Y,Z)X - g(X,Z)Y] \\
&\quad + \alpha\beta[g(X,Z)\varphi Y - g(Y,Z)\varphi X] + \beta[\nabla_Y g(X,Z) - \nabla_X g(Y,Z)]\xi \\
&\quad - \frac{1}{2n}[S(Y,Z)X - 2n\alpha^2\eta(Y)\eta(Z)X + 2n\beta^2 g(Y,Z)X \\
&\qquad - S(X,Z)Y + 2n\alpha^2\eta(X)\eta(Z)Y - 2n\beta^2 g(X,Z)Y] \\
&= P(X,Y)Z + \alpha^2[\eta(X)Y - \eta(Y)X]\eta(Z) \\
&\quad + \beta[\eta(X)\varphi Y - \eta(Y)\varphi X]\eta(Z) + \beta^2[g(Y,Z)X - g(X,Z)Y] \\
&\quad + \alpha\beta[g(X,Z)\varphi Y - g(Y,Z)\varphi X] + \beta[\nabla_Y g(X,Z) - \nabla_X g(Y,Z)]\xi \\
&\quad - [-\alpha^2\eta(Y)\eta(Z)X + \beta^2 g(Y,Z)X + \alpha^2\eta(X)\eta(Z)Y - \beta^2 g(X,Z)Y].
\end{aligned}$$

That implies

$$P^*(X,Y)\xi = P(X,Y)\xi + (\alpha\beta+\beta)[\eta(X)\varphi Y - \eta(Y)\varphi Y] + \beta[\nabla_Y \eta(X) - \nabla_X \eta(Y)]\xi.$$

Again, using (2.5), (2.11) on (2.6) we get

$$\nabla_X \eta(Y) = 0, \tag{5.1}$$

and applying it on the above equation we obtain

$$P^*(X,Y)\xi = P(X,Y)\xi + \beta(\alpha+1)[\eta(X)\varphi Y - \eta(Y)\varphi X] \Rightarrow P^*(X,Y)\xi = P(X,Y)\xi$$

if $\alpha = -1$ or $\beta = 0$ or X, Y are horizontal vector fields. Therefore we can state the following theorem.

Theorem 5.1. *A $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is ξ-projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection if $\alpha = -1$ or $\beta = 0$ or the vector fields are horizontal, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

6. M-projectively flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

In this section, a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is taken and its nature is studied when it is M-projectively flat and ξ-M-projectively flat with respect to Zamkovoy connection. Also a Ricci soliton on M is discussed.

If M is M-projectively flat with respect to Zamkovoy connection, then from (1.5) we have

$$R^*(X,Y)Z = \frac{1}{4n}[S^*(Y,Z)X - S^*(X,Z)Y] + \frac{1}{4n}[g(Y,Z)Q^*X - g(X,Z)Q^*Y]$$

and then

$$S^*(Y,Z) = \frac{r^*}{2n+1} g(Y,Z). \tag{6.1}$$

If $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$, then using (3.5) and (4.1) on (6.1) and by Lemma 2.1 we obtain

$$S(Y,Z) = \frac{r - 2n\alpha^2}{2n+1} g(Y,Z) + 2n\alpha^2 \eta(Y)\eta(Z), \tag{6.2}$$

from which we can conclude the following theorem.

Theorem 6.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α, β) is M-projectively flat with respect to Zamkovoy connection, then M is η-Einstein if $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

Putting $Z = \xi$ in (6.2) we have

$$S(Y, \xi) = \frac{r + 4n^2\alpha^2}{2n+1}\eta(Y). \tag{6.3}$$

Let us consider a Ricci soliton (g, ξ, λ) on M. Then equating (3.8) and (6.3) we get

$$\lambda = -\frac{r + 4n^2\alpha^2}{2n+1},$$

which is $< 0, = 0$ or > 0 according to $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$. Hence, by using Lemma 2.1 we can state the following theorem.

