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CONGRUENCES FOR CERTAIN FAMILIES  
OF APÉRY-LIKE SEQUENCES

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*Abstract.* We systematically investigate the expressions and congruences for both a one-parameter family  $\{G_n(x)\}$  as well as a two-parameter family  $\{G_n(r, m)\}$  of sequences.

*Keywords:* Apéry-like number; congruence; combinatorial identity; Bernoulli polynomial; binary quadratic form

*MSC 2020:* 05A10, 05A19, 11A07, 11B68, 11E25

1. INTRODUCTION

For  $s > 1$  let  $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ . In 1979, in order to prove that  $\zeta(3)$  and  $\zeta(2)$  are irrational, Apéry in [3] introduced the Apéry numbers  $\{A_n\}$  and  $\{A'_n\}$  given by

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \quad \text{and} \quad A'_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}.$$

In 2009 Zagier in [36] studied the Apéry-like numbers  $\{u_n\}$  given by

$$u_0 = 1, \quad u_1 = b \quad \text{and} \quad (n+1)^2 u_{n+1} = (an(n+1) + b)u_n - cn^2 u_{n-1} \quad (n \geq 1),$$

where  $a, b, c \in \mathbb{Z}$ ,  $c \neq 0$  and  $u_n \in \mathbb{Z}$  for  $n = 1, 2, 3, \dots$ . Let

$$G_n = \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{2k}{k}^2 16^{n-k}.$$

According to [36],  $\{A'_n\}$  and  $\{G_n\}$  are Apéry-like sequences with  $(a, b, c) = (11, 3, -1)$  and  $(32, 12, 256)$ , respectively. See the sequences #9 and #19 in [36], and A005258

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and A143583 in [10]. The formula

$$G_n = \sum_{k=0}^n \binom{2k}{k}^2 \binom{2n-2k}{n-k} 4^{n-k}$$

was implicitly given in [2] and also stated by Wang in [34].

In [30],  $\{G_n(x)\}$ , the one-parameter generalization of  $G_n$ , was defined by

$$G_n(x) = \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{x}{k} \binom{-1-x}{k} \quad (n = 0, 1, 2, \dots),$$

where  $\binom{x}{k}$  is the generalized binomial coefficient given by

$$\binom{x}{0} = 1 \quad \text{and} \quad \binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!} \quad (k \geq 1).$$

Let  $\mathbb{Z}^+$  be the set of positive integers. For a prime  $p$  let  $\mathbb{Z}_p$  be the set of rational numbers whose denominator is not divisible by  $p$ . In [30], the author's brother Sun showed that for any prime  $p > 3$  and  $x \in \mathbb{Z}_p$  with  $x \not\equiv -\frac{1}{2} \pmod{p}$ ,

$$\sum_{n=0}^{p-1} G_n(x)^2 \equiv (-1)^{\langle x \rangle_p} p \frac{1 + 2(x - \langle x \rangle_p)/p}{1 + 2x} \pmod{p^2},$$

where  $\langle x \rangle_p \in \{0, 1, \dots, p-1\}$  is given by  $x \equiv \langle x \rangle_p \pmod{p}$ . In [6], Guo showed that

$$\sum_{k=0}^{n-1} (2k+1)G_k(x)^2 \equiv 0 \pmod{n^2} \quad \text{for } n \in \mathbb{Z}^+.$$

When  $n$  is a prime, this result is due to Sun [30].

For  $m \in \mathbb{Z}^+$ , let  $p_1, \dots, p_s$  be all the distinct odd prime divisors of  $m$ , and let

$$\lambda(m) = \begin{cases} 1 & \text{if } m = 1, \\ p_1 \dots p_s m^2 & \text{if } m > 1 \text{ is odd,} \\ 4p_1 \dots p_s m^2 & \text{if } m \text{ is even.} \end{cases}$$

Then clearly  $\lambda(2) = 16$ ,  $\lambda(3) = 27$ ,  $\lambda(4) = 64$  and  $\lambda(6) = 432$ . For  $n \in \{0, 1, 2, \dots\}$ ,  $m \in \mathbb{Z}^+$  and  $r \in \mathbb{Z}$  with  $\gcd(r, m) = 1$ , put

$$G_n(r, m) = \lambda(m)^n G_n\left(\frac{r}{m}\right).$$

The paper is organized as follows. Section 2 is devoted to preliminary notations, facts and lemmas. In Section 3, we state that

$$G_n(x) = \sum_{k=0}^n \binom{x}{k}^2 (-1)^{n-k} \binom{-1-x}{n-k},$$

and  $G_n(r, m)$  is an Apéry-like sequence with  $a = 2\lambda(m)$ ,  $b = \lambda(m)(m^2 + rm + r^2)/m^2$  and  $c = \lambda(m)^2$ . This provides infinitely many examples of Apéry-like sequences. Suppose that  $p > 3$  is a prime. We also deduce congruences for

$$\sum_{k=0}^{p-1} \frac{G_k(x)}{m^k}, \quad \sum_{k=0}^{p-1} \binom{2k}{k} \frac{G_k(x)}{m^k}$$

modulo  $p$  and  $G_{(p-1)/2}(x)$  modulo  $p^2$ . As consequences, we deduce some congruences involving  $G_n$ ,  $G_n(-1, 3)$ ,  $G_n(-1, 4)$  and  $G_n(-1, 6)$ . In Section 4, for  $x \in \mathbb{Z}_p$  with  $x \not\equiv 0, \pm 1, -2 \pmod{p}$ , we obtain the congruences for  $\sum_{n=0}^{p-1} G_n(x)$  and  $\sum_{n=0}^{p-1} nG_n(x)$  modulo  $p^3$ . In Section 5, we prove some congruences for the sums involving  $G_n$  modulo  $p^2$ . In Section 6, we establish the congruences for  $G_p(x)$  and  $G_{p-1}(x)$  modulo  $p^3$ . In Section 7, we pose a lot of challenging conjectures on congruences involving  $G_n$ ,  $G_n(-1, 3)$ ,  $G_n(-1, 4)$  and  $G_n(-1, 6)$  modulo prime powers.

As typical results in this paper for any prime  $p > 3$  we have

$$\begin{aligned} \sum_{n=0}^{p-1} \frac{nG_n}{16^n} &\equiv \frac{15 - 16(-1)^{(p-1)/2}}{9} p^2 \pmod{p^3}, \\ \sum_{n=0}^{p-1} \frac{G_n(-1, 3)}{27^n} &\equiv \begin{cases} p^2 \pmod{p^3} & \text{if } 3 \mid p-1, \\ -8p^2 \pmod{p^3} & \text{if } 3 \nmid p-1, \end{cases} \\ G_p &\equiv 12 + 64(-1)^{(p-1)/2} p^2 E_{p-3} \pmod{p^3}, \\ G_{p-1}(-1, 6) &\equiv (-1)^{(p-1)/2} 186624p^{-1} + \frac{155}{9} p^2 E_{p-3} \pmod{p^3}, \\ G_{(p-1)/2} &\equiv \begin{cases} 4^p x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{4} \text{ and so} \\ & p = x^2 + 4y^2 \text{ for } x, y \in \mathbb{Z}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases} \\ \sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{64^n (n+1)} &\equiv (-1)^{(p-1)/2} \pmod{p^2}, \\ \sum_{k=0}^{p-1} (8k^2 + 8k + 3) \frac{G_k^2}{(-256)^k} &\equiv 3(-1)^{(p-1)/2} p^2 + 25p^4 E_{p-3} \pmod{p^5}, \end{aligned}$$

where  $\{E_n\}$  are the Euler numbers given by

$$E_{2n-1} = 0, \quad E_0 = 1, \quad E_{2n} = - \sum_{k=1}^n \binom{2n}{2k} E_{2n-2k} \quad (n \geq 1).$$

