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DIRECT SUMMANDS OF GOLDIE EXTENDING ELEMENTS IN MODULAR LATTICES

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Abstract. In this paper some results on direct summands of Goldie extending elements are studied in a modular lattice. An element a of a lattice L with 0 is said to be a Goldie extending element if and only if for every $b \leqslant a$ there exists a direct summand c of a such that $b \wedge c$ is essential in both b and c. Some characterizations of decomposition of a Goldie extending element in a modular lattice are obtained.

Keywords: modular lattice; direct summand; Goldie extending element

MSC 2020: 06B10, 06C05

1. Introduction

The notion of extending modules has been studied by several researchers under the names such as extending module or a module with C_1-property or a CS-module. These modules and their generalizations are extensively studied by authors such as Harmanci and Smith (see [7]), Akalan, Birkenmeier and Tercan (see [1]), Dung et al. (see [4]). A module M is called *extending* if every submodule of M is essential in a direct summand of M.

Călugăreanu in [2] studied several concepts from module theory in lattices. In [1], Akalan, Birkenmeier and Tercan defined a Goldie extending module over an associative ring. They posed and answered the following open problem.

Open Problem: Determine necessary and/or sufficient conditions for a direct sum of Goldie extending modules to be Goldie extending.

In [1], Akalan et al. posed five open questions on the notion of Goldie extending module, see also [10]; out of which the following two are answered by Wu and Wang in [11]:

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(1) Determine necessary and/or sufficient conditions on a module M so that M is G-extending if and only if M is G^+-extending.
(2) Determine necessary and/or sufficient conditions for a direct sum of G-extending modules to be G-extending.

As an analogue of the concept of a Goldie extending module, Nimbhorkar and Shroff (see [9]) defined a Goldie extending element in a modular lattice and proved some properties of such elements. They also formulated and answered the following open problem in the context of a modular lattice.

Open Problem: Determine necessary and/or sufficient conditions for a direct sum of Goldie extending elements in a lattice to be Goldie extending.

In this paper, a lattice theoretic analogue is obtained from the results of Wu and Wang (see [11]). Some properties of direct summands of a Goldie extending element are studied in a lattice L with 0 and some properties are obtained in a certain class of modular lattices by using the concept of relative ejectivity and ojectivity. The second section deals with preliminaries required in subsequent sections. In the third section, some properties of direct summands of a Goldie extending element are obtained by using the concept of ejectivity and ojectivity in modular lattices. The second open problem stated above is answered for a certain class of lattices.

2. Preliminaries

The concepts of lattice theory used in this paper are from Grätzer (see [5]) and Crawley and Dilworth (see [3]). Throughout this paper L denotes a lattice with the least element 0. The following definitions are from Călugăreanu, see [2].

Let $a, b \in L$, $a \leqslant b$, a is said to be *essential* in b (or b is an essential extension of a) if there is no nonzero $c \leqslant b$ such that $a \wedge c = 0$. It is denoted by $a \leqslant_e b$. If $a \leqslant_e b$ and there is no $c > b$ such that $a \leqslant_e c$, then b is called a *maximal essential extension* of a. An element $a \in L$ is *closed* (or essentially closed) in b if a has no proper essential extension in b.

The concept of the max-semicomplement is defined by Nimbhorkar and Shroff in [8]. If $a, b \in L$ and b is a maximal element in the set $\{x\colon\ x \in L, a \wedge x = 0\}$, then b is said to be a *max-semicomplement* of a.

Some properties of essential extensions and closed extensions that are used in subsequent sections follow.