Theorem 6.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α, β) is M-projectively flat with respect to Zamkovoy connection, then a Ricci soliton (g, ξ, λ) on M is shrinking, steady or expanding according to $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

Now if $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$, then applying (3.4), (3.5) and (3.6) on (1.5) we have

$$\begin{aligned}\overline{M}^*(X, Y)Z = {} & \overline{M}(X, Y)Z + \alpha^2[\eta(X)Y - \eta(Y)X]\eta(Z) + \beta[\eta(X)\varphi Y - \eta(Y)\varphi X]\eta(Z) \\ & + \alpha\beta[g(X, Z)\varphi Y - g(Y, Z)\varphi X] + \beta[\nabla_Y g(X, Z) - \nabla_X g(Y, Z)]\xi \\ & + \frac{\alpha^2}{2}[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] + \frac{\alpha^2}{2}[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi.\end{aligned}$$

Putting $Z = \xi$ in the above equation and using (5.1) we obtain

$$\overline{M}^*(X, Y)\xi = \overline{M}(X, Y)\xi + \beta(\alpha+1)[\eta(X)\varphi Y - \eta(Y)\varphi X] + \frac{\alpha^2}{2}[\eta(X)Y - \eta(Y)X],$$

which implies that $\overline{M}^*(X, Y)\xi = \overline{M}(X, Y)\xi$ if X, Y are horizontal vector fields. Thus, we have the following theorem.

Theorem 6.3. *A $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α, β) is ξ-M-projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection if the vector fields are horizontal, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

7. Pseudo projectively flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

This section deals with the study of a pseudo projectively flat anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α, β) admitting Zamkovoy connection along with a Ricci soliton on it. Also the condition is established under which M is ξ-pseudo projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection.

Since M is pseudo projectively flat with respect to Zamkovoy connection, from (1.6) we have

$$\begin{aligned} aR^*(X,Y)Z &= b[S^*(X,Z)Y - S^*(Y,Z)X] + cr^*[g(X,Z)Y - g(Y,Z)X] \\ &\Rightarrow (a+2nb)S^*(Y,Z) = -2cnr^*g(Y,Z). \end{aligned} \tag{7.1}$$

Applying (1.7), (3.5), (4.1) and the condition $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$ of Lemma 2.1 on (7.1) we obtain

$$S(Y,Z) = 2n\alpha^2\eta(Y)\eta(Z) + \frac{r - 2n\alpha^2}{2n+1} g(Y,Z). \tag{7.2}$$

Thus, we can state the following theorem.

Theorem 7.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α, β) is pseudo projectively flat with respect to Zamkovoy connection, then M is η-Einstein if $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

Setting $Z = \xi$ in (7.2) we have

$$S(Y,\xi) = \frac{r + 4n^2\alpha^2}{2n+1}\eta(Y). \tag{7.3}$$

Now, considering a Ricci soliton (g, ξ, λ) on M we have (3.8) and then equating it with (7.3) we get

$$\lambda = -\frac{r + 4n^2\alpha^2}{2n+1},$$

which is $< 0, = 0$ or > 0 according to $r > -4n^2\alpha^2, r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$. Thus, we get the following theorem.

Theorem 7.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α, β) is pseudo projectively flat with respect to Zamkovoy connection, then a Ricci soliton (g, ξ, λ) on M is shrinking, steady or expanding according to $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

Now, if $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$, then putting $Z = \xi$ in (1.6) and using (1.7), (3.4), (3.5), (4.1), (5.1) we obtain

$$\begin{aligned}\overline{P}^*(X,Y)\xi = \overline{P}(X,Y)\xi &+ a\beta(\alpha+1)[\eta(X)\varphi Y - \eta(Y)\varphi X] \\ &+ \frac{2n}{2n+1}(a+2nb)\alpha^2[\eta(X)Y - \eta(Y)X]\end{aligned}$$

implies $\overline{P}^*(X,Y)\xi = \overline{P}(X,Y)\xi$ if X, Y are horizontal vector fields. Hence, we can state the following theorem.