## 2. PRELIMINARIES

In addition to the notation in Section 1, throughout this paper we also use the following notations. Let  $[x]$  be the greatest integer not exceeding  $x$ . For  $a \in \mathbb{Z}$  and a given odd prime  $p$  let  $\left(\frac{a}{p}\right)$  be the Legendre symbol and  $q_p(a) = (a^{p-1} - 1)/p$ . For positive integers  $a, b$  and  $n$ , if  $n = ax^2 + by^2$  for some integers  $x$  and  $y$ , we briefly write that  $n = ax^2 + by^2$ . For  $n, r \in \mathbb{Z}^+$  define  $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$  and  $H_n^{(r)} = 1 + 2^{-r} + \dots + n^{-r}$ . For convenience, we also assume that  $H_0 = H_0^{(r)} = 0$ . The Bernoulli numbers  $\{B_n\}$  and the sequence  $\{U_n\}$  are defined by

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n \geq 2),$$

$$U_{2n-1} = 0, \quad U_0 = 1, \quad U_{2n} = -2 \sum_{k=1}^n \binom{2n}{2k} U_{2n-2k} \quad (n \geq 1).$$

For the congruences involving  $B_n, E_n$  and  $U_n$ , see [11], [12], [14]. The Bernoulli polynomials  $\{B_n(x)\}$  and Euler polynomials  $\{E_n(x)\}$  are given by

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \quad \text{and} \quad E_n(x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} (2x-1)^{n-k} E_k.$$

For any nonnegative integer  $m$  and real number  $x$ , it is well known [4], (1.5) that

$$(2.1) \quad \sum_{k=0}^m \binom{x}{k} (-1)^k = (-1)^m \binom{x-1}{m} = \binom{m-x}{m}.$$

From [4], (1.52),

$$(2.2) \quad \sum_{n=k}^m \binom{n}{k} = \binom{m+1}{k+1}.$$

By [22], (11),

$$(2.3) \quad \sum_{n=k}^{p-1} n \binom{n}{k} = \left( \frac{p^2+p}{k+2} - \frac{p}{k+1} \right) \binom{p-1}{k}.$$

In 1952 Ljunggren proved (see [5]) that for any prime  $p > 3$  and  $m, n \in \mathbb{Z}^+$ ,

$$(2.4) \quad \binom{mp}{np} \equiv \binom{m}{n} \pmod{p^3}.$$

For any odd prime  $p$  one can easily prove (see [12], Lemma 2.9) that for  $k = 1, 2, \dots, p-1$ ,

$$(2.5) \quad (-1)^k \binom{p-1}{k} \equiv 1 - pH_k + \frac{p^2}{2}(H_k^2 - H_k^{(2)}) \pmod{p^3}.$$

For  $k = 0, 1, 2, \dots$  it is easily seen (see, for example, [15]–[18]) that

$$(2.6) \quad \begin{aligned} \binom{-\frac{1}{2}}{k} &= \frac{\binom{2k}{k}}{(-4)^k}, & \binom{-\frac{1}{3}}{k} \binom{-\frac{2}{3}}{k} &= \frac{\binom{2k}{k} \binom{3k}{k}}{27^k}, \\ \binom{-\frac{1}{4}}{k} \binom{-\frac{3}{4}}{k} &= \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k}, & \binom{-\frac{1}{6}}{k} \binom{-\frac{5}{6}}{k} &= \frac{\binom{3k}{k} \binom{6k}{3k}}{432^k}. \end{aligned}$$

Since  $\binom{x}{k} = x/(x-k)\binom{x-1}{k}$ , we see that

$$(2.7) \quad \binom{\frac{1}{2}}{k} \binom{-1-\frac{1}{2}}{k} = \frac{1+2k}{1-2k} \cdot \frac{\binom{2k}{k}^2}{16^k} = -2(2k+1) \frac{\binom{2k}{k} C_{k-1}}{16^k},$$

where  $C_n = (n+1)^{-1} \binom{2n}{n}$  is the  $n$ th Catalan number. Let  $p$  be an odd prime. By [18], Theorem 2.4, for  $x, u_0, u_1, \dots, u_{p-1} \in \mathbb{Z}_p$ ,

$$(2.8) \quad \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \left( (-1)^{\langle x \rangle_p} u_k - \sum_{r=0}^k \binom{k}{r} (-1)^r u_r \right) \equiv 0 \pmod{p^2}.$$

Suppose that  $p$  is a prime,  $p > 3$ . By [21], Lemma 2.2 (with  $k = 2$ ), for  $x \in \mathbb{Z}_p$  with  $x \not\equiv 0 \pmod{p}$ ,

$$(2.9) \quad \sum_{k=1}^{\langle x \rangle_p} \frac{(-1)^k}{k^2} \equiv \frac{1}{2} (-1)^{\langle x \rangle_p} E_{p-3}(-x) \pmod{p}.$$

From [21], Theorem 2.1, for  $x \in \mathbb{Z}_p$  and  $x' = (x - \langle x \rangle_p)/p$ ,

$$(2.10) \quad \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \equiv (-1)^{\langle x \rangle_p} + p^2 x' (x' + 1) E_{p-3}(-x) \pmod{p^3}.$$

By [21], pages 3300–3301,

$$(2.11) \quad E_{p-3}\left(\frac{1}{2}\right) \equiv 4E_{p-3} \pmod{p}, \quad E_{p-3}\left(\frac{1}{6}\right) \equiv 20E_{p-3} \pmod{p},$$

$$(2.12) \quad E_{p-3}\left(\frac{1}{3}\right) \equiv 9E_{p-3} \pmod{p}, \quad E_{p-3}\left(\frac{1}{4}\right) \equiv 16E_{p-3} \pmod{p},$$

where  $\{s_n\}$  is given by

$$(2.13) \quad s_0 = 1 \quad \text{and} \quad s_n = 1 - \sum_{k=0}^{n-1} \binom{n}{k} 2^{2n-1-2k} s_k \quad (n \geq 1).$$

From [19], Lemma 2.3 and Theorem 2.1, for  $x \in \mathbb{Z}_p$ ,  $x \not\equiv 0 \pmod{p}$  and  $x' = (x - \langle x \rangle_p)/p$ ,

$$(2.14) \quad \sum_{k=1}^{p-1} \frac{\binom{x}{k} \binom{-1-x}{k}}{k} \equiv -2H_{\langle x \rangle_p} + 2px'H_{\langle x \rangle_p}^{(2)} \\ \equiv -2 \frac{B_{p^2(p-1)}(-x) - B_{p^2(p-1)}}{p^2(p-1)} \pmod{p^2}.$$

By [31],

$$(2.15) \quad \sum_{k=1}^{p-1} \frac{\binom{x}{k} \binom{-1-x}{k}}{k^2} \equiv -\frac{1}{2} \left( \sum_{k=1}^{p-1} \frac{\binom{x}{k} \binom{-1-x}{k}}{k} \right)^2 \pmod{p}.$$

By [19], (2.5) and [11], Theorem 5.2,

$$(2.16) \quad \frac{B_{p^2(p-1)}(\frac{1}{2}) - B_{p^2(p-1)}}{p^2(p-1)} \equiv H_{(p-1)/2} + \frac{p}{2} H_{(p-1)/2}^{(2)} \\ \equiv -2q_p(2) + pq_p(2)^2 \pmod{p^2}.$$