Lemma 2.1 ([6], Lemma 2). *In a lattice L with 0, the following statements hold.*
(1) *If $a, b, c \in L$, then $a \leqslant_e b$ implies $a \wedge c \leqslant_e b \wedge c$.*
(2) *If $a \leqslant b \leqslant c$, then $a \leqslant_e b$, $b \leqslant_e c$ if and only if $a \leqslant_e c$.*

The concept of direct summands in lattices is introduced by Nimbhorkar and Shroff in [9]. If $a, b, c \in L$ are such that $a \vee b = c$ and $a \wedge b = 0$, then a and b are called *direct summands* of c and it is denoted by $c = a \oplus b$. Here c is a direct sum of a and b. In a modular lattice L, if $a, b, c \in L$ are such that $c = a \oplus b$, then a is a max-semicomplement of b in c. Hence, the direct summands of c are closed in c. Also, in a modular lattice L if $a, b, c \in L$ are such that $a \leqslant b \leqslant c$ and a is a direct summand of c, then a is also a direct summand of b.

Let $a \in L$. If for any two direct summands b, c of a, $b \vee c$ is a direct summand of a, then a satisfies the *summand sum property.* Also, if for any two direct summands b, c of a with $b \wedge c \neq 0$, $b \wedge c$ is a direct summand of a, then a satisfies the *summand intersection property.*

In a modular lattice L if an element a satisfies the summand sum (intersection) property, then every direct summand of a satisfies the summand sum (intersection) property.

Lemma 2.2 ([6], Lemma 3)**.** *Let L be a modular lattice with 0. Suppose that $a, b, c, d \in L$ are such that $a \leqslant b$, $c \leqslant d$ and $b \wedge d = 0$. Then $a \leqslant_e b$, $c \leqslant_e d$ if and only if $a \oplus c \leqslant_e b \oplus d$.*

Throughout this paper, wherever necessary, it is assumed that L satisfies one or more of the following conditions:

Condition (1): For any $a \leqslant b$ there exists a maximal essential extension of a in b.

Condition (2): For any $a \leqslant b$ and for any $c \leqslant b$ with $c \wedge a = 0$, there exists a max-semicomplement $d \geqslant c$ of a in b.

A familiar and important class of lattices with these properties is the class of upper continuous modular lattices, in particular, the lattices of ideals of a modular lattice with 0.

Lemma 2.3 ([9])**.** *Let L be a modular lattice with 0 satisfying condition (2). Let $a, b \in L$ and $a \leqslant b$. Then a is closed in b if and only if a is a max-semicomplement of some $c \leqslant b$.*

Proposition 2.1 ([2], Proposition 4.4)**.** *Let L be a modular lattice with 0 satisfying condition (2) and $a, b \in L$ be such that $a \wedge b = 0$. Then a is a max-semicomplement of b in L if and only if a is closed in L and $a \vee b$ is essential in L.*

An element a of a lattice L is called *extending* if every nonzero $b \leqslant a$ is essential in a direct summand of a.

Note that in a modular lattice L satisfying condition (1), every nonzero $a \leqslant b$ has a maximal essential extension in b and a maximal essential extension is closed. Hence, a nonzero $a \in L$ is extending if every $b \leqslant a$, which is closed in a, is a direct summand of a.

Let L be a modular lattice satisfying conditions (1) and (2) and $a \leqslant b$. Then b is extending if every max-semicomplement in b is a direct summand of b. Also, if $a \in L$ is extending, then every direct summand of a is extending.

As a generalization of an extending element, Nimbhorkar and Shroff in [9] defined a Goldie extending element in a lattice. They have also defined the following relations and an a-ejective element in a lattice.

Let $a, b \in L$. Then

(1) $a \,\alpha\, b$ if and only if there exists $c \in L$ such that $a \leqslant_e c$ and $b \leqslant_e c$.

(2) $a \,\beta\, b$ if and only if $a \wedge b \leqslant_e a$ and $a \wedge b \leqslant_e b$.

Note that $a \,\alpha\, b$ implies $a \,\beta\, b$, but the converse need not hold. $a \,\beta\, b$ is an equivalence relation on L. By using the condition β, a Goldie extending element in a lattice L with 0 is defined.