Theorem 7.3. *A $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is ξ-pseudo projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection if the vector fields are horizontal, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

From Theorems 3.1, 4.1, 6.1 and 7.1 we make the following conclusion.

Conclusion 7.1.
(i) Ricci flat,
(ii) concircularly flat,
(iii) M-projectively flat or
(iv) pseudo projectively flat
$(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) admitting Zamkovoy connection is η-Einstein if $\varphi(\operatorname{grad}\alpha) = (2n-1)\times(\operatorname{grad}\beta)$.

Next, observing Theorems 4.2, 6.2 and 7.2 we reach the following interesting conclusion.

Conclusion 7.2. If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is
(i) concircularly flat,
(ii) M-projectively flat or
(iii) pseudo projectively flat
with respect to Zamkovoy connection, then a Ricci soliton (g,ξ,λ) on M is shrinking, steady or expanding according to $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.

Again, observing Theorems 5.1, 6.3 and 7.3 we can conclude the following.

Conclusion 7.3. For horizontal vector fields, a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is
(i) ξ-projectively flat,
(ii) ξ-M-projectively flat and
(iii) ξ-pseudo projectively flat

with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.

Now we give the following example.

Example 7.1. Unit sphere S^5 is a trans-Sasakian manifold of type $(-1,0)$ (see [27]). We here state an example of an anti-invariant submanifold of S^5 from [31] as:

Let $J = (a_{ts})$ $(t, s = 1,2,3,4,5,6)$ be the almost complex structure of $\mathbb{C}^3$ such that $a_{2i,2i-1} = 1$, $a_{2i-1,2i} = -1$ $(i = 1,2,3)$ and all the other components are 0. Let $S^1\bigl(\frac{1}{\sqrt{3}}\bigr) = \{z \in \mathbb{C}\colon\ |z|^2 = \frac{1}{3}\}$. We consider $S^1\bigl(\frac{1}{\sqrt{3}}\bigr) \times S^1\bigl(\frac{1}{\sqrt{3}}\bigr) \times S^1\bigl(\frac{1}{\sqrt{3}}\bigr)$ in S^5 in $\mathbb{C}^3$. The position vector X of $S^1 \times S^1 \times S^1$ in S^5 in $\mathbb{C}^3$ has components given by

$$X = \frac{1}{\sqrt{3}}(\cos u^1, \sin u^1, \cos u^2, \sin u^2, \cos u^3, \sin u^3),$$

where u^1, u^2, u^3 are parameters on each $S^1\bigl(\frac{1}{\sqrt{3}}\bigr)$.

Let $X_i = \frac{\partial X}{\partial u^i}$, then we have

$$\begin{aligned}
X_1 &= \frac{1}{\sqrt{3}}(-\sin u^1, \cos u^1, 0, 0, 0, 0),\\
X_2 &= \frac{1}{\sqrt{3}}(0, 0, -\sin u^2, \cos u^2, 0, 0),\\
X_3 &= \frac{1}{\sqrt{3}}(0, 0, 0, 0, -\sin u^3, \cos u^3).
\end{aligned}$$

The vector field ξ on S^5 is given by

$$\xi = JX = \frac{1}{\sqrt{3}}(-\sin u^1, \cos u^1, -\sin u^2, \cos u^2, -\sin u^3, \cos u^3).$$

Since $\xi = X_1 + X_2 + X_3$, ξ is tangent to $S^1 \times S^1 \times S^1$. Also the structure tensors (φ, ξ, η) of S^5 satisfy

$$\varphi X_i = JX_i + \eta(X_i)X, \quad i = 1,2,3,$$

which shows that φX_i is normal to $S^1 \times S^1 \times S^1$ for all i. Thus, $S^1 \times S^1 \times S^1$ is an anti-invariant submanifold of S^5.

Now the results proved in this paper can be verified in the example given above very easily.

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