By [12], p. 287,

$$(2.17) \quad \frac{B_{p^2(p-1)}(\frac{1}{3}) - B_{p^2(p-1)}}{p^2(p-1)} \equiv -\frac{3}{2}q_p(3) + \frac{3}{4}pq_p(3)^2 \pmod{p^2},$$

$$(2.18) \quad \frac{B_{p^2(p-1)}(\frac{1}{4}) - B_{p^2(p-1)}}{p^2(p-1)} \equiv -3q_p(2) + \frac{3}{2}pq_p(2)^2 \pmod{p^2},$$

$$(2.19) \quad \frac{B_{p^2(p-1)}(\frac{1}{6}) - B_{p^2(p-1)}}{p^2(p-1)} \\ \equiv -2q_p(2) - \frac{3}{2}q_p(3) + p(q_p(2)^2 + \frac{3}{4}q_p(3)^2) \pmod{p^2}.$$

**Lemma 2.1.** *Let  $p$  be an odd prime and  $m \in \mathbb{Z}_p$  with  $m \not\equiv 0, 1 \pmod{p}$ . Suppose that  $u_0, u_1, \dots, u_{p-1} \in \mathbb{Z}_p$  and  $v_n = \sum_{k=0}^n \binom{n}{k} u_k$  ( $n \geq 0$ ). Then*

$$\sum_{k=0}^{p-1} \frac{v_k}{m^k} \equiv \sum_{k=0}^{p-1} \frac{u_k}{(m-1)^k} \pmod{p}.$$

Also,

$$\sum_{k=0}^{p-1} v_k \equiv \sum_{k=0}^{p-1} \frac{(-1)^k p}{k+1} u_k - p^2 \sum_{k=0}^{p-2} \frac{(-1)^k H_k}{k+1} u_k \pmod{p^3}.$$

**Proof.** It is clear that

$$\begin{aligned}
 \sum_{k=0}^{p-1} \frac{v_k}{m^k} &= \sum_{k=0}^{p-1} \frac{1}{m^k} \sum_{s=0}^k \binom{k}{s} u_s = \sum_{s=0}^{p-1} \sum_{k=s}^{p-1} \frac{1}{m^k} \binom{-1-s}{k-s} (-1)^{k-s} u_s \\
 &= \sum_{s=0}^{p-1} \frac{u_s}{m^s} \sum_{k=s}^{p-1} \binom{-1-s}{k-s} \frac{1}{(-m)^{k-s}} = \sum_{s=0}^{p-1} \frac{u_s}{m^s} \sum_{r=0}^{p-1-s} \binom{-1-s}{r} \left(-\frac{1}{m}\right)^r \\
 &\equiv \sum_{s=0}^{p-1} \frac{u_s}{m^s} \sum_{r=0}^{p-1-s} \binom{p-1-s}{r} \left(-\frac{1}{m}\right)^r = \sum_{s=0}^{p-1} \frac{u_s}{m^s} \left(1 - \frac{1}{m}\right)^{p-1-s} \\
 &\equiv \sum_{s=0}^{p-1} \frac{u_s}{(m-1)^s} \pmod{p}.
 \end{aligned}$$

For  $m = 1$ , from the above and (2.1) we see that

$$\begin{aligned}
 \sum_{k=0}^{p-1} v_k &= \sum_{s=0}^{p-1} u_s \sum_{r=0}^{p-1-s} \binom{-1-s}{r} (-1)^r = \sum_{s=0}^{p-1} \binom{p}{p-1-s} u_s \\
 &= \sum_{s=0}^{p-1} \frac{p}{s+1} \binom{p-1}{s} u_s \equiv u_{p-1} + \sum_{s=0}^{p-2} \frac{p}{s+1} (-1)^s (1 - pH_s) u_s \pmod{p^3}.
 \end{aligned}$$

This yields the remaining part. □

**Lemma 2.2** ([29], Theorem 2.2). *Let  $p$  be an odd prime,  $u_0, u_1, \dots, u_{p-1} \in \mathbb{Z}_p$  and  $v_n = \sum_{k=0}^n \binom{n}{k} (-1)^k u_k$  for  $n \geq 0$ . For  $m \in \mathbb{Z}_p$  with  $m \not\equiv 0, 4 \pmod{p}$ ,*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{v_k}{m^k} \equiv \left(\frac{m(m-4)}{p}\right) \sum_{k=0}^{p-1} \binom{2k}{k} \frac{u_k}{(4-m)^k} \pmod{p}.$$

**Lemma 2.3.** *Let  $k, m \in \mathbb{Z}^+$  and  $r \in \mathbb{Z}$ . Then  $\binom{r/m}{k} \binom{-1-r/m}{k} \lambda(m)^k \in \mathbb{Z}$ .*

**Proof.** Suppose that  $p$  is a prime. For  $n \in \mathbb{Z}^+$  let  $\text{ord}_p n$  be the unique nonnegative integer  $\alpha$  such that  $p^\alpha \mid n$  but  $p^{\alpha+1} \nmid n$ . It is well known that

$$\text{ord}_p k! = \left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \left[\frac{k}{p^3}\right] + \dots$$

Thus,

$$\text{ord}_p k! \leq \frac{k}{p} + \frac{k}{p^2} + \frac{k}{p^3} + \dots = \frac{k}{p-1} \quad \text{and so} \quad \text{ord}_p k!^2 \leq \frac{2k}{p-1}.$$



If  $p \nmid m$ , then clearly  $r/m \in \mathbb{Z}_p$  and so  $\binom{r/m}{k} \binom{-1-r/m}{k} \lambda(m)^k \in \mathbb{Z}_p$ . Now assume that  $p \mid m$ . Since

$$\begin{aligned} & \binom{r/m}{k} \binom{-1-r/m}{k} \lambda(m)^k \\ &= r(r-m) \dots (r-(k-1)m)(-r-m) \dots (-r-km) \left(\frac{\lambda(m)}{m^2}\right)^k \cdot \frac{1}{k!^2} \end{aligned}$$

and

$$\text{ord}_p\left(\frac{\lambda(m)}{m^2}\right)^k = \begin{cases} k \geq \frac{2k}{p-1} \geq \text{ord}_p k!^2 & \text{if } p > 2, \\ 2k = \frac{2k}{p-1} \geq \text{ord}_p k!^2 & \text{if } p = 2, \end{cases}$$

we also have  $\binom{r/m}{k} \binom{-1-r/m}{k} \lambda(m)^k \in \mathbb{Z}_p$  when  $p \mid m$ . Hence,

$$\binom{r/m}{k} \binom{-1-r/m}{k} \lambda(m)^k \in \mathbb{Z}_p$$

for any prime  $p$ , which yields  $\binom{r/m}{k} \binom{-1-r/m}{k} \lambda(m)^k \in \mathbb{Z}$  as claimed. □

**Lemma 2.4** ([24], Theorem 4.1). *Let  $p$  be an odd prime,  $b, x \in \mathbb{Z}_p$ ,  $bx \not\equiv 0 \pmod{p}$  and  $\langle b \rangle_p \leq \min\{\langle x \rangle_p, p-1-\langle x \rangle_p\}$ . Assume that  $x' = (x - \langle x \rangle_p)/p$  and  $b' = (b - \langle b \rangle_p)/p \not\equiv -1 \pmod{p}$ . Then*

$$\sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{1}{k+b} \equiv \frac{p(b'+x'+1)(b'-x')}{b^2(b'+1) \binom{\langle x \rangle_p}{\langle b \rangle_p} \binom{p-1-\langle x \rangle_p}{\langle b \rangle_p}} \pmod{p^2}.$$