Definition 2.1. Let L be a lattice and $a \in L$. If for every $b \leqslant a$ there exists a direct summand c of a such that $b \,\beta\, c$, then a is said to be a *Goldie extending* (*G-extending*) element.

Equivalently, a G-extending element in a modular lattice L can be defined as follows.

An element $a \in L$ is called a *Goldie extending element* if for every closed element $b \leqslant a$ there exists a direct summand c of a such that $b \,\beta\, c$ holds.

In the following result a necessary and sufficient condition for an element to be Goldie extending is given.

Lemma 2.4. *Let L be a lattice and $a \in L$. Then the following statements are equivalent.*

(1) *a is a G-extending element.*

(2) *For every $b \leqslant a$ there exists $c \leqslant a$ and a direct summand d of a such that $c \leqslant_e b$ and $c \leqslant_e d$.*

3. Direct summands of Goldie extending elements

In this section, the properties of direct summands of G-extending elements are studied in the context of summand sum property, relative ojectivity and relative ejectivity. The notion of a module N being M-injective is generalized to M-ejective by Akalan et al. (see [1], Definition 2.1). An analogue of the same is defined by Nimbhorkar and Shroff in [9] as follows:

Definition 3.1. Let $a, b, c \in L$ be such that $a = b \oplus c$. Then b is said to be *c-ejective* in a if for every $d \leqslant a$ such that $d \wedge b = 0$ there exists an $f \leqslant a$ such that $a = b \oplus f$ and $d \wedge f \leqslant_e d$. If b is c-ejective and c is b-ejective, then b and c are said to be *relatively ejective.*

Lemma 3.1. *Let L be a modular lattice satisfying conditions* (1) *and* (2) *and $a, a_1, a_2 \in L$. Suppose that $a = a_1 \oplus a_2$ and a is G-extending and a satisfies the summand sum property. Then a_1 and a_2 are relatively ejective.*

Proof. Let $b \in L$ be such that $b \leqslant a$ and $b \wedge a_1 = 0$. Since a is G-extending, there exists a direct summand d of a such that $b \wedge d \leqslant_e b$ and $b \wedge d \leqslant_e d$. Now $b \wedge a_1 = 0$ implies that $(b \wedge d) \wedge a_1 = 0$ and $b \wedge (d \wedge a_1) = 0$, $b \wedge d \leqslant_e d$ implies that $d \wedge a_1 = 0$.

Since a satisfies the summand sum property, $d \wedge a_1 = 0$ implies that $d \oplus a_1$ is a direct summand of a. Hence, there exists a direct summand c of a such that $a = a_1 \oplus d \oplus c$. Now it remains to show that $b \wedge (d \oplus c) \leqslant_e b$. Since $b \wedge d \leqslant b \wedge (d \oplus c) \leqslant b$ and $b \wedge d \leqslant_e b$, by Lemma 2.1 (2), $b \wedge (d \oplus c) \leqslant_e b$. Hence a_1 is a_2-ejective.

Similarly, it can be proved that a_2 is a_1-ejective. □

The concept of an ojective ideal in a lattice is defined by Nimbhorkar and Shroff in [8] as follows:

Definition 3.2. Let $I, J, K \in I(L)$ be such that $K = I \oplus J$. The ideal J is said to be *I-ojective* if for any max-semicomplement C of J in K, K can be decomposed as $K = C \oplus I' \oplus J'$ with $I' \subseteq I$ and $J' \subseteq J$.

An analogue of this concept is formulated as an ojective element in a lattice as follows:

Definition 3.3. Let $a, b, c \in L$ be such that $c = a \oplus b$. The element b is said to be *a-ojective* if for any max-semicomplement k of b in c, c can be decomposed as $c = k \oplus a' \oplus b'$ with $a' \subseteq a$ and $b' \subseteq b$.