### 3. BASIC PROPERTIES OF $G_n(x)$

Recall that for  $n = 0, 1, 2, \dots$ ,

$$G_n(x) = \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{x}{k} \binom{-1-x}{k} \quad \text{and} \quad G_n(r, m) = \lambda(m)^n G_n\left(\frac{r}{m}\right),$$

where  $m \in \mathbb{Z}^+$ ,  $r \in \mathbb{Z}$  and  $\gcd(r, m) = 1$ . Applying the binomial inversion formula,

$$\sum_{k=0}^n \binom{n}{k} (-1)^k G_k(x) = \binom{x}{n} \binom{-1-x}{n}.$$

**Theorem 3.1.** *Let  $m \in \mathbb{Z}^+$ ,  $r \in \mathbb{Z}$  and  $\gcd(r, m) = 1$ . Then  $G_n(r, m)$  is an Apéry-like sequence with  $a = 2\lambda(m)$ ,  $b = \lambda(m)(m^2 + rm + r^2)/m^2$  and  $c = \lambda(m)^2$ .*

*Proof.* By Lemma 2.3 for  $n = 0, 1, 2, \dots$ ,

$$G_n(r, m) = \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{r/m}{k} \binom{-1 - r/m}{k} \lambda(m)^n \in \mathbb{Z}.$$

Clearly,  $G_0(r, m) = 1$  and

$$G_1(r, m) = \lambda(m) \left( 1 - \frac{r}{m} \left( -1 - \frac{r}{m} \right) \right) = \frac{\lambda(m)}{m^2} (m^2 + rm + r^2).$$

Using Zeilberger's algorithm or sumtools in Maple (see [9]),

$$(3.1) \quad (n+1)^2 G_{n+1}(x) = (2n(n+1) + x^2 + x + 1)G_n(x) - n^2 G_{n-1}(x) \quad (n \geq 1).$$

Thus,

$$\begin{aligned} (n+1)^2 G_{n+1}(r, m) &= (n+1)^2 \lambda(m)^{n+1} G_{n+1}\left(\frac{r}{m}\right) \\ &= \left( 2n(n+1) + \frac{r^2}{m^2} + \frac{r}{m} + 1 \right) \lambda(m)^{n+1} G_n\left(\frac{r}{m}\right) - n^2 \lambda(m)^{n+1} G_{n-1}\left(\frac{r}{m}\right) \\ &= \left( 2\lambda(m)n(n+1) + \frac{\lambda(m)}{m^2} (m^2 + rm + r^2) \right) G_n(r, m) - \lambda(m)^2 n^2 G_{n-1}(r, m). \end{aligned}$$

This completes the proof. □

**Theorem 3.2.** *For  $n = 0, 1, 2, \dots$  we have*

$$G_n(x) = \sum_{k=0}^n \binom{x}{k}^2 (-1)^{n-k} \binom{-1-x}{n-k}.$$

*Proof.* Set  $G'_n(x) = \sum_{k=0}^n \binom{x}{k}^2 (-1)^{n-k} \binom{-1-x}{n-k}$ . Then  $G'_0(x) = 1 = G_0(x)$  and  $G'_1(x) = x^2 + x + 1 = G_1(x)$ . Using Zeilberger's algorithm or sumtools in Maple (see [9]),

$$(n+1)^2 G'_{n+1}(x) = (2n(n+1) + x^2 + x + 1)G'_n(x) - n^2 G'_{n-1}(x) \quad (n \geq 1).$$

Thus,  $G'_n(x) = G_n(x)$  for  $n = 0, 1, 2, \dots$  by (3.1). □

**Remark 3.1.** From the definition of  $G_n(r, m)$ , (2.6) and (2.7) we know that for  $n = 0, 1, 2, \dots$ ,

$$\begin{aligned} G_n(1, 2) &= 16^n - 2 \sum_{k=1}^n \binom{n}{k} (-1)^k (2k+1) \binom{2k}{k} C_{k-1} 16^{n-k}, \\ G_n(-1, 2) &= \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{2k}{k}^2 16^{n-k} = G_n, \\ G_n(-1, 3) &= \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{2k}{k} \binom{3k}{k} 27^{n-k}, \\ G_n(-1, 4) &= \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{2k}{k} \binom{4k}{2k} 64^{n-k}, \\ G_n(-1, 6) &= \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{3k}{k} \binom{6k}{3k} 432^{n-k}. \end{aligned}$$

These are *Apéry-like sequences* with  $(a, b, c) = (32, 28, 256)$ ,  $(32, 12, 256)$ ,  $(54, 21, 729)$ ,  $(128, 52, 4096)$  and  $(864, 372, 186624)$ , respectively. We note that  $G_n(-1, 3)$  is the sequence #25 in [36], and  $G_n(-1, 4)$  and  $G_n(-1, 6)$  were first introduced in [33]. See also [2].

**Theorem 3.3.** *Let  $p$  be an odd prime and  $m, x \in \mathbb{Z}_p$ . Then*

$$\sum_{k=0}^{p-1} \frac{G_k(x)}{m^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{x}{k} \binom{-1-x}{k}}{(1-m)^k} \pmod{p} \quad \text{for } m \not\equiv 0, 1 \pmod{p}$$

and

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{G_k(x)}{m^k} \equiv \left( \frac{m(m-4)}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{x}{k} \binom{-1-x}{k}}{(4-m)^k} \pmod{p}$$

for  $m \not\equiv 0, 4 \pmod{p}$ .

**Proof.** Putting  $u_k = (-1)^k \binom{x}{k} \binom{-1-x}{k}$  and  $v_k = G_k(x)$  in Lemma 2.1 yields the first congruence. Putting  $u_k = \binom{x}{k} \binom{-1-x}{k}$  and  $v_k = G_k(x)$  in Lemma 2.2 yields the second congruence.  $\square$

**Theorem 3.4.** *Let  $p$  be an odd prime. Then*

$$\begin{aligned} \sum_{n=0}^{p-1} \frac{G_n}{(-16)^n} &\equiv (-1)^{(p-1)/4} \sum_{n=0}^{p-1} \frac{G_n}{8^n} \equiv (-1)^{(p-1)/4} \sum_{n=0}^{p-1} \frac{G_n}{32^n} \\ &\equiv \begin{cases} 2x \pmod{p} & \text{if } 4 \mid p-1 \text{ and so } p = x^2 + 4y^2 \text{ with } 4 \mid x-1, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

*Proof.* Taking  $x = -\frac{1}{2}$  and  $m = -1, \frac{1}{2}, 2$  in Theorem 3.3 and then applying [13], Theorems 2.2, 2.9 and Corollary 2.3 yields the result.  $\square$

**Theorem 3.5.** *Let  $p > 3$  be a prime. Then*

$$\sum_{n=0}^{p-1} \frac{G_n(-1, 3)}{(-27)^n} \equiv \begin{cases} 2x \pmod{p} & \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3} \text{ and } 3 \mid x - 1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

$$\sum_{n=0}^{p-1} \frac{G_n(-1, 3)}{3^n} \equiv \sum_{n=0}^{p-1} \frac{G_n(-1, 3)}{243^n}$$

$$\equiv \begin{cases} -L \pmod{p} & \text{if } 3 \mid p - 1, 4p = L^2 + 27M^2 \text{ and } 3 \mid L - 1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

*Proof.* Putting  $x = -\frac{1}{3}$  in Theorem 3.3 yields

$$\sum_{k=0}^{p-1} \frac{G_k(-1, 3)}{(27m)^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{(27(1-m))^k} \pmod{p}.$$

Now, taking  $m = -1, \frac{1}{9}, 9$  and then applying [18], Theorem 3.4 and [16], Theorem 3.2 yields the result.  $\square$

**Theorem 3.6.** *Let  $p > 3$  be a prime. Then*

$$\left(\frac{-3}{p}\right) \sum_{n=0}^{p-1} G_n(-1, 4) \equiv \left(\frac{6}{p}\right) \sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{4096^n}$$

$$\equiv \begin{cases} 2x \pmod{p} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod{7} \text{ and } \left(\frac{x}{7}\right) = 1, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7}, \end{cases}$$

$$\left(\frac{6}{p}\right) \sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{(-8)^n} \equiv \left(\frac{3}{p}\right) \sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{(-512)^n}$$

$$\equiv \begin{cases} 2x \pmod{p} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod{4} \text{ and } 4 \mid x - 1, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$\sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{16^n} \equiv (-1)^{\lfloor p/4 \rfloor} \sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{256^n}$$

$$\equiv \begin{cases} 2x \pmod{p} & \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3} \text{ and } 3 \mid x - 1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

$$\sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{(-64)^n} \equiv \begin{cases} (-1)^{\lfloor p/8 \rfloor + (p-1)/2} 2x \pmod{p} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8} \\ & \text{and } 4 \mid x - 1, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Proof. From Theorem 3.3 (with  $x = -\frac{1}{4}$ ),

$$\sum_{k=0}^{p-1} \frac{G_k(-1, 4)}{n^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{(64-n)^k} \pmod{p}.$$

Now taking  $n = 1, 4096, -8, -512, 16, 256, -64$  and then applying [15], Theorems 2.2–2.4, 4.1 and 4.3 yields the result.  $\square$

**Theorem 3.7.** *Let  $p$  be an odd prime,  $x \in \mathbb{Z}_p$  and  $x \not\equiv 0 \pmod{p}$ . Then*

$$G_{(p-1)/2}(x) \equiv \begin{cases} \left(\frac{\langle x \rangle_p}{2}\right)^2 \frac{1}{4^{\langle x \rangle_p}} \pmod{p} & \text{if } \langle x \rangle_p \text{ is even,} \\ 0 \pmod{p} & \text{if } \langle x \rangle_p \text{ is odd.} \end{cases}$$

Moreover,

$$G_{(p-1)/2}(x) \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{x}{k} \binom{-1-x}{k}}{4^k} \pmod{p^2} \quad \text{for } \langle x \rangle_p \equiv 0 \pmod{2}.$$

In particular, for  $p > 3$ ,

$$\begin{aligned} G_{(p-1)/2} &\equiv \begin{cases} 4^p x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod{4}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases} \\ G_{(p-1)/2}(-1, 3) &\equiv \begin{cases} 27^{(p-1)/2} (4x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}, \end{cases} \\ G_{(p-1)/2}(-1, 4) &\equiv \begin{cases} 8^{p-1} \cdot 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ and so } p = x^2 + 2y^2, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}, \end{cases} \\ G_{(p-1)/2}(-1, 6) &\equiv \begin{cases} 432^{(p-1)/2} \binom{p}{3} \cdot 4x^2 - 2p \pmod{p^2} & \text{if } 4 \mid p-1 \text{ and so } p = x^2 + 4y^2, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Proof. By [13], Lemma 2.4, for  $k = 1, 2, \dots, (p-1)/2$ ,

$$\binom{\frac{1}{2}(p-1)}{k} (-1)^k \equiv \frac{\binom{2k}{k}}{4^k} \left(1 - p \sum_{i=1}^k \frac{1}{2i-1}\right) = \frac{\binom{2k}{k}}{4^k} \left(1 - p \left(H_{2k} - \frac{1}{2}H_k\right)\right) \pmod{p^2}.$$

Thus,

$$\begin{aligned}
 G_{(p-1)/2}(x) &= \sum_{k=0}^{(p-1)/2} \binom{\frac{1}{2}(p-1)}{k} (-1)^k \binom{x}{k} \binom{-1-x}{k} \\
 &\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{4^k} \binom{x}{k} \binom{-1-x}{k} \\
 &\quad - p \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{4^k} \binom{x}{k} \binom{-1-x}{k} \left( H_{2k} - \frac{1}{2} H_k \right) \\
 &\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(-4)^k} \binom{x}{k} \binom{x+k}{k} \\
 &\quad - p \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{(-4)^k} \binom{x}{k} \binom{x+k}{k} \left( H_{2k} - \frac{1}{2} H_k \right) \pmod{p^2}.
 \end{aligned}$$

Using the WZ method or the summation package Sigma one can prove that

$$(3.2) \quad \sum_{k=0}^n \frac{\binom{2k}{k}}{(-4)^k} \binom{n}{k} \binom{n+k}{k} \left( H_{2k} - \frac{2 - (-1)^n}{2} H_k \right) = 0.$$

When  $n$  is even, this identity follows from [32], (4.5) and (4.6). Thus,

$$\begin{aligned}
 &\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{(-4)^k} \binom{x}{k} \binom{x+k}{k} \left( H_{2k} - \frac{1}{2} H_k \right) \\
 &\equiv \sum_{k=0}^{\langle x \rangle_p} \frac{\binom{2k}{k}}{(-4)^k} \binom{\langle x \rangle_p}{k} \binom{\langle x \rangle_p + k}{k} \left( H_{2k} - \frac{1}{2} H_k \right) \\
 &= \begin{cases} 0 \pmod{p} & \text{if } \langle x \rangle_p \text{ is even,} \\ \sum_{k=0}^{\langle x \rangle_p} \frac{\binom{2k}{k}}{(-4)^k} \binom{\langle x \rangle_p}{k} \binom{\langle x \rangle_p + k}{k} H_k \pmod{p} & \text{if } \langle x \rangle_p \text{ is odd.} \end{cases}
 \end{aligned}$$

Hence, for  $\langle x \rangle_p \equiv 0 \pmod{2}$  we have

$$G_{(p-1)/2}(x) \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{x}{k} \binom{-1-x}{k}}{4^k} \pmod{p^2}.$$

By the identity due to Bell (see [4], (6.35)),

$$(3.3) \quad \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \frac{\binom{2k}{k}}{(-4)^k} = \begin{cases} \left( \frac{n}{2} \right)^2 \frac{1}{4^n} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

In view of the above,

$$\begin{aligned}
 G_{(p-1)/2}(x) &\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(-4)^k} \binom{x}{k} \binom{x+k}{k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{(-4)^k} \binom{x}{k} \binom{x+k}{k} \\
 &\equiv \sum_{k=0}^{\langle x \rangle_p} \frac{\binom{2k}{k}}{(-4)^k} \binom{\langle x \rangle_p}{k} \binom{\langle x \rangle_p + k}{k} \\
 &= \begin{cases} \left( \frac{\langle x \rangle_p}{2} \right)^2 \frac{1}{4^{\langle x \rangle_p}} \pmod{p} & \text{if } \langle x \rangle_p \text{ is even,} \\ 0 \pmod{p} & \text{if } \langle x \rangle_p \text{ is odd.} \end{cases}
 \end{aligned}$$

By [8] and [26],

$$\begin{aligned}
 \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \end{cases} \\
 \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ and } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}, \end{cases} \\
 \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{1728^k} &\equiv \begin{cases} \left( \frac{p}{3} \right) (4x^2 - 2p) \pmod{p^2} & \text{if } 4 \mid p-1 \text{ and } p = x^2 + 4y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}
 \end{aligned}$$