If a is b-ojective and b is a-ojective for some $c = a \oplus b$, then a and b are called *mutually ojective.*

E x a m p l e 3.1. In the lattice shown in Figure 1, consider the elements k, f, j such that $k = f \oplus j$. Also, j has a max-semicomplement c and there exist elements $a \leqslant f$ and $d \leqslant j$ such that $k = f \oplus j = a \oplus d \oplus c$. Hence j is f-ojective.

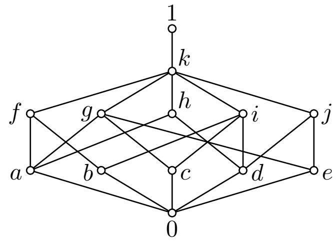

Figure 1.

The following proposition is proved by Nimbhorkar and Shroff in [8] for ideals of a lattice. However, it can be similarly proved for elements of a lattice satisfying conditions (1) and (2).

Proposition 3.1. *Let L be a modular lattice and $I, J, K \in \mathrm{Id}(L)$ be such that $K = I \oplus J$. Let I_1 and J_1 be direct summands of I and J, respectively. If J is I-ojective, then:*

(1) *J_1 is I-ojective,*
(2) *J is I_1-ojective,*
(3) *J_1 is I_1-ojective.*

Lemma 3.2. *Let L be a modular lattice satisfying conditions* (1) *and* (2) *and $a, b, c \in L$ be such that $c = a \oplus b$. Suppose that b is a-ojective and a is G-extending. If $e \in L$ be such that $e \leqslant a$ and $e \wedge b = 0$, then there exists a direct summand d of c such that $e \,\beta\, d$ and $c = d \oplus a' \oplus b'$ for $a' \leqslant a$, $b' \leqslant b$.*

P r o o f. Let $k = (e \oplus b) \wedge a$. Then by modularity of L,

$$b \oplus k = b \oplus [(e \oplus b) \wedge a] = (b \oplus a) \wedge (e \oplus b) = c \wedge (e \oplus b) = e \oplus b.$$

Since a is G-extending, there exist a direct summand a_1 of a such that

$$k \wedge a_1 \leqslant_e k \quad \text{and} \quad k \wedge a_1 \leqslant_e a_1.$$

Put $f = a_1 \oplus b$. Then

$$(k \wedge a_1) \oplus b \leqslant_e a_1 \oplus b = f \quad \text{and} \quad (k \wedge a_1) \oplus b \leqslant_e k \oplus b = e \oplus b.$$

Now,

$$k \wedge a_1 = (e \oplus b) \wedge a \wedge a_1 = (e \oplus b) \wedge a_1.$$

By modularity of L,

$$(k \wedge a_1) \oplus b = [(e \oplus b) \wedge a_1] \oplus b = (e \oplus b) \wedge (a_1 \oplus b) = (e \oplus b) \wedge f.$$

Hence $(k \wedge a_1) \oplus b \leqslant_e (e \oplus b)$. This implies that $(e \oplus b) \wedge f \leqslant_e (e \oplus b)$.

Again, by modularity for $b \leqslant f$, $(e \oplus b) \wedge f = b \oplus (e \wedge f)$. Hence, by Lemma 2.2, $e \wedge f \leqslant_e e$.

Now $e \wedge f \leqslant e$ implies that there exists a closed element $f_1 \leqslant f$ which is a maximal essential extension of $e \wedge f$ such that $e \wedge f \leqslant_e f_1$.