Now, from the above congruences for  $G_{(p-1)/2}(x)$  (with  $x = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ ) and (2.6) we deduce the congruences for  $G_{(p-1)/2}(-1, 3)$ ,  $G_{(p-1)/2}(-1, 4)$  and  $G_{(p-1)/2}(-1, 6)$ . It is well known (see [1]) that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{4} \text{ and so } p = x^2 + 4y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

By [32], Theorem 5.1, for  $p \equiv 3 \pmod{4}$ ,

$$\begin{aligned}
 \sum_{k=0}^{(p-1)/2} \binom{\frac{1}{2}(p-1)}{k} \binom{\frac{1}{2}(p-1)+k}{k} \frac{\binom{2k}{k}}{(-4)^k} H_k \\
 \equiv \sum_{k=0}^{(p-1)/2} \binom{-\frac{1}{2}}{k} \frac{\binom{2k}{k}}{4^k} H_k \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} H_k \equiv 0 \pmod{p}.
 \end{aligned}$$

Now, putting  $x = -\frac{1}{2}$  in the previous congruences for  $G_{(p-1)/2}(x)$  modulo  $p^2$  and then applying the above yields the congruence for  $G_{(p-1)/2}$  modulo  $p^2$ . This completes the proof.  $\square$

4. CONGRUENCES FOR  $\sum_{n=0}^{p-1} G_n(x)$  AND  $\sum_{n=0}^{p-1} nG_n(x)$  MODULO  $p^3$

**Theorem 4.1.** *Let  $p$  be an odd prime,  $x \in \mathbb{Z}_p$ ,  $x \not\equiv 0, -1 \pmod{p}$  and  $x' = (x - \langle x \rangle_p)/p$ . Then*

$$\sum_{n=0}^{p-1} G_n(x) \equiv p^2 \frac{x'(x'+1) + 1 - (-1)^{\langle x \rangle_p}}{x(x+1)} \pmod{p^3}.$$

Hence, for  $p > 3$ ,

$$\begin{aligned} \sum_{n=0}^{p-1} \frac{G_n(-1, 3)}{27^n} &\equiv \begin{cases} p^2 \pmod{p^3} & \text{if } p \equiv 1 \pmod{3}, \\ -8p^2 \pmod{p^3} & \text{if } p \equiv 2 \pmod{3}, \end{cases} \\ \sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{64^n} &\equiv \begin{cases} p^2 \pmod{p^3} & \text{if } p \equiv 1, 3 \pmod{8}, \\ -\frac{29}{3}p^2 \pmod{p^3} & \text{if } p \equiv 5, 7 \pmod{8}, \end{cases} \\ \sum_{n=0}^{p-1} \frac{G_n(-1, 6)}{432^n} &\equiv \begin{cases} p^2 \pmod{p^3} & \text{if } p \equiv 1 \pmod{4}, \\ -\frac{67}{5}p^2 \pmod{p^3} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

**Proof.** Using (2.2),

$$\begin{aligned} \sum_{n=0}^{p-1} G_n(x) &= \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} (-1)^k \sum_{n=k}^{p-1} \binom{n}{k} \\ &= \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} (-1)^k \frac{p}{k+1} \binom{p-1}{k}. \end{aligned}$$

Since  $x \not\equiv p, p-1 \pmod{p}$  we see that  $\binom{x}{p-1} \binom{-1-x}{p-1} \equiv \binom{\langle x \rangle_p}{p-1} \binom{p-1-\langle x \rangle_p}{p-1} \equiv 0 \pmod{p^2}$ . From the above and (2.5),

$$\sum_{k=0}^{p-1} G_k(x) \equiv \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{p}{k+1} - p^2 \sum_{k=0}^{p-2} \binom{x}{k} \binom{-1-x}{k} \frac{H_k}{k+1} \pmod{p^3}.$$

From Lemma 2.4 (with  $b = 1$ ),

$$(4.1) \quad \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{1}{k+1} \equiv p \frac{x'(x'+1)}{x(x+1)} \pmod{p^2}.$$



Using the symbolic summation package Sigma in Mathematica, Liu and Ni in [7] found the identity

$$(4.2) \quad \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^k \frac{H_k}{k+1} = \frac{(-1)^n - 1}{n(n+1)}.$$

Hence

$$\begin{aligned} \sum_{k=0}^{p-2} \binom{x}{k} \binom{-1-x}{k} \frac{H_k}{k+1} &= \sum_{k=0}^{p-2} \binom{x}{k} \binom{x+k}{k} (-1)^k \frac{H_k}{k+1} \\ &\equiv \sum_{k=0}^{p-2} \binom{\langle x \rangle_p}{k} \binom{\langle x \rangle_p + k}{k} (-1)^k \frac{H_k}{k+1} \\ &= \sum_{k=0}^{\langle x \rangle_p} \binom{\langle x \rangle_p}{k} \binom{\langle x \rangle_p + k}{k} (-1)^k \frac{H_k}{k+1} = \frac{(-1)^{\langle x \rangle_p} - 1}{\langle x \rangle_p (\langle x \rangle_p + 1)} \\ &\equiv \frac{(-1)^{\langle x \rangle_p} - 1}{x(x+1)} \pmod{p}. \end{aligned}$$

Therefore,

$$\sum_{k=0}^{p-1} G_k(x) \equiv p^2 \frac{x'(x'+1)}{x(x+1)} - p^2 \frac{(-1)^{\langle x \rangle_p} - 1}{x(x+1)} \pmod{p^3}.$$

Taking  $x = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$  yields the remaining results. □

**Remark 4.1.** Let  $p$  be an odd prime. Taking  $x = -\frac{1}{2}$  in Theorem 4.1 gives

$$(4.3) \quad \sum_{n=0}^{p-1} \frac{G_n}{16^n} \equiv (4(-1)^{(p-1)/2} - 3)p^2 \pmod{p^3},$$

which was conjectured by the author earlier and later proved by Liu and Ni in [7].

**Theorem 4.2.** Let  $p$  be an odd prime,  $x \in \mathbb{Z}_p$ ,  $x \not\equiv 0, \pm 1, -2 \pmod{p}$  and  $x' = (x - \langle x \rangle_p)/p$ . Then

$$\sum_{n=0}^{p-1} nG_n(x) \equiv \begin{cases} p^2 \frac{x'(x'+1)(1-x(x+1)) - x(x+1)}{(x-1)x(x+1)(x+2)} \pmod{p^3} & \text{if } 2 \mid \langle x \rangle_p, \\ p^2 \frac{x'(x'+1)(1-x(x+1)) - (x-1)(x+2)}{(x-1)x(x+1)(x+2)} \pmod{p^3} & \text{if } 2 \nmid \langle x \rangle_p. \end{cases}$$

Hence, for  $p > 3$ ,

$$\sum_{n=0}^{p-1} \frac{nG_n}{16^n} \equiv \begin{cases} -\frac{1}{9}p^2 \pmod{p^3} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{31}{9}p^2 \pmod{p^3} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$\sum_{n=0}^{p-1} \frac{nG_n(-1, 3)}{27^n} \equiv \begin{cases} -\frac{1}{10}p^2 \pmod{p^3} & \text{if } p \equiv 1 \pmod{3}, \\ \frac{79}{20}p^2 \pmod{p^3} & \text{if } p \equiv 2 \pmod{3} \text{ and } p \neq 5, \end{cases}$$

$$\sum_{n=0}^{p-1} \frac{nG_n(-1, 4)}{64^n} \equiv \begin{cases} -\frac{3}{35}p^2 \pmod{p^3} & \text{if } p \equiv 1, 3 \pmod{8}, \\ \frac{503}{105}p^2 \pmod{p^3} & \text{if } p \equiv 5, 7 \pmod{8} \text{ and } p \neq 5, 7, \end{cases}$$

$$\sum_{n=0}^{p-1} \frac{nG_n(-1, 6)}{432^n} \equiv \begin{cases} -\frac{5}{77}p^2 \pmod{p^3} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{2567}{385}p^2 \pmod{p^3} & \text{if } p \equiv 3 \pmod{4} \text{ and } p \neq 7, 11. \end{cases}$$