Now, by using modularity,

$$(k \wedge a_1) \oplus b \leqslant_e f \Rightarrow (e \oplus b) \wedge f \leqslant_e f \Rightarrow b \oplus (e \wedge f) \leqslant_e f.$$

Since f_1 is a maximal essential extension of $e \wedge f$ in f and $b \oplus (e \wedge f) \leqslant_e f$, it follows that f_1 is a max-semicomplement of b in f. But b is a-ojective and therefore b is a_1-ojective, which yields

$$f = f_1 \oplus a_1' \oplus b', \quad a_1' \leqslant a_1, b' \leqslant b.$$

Now, there exists a direct summand a_2 of a such that $a = a_1 \oplus a_2$. Therefore

$$c = a_1 \oplus a_2 \oplus b = f \oplus a_2 = f_1 \oplus a_1' \oplus b' \oplus a_2, \quad a_1' \oplus a_2 \leqslant a, \quad b' \leqslant b.$$

Also, $e \wedge f \leqslant f_1 \leqslant f$ and so $e \wedge f = e \wedge f_1$. Hence, $e \wedge f_1 \leqslant_e e$, $e \wedge f_1 \leqslant_e f_1$, and so $e \,\beta\, f_1$ holds. This completes the proof. $\square$

Lemma 3.3. *Let L be a modular lattice satisfying conditions* (1) *and* (2) *and $a, a_1, a_2 \in L$ be such that $a = a_1 \oplus a_2$. Suppose that a satisfies the summand sum property. If a_1 is a_2-ojective (or a_2 is a_1-ojective) and a_1 (or a_2) is G-extending, then a is G-extending.*

Proof. Let $b \in L$ be such that $b \leqslant a$. If $b \wedge a_1 = 0$, then by Lemma 3.2, there exists a direct summand d of a such that $b \,\beta\, d$. So a is G-extending.

If $b \wedge a_1 \neq 0$, then there exists a maxsemicomplement $k \leqslant b$ of $b \wedge a_1$ such that $(b \wedge a_1) \oplus k \leqslant_e b$. It is clear that $k \wedge a_1 = 0$. Then by Lemma 3.2, there exists a direct summand c of a such that $k \,\beta\, c$, i.e. $k \wedge c \leqslant_e k$, $k \wedge c \leqslant_e c$. Note that $(k \wedge c) \wedge a_1 = 0$.

Since a_1 is G-extending by Lemma 2.4, for $b \wedge a_1$ there exists some $m \leqslant a_1$ and a direct summand d_1 of a_1 such that $m \leqslant_e b \wedge a_1$ and $m \leqslant_e d_1$. Then $m \leqslant_e b \wedge a_1$, $(b \wedge a_1) \wedge k = 0$, $k \wedge c \leqslant_e k$ together imply that $(k \wedge c) \oplus m \leqslant_e (b \wedge a_1) \oplus k \leqslant_e b$. Hence $(k \wedge c) \oplus m \leqslant_e b$. Also, $k \wedge c \leqslant_e c$ and $m \leqslant_e d_1$, $c \wedge a_1 = 0$ together imply that $(k \wedge c) \oplus m \leqslant_e c \oplus d_1$.

Since a satisfies the summand sum property, $c \oplus d_1$ is a direct summand of a. Hence by Lemma 2.4, a is G-extending. $\square$

Theorem 3.1. *Let L be a modular lattice satisfying conditions* (1) *and* (2) *and $a, a_1, a_2 \in L$ be such that $a = a_1 \oplus a_2$. Suppose that a satisfies the summand sum property. If a_1 is a_2-ojective (or a_2 is a_1-ojective) and a_1 and a_2 are G-extending, then a_1 and a_2 are relatively ejective.*

P r o o f. Follows from Lemma 3.1 and Lemma 3.3. □

Theorem 3.2. *Let L be a modular lattice satisfying conditions* (1) *and* (2)*. Let $a, a_i \in L$ for an indexing set $i \in I$ be such that $a = \bigoplus_{i\in I} a_i$ with $|I| \geqslant 2$. Suppose that a satisfies the summand sum property. Then the following statements are equivalent.*

(1) *a is G-extending.*
(2) *If there exists $i \neq j$ in I such that for every $b \leqslant a$ with $b \wedge a_i = 0$ or $b \wedge a_j = 0$, then there exists a direct summand d of a such that $b \,\beta\, d$.*
(3) *If there exists $i \neq j$ in I such that for every $b \leqslant a$ with $b \wedge a_i \leqslant_e b$ or $b \wedge a_j \leqslant_e b$ or $b \wedge a_i = b \wedge a_j = 0$, then there exists a direct summand d of a such that $b \,\beta\, d$.*

P r o o f. (1) $\Rightarrow$ (3): Let a be a G-extending element in L. Then statement (3) holds by the definition.