**Proof.** By the definition of  $G_n(x)$  and (2.3),

$$\begin{aligned} \sum_{n=0}^{p-1} nG_n(x) &= \sum_{n=0}^{p-1} n \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{x}{k} \binom{-1-x}{k} \\ &= \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} (-1)^k \sum_{n=k}^{p-1} n \binom{n}{k} \\ &= \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} (-1)^k \left( \frac{p^2+p}{k+2} - \frac{p}{k+1} \right) \binom{p-1}{k}. \end{aligned}$$

For  $k = p-2$  or  $p-1$ , we have  $\binom{x}{k} \equiv \binom{x}{k}^p = 0 \pmod{p}$ . Thus,

$$\binom{x}{k} \binom{-1-x}{k} \frac{1}{k+1}, \quad \binom{x}{k} \binom{-1-x}{k} \frac{1}{k+2} \in \mathbb{Z}_p \quad \text{for } k = 0, 1, \dots, p-1.$$

Since  $\binom{p-1}{k} (-1)^k \equiv 1 - pH_k \pmod{p^2}$  by (2.5), we see that

$$\begin{aligned} \sum_{n=0}^{p-1} nG_n(x) &\equiv \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \left( \frac{p^2+p}{k+2} - \frac{p}{k+1} \right) (1 - pH_k) \\ &\equiv (p^2+p) \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{1}{k+2} - p \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{1}{k+1} \\ &\quad + p^2 \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{H_k}{k+1} - p^2 \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{H_k}{k+2} \pmod{p^3}. \end{aligned}$$

By Lemma 2.4 (with  $b = 2$ ),

$$(4.4) \quad \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{1}{k+2} \equiv -\frac{px'(x'+1)}{(x-1)x(x+1)(x+2)} \pmod{p^2}.$$

By the proof of Theorem 4.1,

$$\sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{H_k}{k+1} \equiv \frac{(-1)^{\langle x \rangle_p} - 1}{x(x+1)} \pmod{p}.$$

Using the WZ method or the symbolic summation package Sigma in Mathematica, one can prove that for  $n \geq 2$ ,

$$(4.5) \quad \sum_{k=1}^n \binom{n}{k} \binom{n+k}{k} (-1)^k \frac{H_k}{k+2} = \begin{cases} \frac{1}{(n-1)(n+2)} & \text{if } n \text{ is even,} \\ -\frac{1}{n(n+1)} & \text{if } n \text{ is odd.} \end{cases}$$

Hence,

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{H_k}{k+2} \\ &= \sum_{k=0}^{p-1} \binom{x}{k} \binom{x+k}{k} (-1)^k \frac{H_k}{k+2} \equiv \sum_{k=0}^{\langle x \rangle_p} \binom{\langle x \rangle_p}{k} \binom{\langle x \rangle_p + k}{k} (-1)^k \frac{H_k}{k+2} \\ &= \begin{cases} \frac{1}{(\langle x \rangle_p - 1)(\langle x \rangle_p + 2)} \equiv \frac{1}{(x-1)(x+2)} \pmod{p} & \text{if } \langle x \rangle_p \text{ is even,} \\ -\frac{1}{\langle x \rangle_p (\langle x \rangle_p + 1)} \equiv -\frac{1}{x(x+1)} \pmod{p} & \text{if } \langle x \rangle_p \text{ is odd.} \end{cases} \end{aligned}$$

If  $\langle x \rangle_p$  is even, combining the above with (4.1) gives

$$\begin{aligned} \sum_{n=0}^{p-1} nG_n(x) &\equiv -\frac{(p^2+p)px'(x'+1)}{(x-1)x(x+1)(x+2)} - p^2 \frac{x'(x'+1)}{x(x+1)} - p^2 \frac{1}{(x-1)(x+2)} \\ &\equiv p^2 \frac{x'(x'+1)(1-x(x+1)) - x(x+1)}{(x-1)x(x+1)(x+2)} \pmod{p^3}. \end{aligned}$$

If  $\langle x \rangle_p$  is odd, from the above and (4.1) we deduce that

$$\begin{aligned} \sum_{n=0}^{p-1} nG_n(x) &\equiv -\frac{(p^2+p)px'(x'+1)}{(x-1)x(x+1)(x+2)} - p^2 \frac{x'(x'+1)}{x(x+1)} - \frac{2p^2}{x(x+1)} + \frac{p^2}{x(x+1)} \\ &\equiv p^2 \frac{x'(x'+1)(1-x(x+1)) - (x-1)(x+2)}{(x-1)x(x+1)(x+2)} \pmod{p^3}. \end{aligned}$$

Now taking  $x = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$  in the above congruences yields the remaining results.  $\square$

5. CONGRUENCES FOR SUMS INVOLVING  $G_n$

**Lemma 5.1** ([28], Lemma 2.2). For  $n = 0, 1, 2, \dots$ ,

$$\binom{2n}{n} G_n = \sum_{k=0}^n \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{n-k} (-64)^{n-k}.$$

**Theorem 5.1.** Let  $p$  be an odd prime,  $m \in \mathbb{Z}_p$  and  $m \not\equiv 0 \pmod{p}$ . Then

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{m^n} \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{m-64}{m^2}\right)^k \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{m^k}\right)^2 \pmod{p^2}$$

and

$$\sum_{n=0}^{p-1} \frac{n \binom{2n}{n} G_n}{m^n} \equiv \begin{cases} -\frac{1536}{m^2} - \frac{219}{16384} (m-64) \pmod{p^2} & \text{if } p \mid m-64, \\ \frac{m-128}{m-64} \sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^2 \binom{4k}{2k}}{(m^2/(m-64))^k} \pmod{p^2} & \text{if } p \nmid m-64. \end{cases}$$

*Proof.* Note that  $p \mid \binom{2k}{k}$  for  $\frac{1}{2}p < k < p$ . By Lemma 5.1 for  $c_0, c_1, \dots, c_{p-1} \in \mathbb{Z}_p$ ,

$$\begin{aligned} (5.1) \quad \sum_{n=0}^{p-1} \frac{\binom{2n}{n} c_n G_n}{m^n} &= \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \sum_{n=k}^{p-1} \binom{k}{n-k} c_n \left(-\frac{64}{m}\right)^{n-k} \\ &= \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \sum_{r=0}^{p-1-k} \binom{k}{r} c_{k+r} \left(-\frac{64}{m}\right)^r \\ &\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \sum_{r=0}^k \binom{k}{r} c_{k+r} \left(-\frac{64}{m}\right)^r \pmod{p^2}. \end{aligned}$$

Thus, applying [15], Theorem 4.1,

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{m^n} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \left(1 - \frac{64}{m}\right)^k \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{m^k}\right)^2 \pmod{p^2}.$$