(3) $\Rightarrow$ (2): Let (3) hold and let $b \leqslant a$ with $b \wedge a_j = 0$. Then there exists a max-semicompliment $m \leqslant b$ of $b \wedge a_i$, $i \neq j$ such that

$$(b \wedge a_i) \oplus m \leqslant_e b.$$

It is clear that $m \wedge a_i = m \wedge a_j = 0$, $i \neq j$. Then by (3), there exists a direct summand d of a such that $m \,\beta\, d$, that is

$$m \wedge d \leqslant_e m, \quad m \wedge d \leqslant_e d.$$

Now it is clear that $(b \wedge a_i) \wedge a_i \leqslant_e b \wedge a_i$. Therefore by (3), there exists a direct summand k of a such that $(b \wedge a_i) \,\beta\, k$, that is

$$(b \wedge a_i) \wedge k \leqslant_e b \wedge a_i \quad \text{and} \quad (b \wedge a_i) \wedge k \leqslant_e k.$$

Hence by Lemma 2.2,

$$(b \wedge a_i) \wedge k \leqslant_e b \wedge a_i, \quad m \wedge d \leqslant_e m$$

implies that

$$[(b \wedge a_i) \wedge k] \oplus (m \wedge d) \leqslant_e (b \wedge a_i) \oplus m \leqslant_e b$$

and

$$(b \wedge a_i) \wedge k \leqslant_e k, \quad m \wedge d \leqslant_e d.$$

This implies that

$$[(b \wedge a_i) \wedge k] \oplus (m \wedge d) \leqslant_e (k \oplus d).$$

Since a satisfies summand sum property, $k \oplus d$ is a direct summand of a. Thus $b \, \beta \, (k \oplus d)$.

The case $b \wedge a_i = 0$ is analogous. Hence (2) holds.

(2) $\Rightarrow$ (1): Let (2) hold and let $b \leqslant a$ be such that $b \wedge a_i \neq 0$. Then there exists a max-semicompliment $m \leqslant b$ of $b \wedge a_i$ such that $(b \wedge a_i) \oplus m \leqslant_e b$. It is clear that $m \wedge a_i = 0$. Then by (3), there exists a direct summand d of a such that $m \, \beta \, d$, that is,

$$m \wedge d \leqslant_e m, \quad m \wedge d \leqslant_e d.$$

Now it is clear that $(b \wedge a_i) \wedge a_j = 0$, $i \neq j$. Therefore by (2), there exists a direct summand k of a such that

$$(b \wedge a_i) \, \beta \, k,$$

that is,

$$(b \wedge a_i) \wedge k \leqslant_e b \wedge a_i \quad \text{and} \quad (b \wedge a_i) \wedge k \leqslant_e k.$$

Hence by Lemma 2.2,

$$(b \wedge a_i) \wedge k \leqslant_e b \wedge a_i, \quad m \wedge d \leqslant_e m.$$

This implies that

$$[(b \wedge a_i) \wedge k] \oplus (m \wedge d) \leqslant_e (b \wedge a_i) \oplus m \leqslant_e b$$

and

$$(b \wedge a_i) \wedge k \leqslant_e k, \quad m \wedge d \leqslant_e d.$$

This implies

$$[(b \wedge a_i) \wedge k] \oplus (m \wedge d) \leqslant_e (k \oplus d).$$

Since a satisfies summand sum property, $k \oplus d$ is a direct summand of a. Thus, by Lemma 2.4, a is G-extending. $\square$

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