Since

$$\begin{aligned} \sum_{r=0}^k \binom{k}{r} (k+r)x^r &= k \sum_{r=0}^k \binom{k}{r} x^r + k \sum_{r=1}^k \binom{k-1}{r-1} x^r \\ &= k(1+x)^k + kx(1+x)^{k-1} = (1+2x) \cdot k(1+x)^{k-1}, \end{aligned}$$

taking  $c_n = n$  in (5.1) we deduce that

$$\begin{aligned} \sum_{n=0}^{p-1} \frac{n \binom{2n}{n} G_n}{m^n} &\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \sum_{r=0}^k \binom{k}{r} (k+r) \left(-\frac{64}{m}\right)^r \\ &= \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \cdot \left(1 - \frac{128}{m}\right) \cdot k \left(1 - \frac{64}{m}\right)^{k-1} \\ &\equiv \begin{cases} \left(1 - \frac{128}{m}\right) \left(\frac{\binom{2}{1}^2 \binom{4}{2}}{m} + \frac{\binom{4}{2}^2 \binom{8}{4}}{m^2} \cdot 2 \left(1 - \frac{64}{m}\right)\right) \pmod{p^2} & \text{if } p \mid m - 64, \\ \frac{m - 128}{m - 64} \sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^2 \binom{4k}{2k}}{(m^2/(m - 64))^k} \pmod{p^2} & \text{if } p \nmid m - 64. \end{cases} \end{aligned}$$

This yields the remaining part.  $\square$

**Theorem 5.2.** *Let  $p$  be an odd prime. Then*

$$\begin{aligned} \sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{128^n} &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ and so } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}, \end{cases} \\ \sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{64^n (n+1)} &\equiv (-1)^{(p-1)/2} \pmod{p^2}. \end{aligned}$$

*Proof.* Taking  $m = 128$  in Theorem 5.1 and then applying [15], Theorem 4.3, (1.6) and Lemma 2.2 yields the first congruence. It is well known [4], (1.47) that

$$(5.2) \quad \sum_{r=0}^k \binom{k}{r} (-1)^r \frac{1}{r+x} = \frac{1}{x \binom{x+k}{k}} \quad \text{for } x \notin \{0, -1, \dots, -k\}.$$

For  $\frac{1}{2}p < k < p$  we see that

$$\binom{2k}{k} \binom{k}{p-1-k} \frac{(-1)^k}{p} = \binom{p-1}{k} (-1)^k \binom{2k}{p-1} \frac{1}{p} \equiv \frac{1}{2k+1} \pmod{p}.$$

Now, taking  $m = 64$  and  $c_n = (n+1)^{-1}$  in (5.1) and then applying the above and [20], Theorem 2.2 gives

$$\begin{aligned} \sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{64^n (n+1)} &= \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{64^k} \sum_{r=0}^k \binom{k}{r} \frac{(-1)^r}{k+r+1} \\ &\quad + \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{64^k} \sum_{r=0}^{p-1-k} \binom{k}{r} \frac{(-1)^r}{k+r+1} \end{aligned}$$

$$\begin{aligned}
&\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{64^k} \cdot \frac{1}{(k+1) \binom{2k+1}{k}} + \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{64^k} \binom{k}{p-1-k} \frac{(-1)^k}{p} \\
&\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k (2k+1)} + \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k (2k+1)} \equiv (-1)^{(p-1)/2} \pmod{p^2}.
\end{aligned}$$

This completes the proof. □

**Theorem 5.3.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned}
\sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{72^n} &\equiv \sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{576^n} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod{4}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases} \\
\sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{48^n} &\equiv \sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{(-192)^n} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \end{cases} \\
\sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{63^n} &\equiv \sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{(-4032)^n} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod{7}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}
\end{aligned}$$

*Proof.* This is immediate from Theorem 5.1, [15], Theorem 5.1 and [35], Corollaries 5.1 and 5.2. □

## 6. CONGRUENCES FOR $G_{p-1}(x)$ AND $G_p(x)$ MODULO $p^3$

**Theorem 6.1.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned}
G_{p-1}(-1, 3) &\equiv (-1)^{\lfloor p/3 \rfloor} 729^{p-1} + 7p^2 U_{p-3} \pmod{p^3}, \\
G_{p-1}(-1, 6) &\equiv (-1)^{(p-1)/2} 186624^{p-1} + \frac{155}{9} p^2 E_{p-3} \pmod{p^3}, \\
G_{p-1}(-1, 4) &\equiv (-1)^{\lfloor p/4 \rfloor} 4096^{p-1} + 13p^2 s_{p-3} \pmod{p^3},
\end{aligned}$$

where  $\{s_n\}$  is given by (2.13).

Proof. Suppose that  $x \in \mathbb{Z}_p$  and  $x' = (x - \langle x \rangle_p)/p$ . Since  $\binom{-1-x}{k} = (-1)^k \binom{x+k}{k}$ , appealing to (2.5),

$$\begin{aligned} G_{p-1}(x) &= \sum_{k=0}^{p-1} \binom{p-1}{k} (-1)^k \binom{x}{k} \binom{-1-x}{k} \\ &\equiv \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} - p \sum_{k=1}^{p-1} \binom{x}{k} \binom{-1-x}{k} H_k \\ &\quad + \frac{p^2}{2} \sum_{k=0}^{p-1} \binom{x}{k} \binom{x+k}{k} (-1)^k (H_k^2 - H_k^{(2)}) \pmod{p^3}. \end{aligned}$$

It is well known (see [4], (1.45)) that  $\sum_{r=1}^n \binom{n}{r} (-1)^r r^{-1} = -H_n$ . Thus, appealing to (2.8) and (2.14),

$$\begin{aligned} -(-1)^{\langle x \rangle_p} \sum_{k=1}^{p-1} \binom{x}{k} \binom{-1-x}{k} H_k &\equiv \sum_{k=1}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{1}{k} \\ &\equiv -2H_{\langle x \rangle_p} + 2px' H_{\langle x \rangle_p}^{(2)} \\ &\equiv -2 \frac{B_{p^2(p-1)}(-x) - B_{p^2(p-1)}}{p^2(p-1)} \pmod{p^2}. \end{aligned}$$

Osburn and Schneider proved (see [32]) that

$$(6.1) \quad \sum_{k=1}^n \binom{n}{k} \binom{n+k}{k} (-1)^k H_k^{(2)} = -2(-1)^n \sum_{k=1}^n \frac{(-1)^k}{k^2}.$$

By [7],

$$(6.2) \quad \sum_{k=1}^n \binom{n}{k} \binom{n+k}{k} (-1)^k H_k^2 = 4(-1)^n H_n^2 + 2(-1)^n \sum_{k=1}^n \frac{(-1)^k}{k^2}.$$

Thus,

$$\begin{aligned} &\sum_{k=0}^{p-1} \binom{x}{k} \binom{x+k}{k} (-1)^k (H_k^2 - H_k^{(2)}) \\ &\equiv \sum_{k=0}^{\langle x \rangle_p} \binom{\langle x \rangle_p}{k} \binom{\langle x \rangle_p + k}{k} (-1)^k (H_k^2 - H_k^{(2)}) \\ &= 4(-1)^{\langle x \rangle_p} \left( H_{\langle x \rangle_p}^2 + \sum_{k=1}^{\langle x \rangle_p} \frac{(-1)^k}{k^2} \right) \pmod{p}. \end{aligned}$$

From (2.9), (2.10) and the above,

$$\begin{aligned}
 (6.3) \quad G_{p-1}(x) &\equiv (-1)^{\langle x \rangle_p} + p^2 x'(x' + 1)E_{p-3}(-x) \\
 &\quad - 2p(-1)^{\langle x \rangle_p} \frac{B_{p^2(p-1)}(-x) - B_{p^2(p-1)}}{p^2(p-1)} \\
 &\quad + 2p^2(-1)^{\langle x \rangle_p} \left( \left( \frac{B_{p^2(p-1)}(-x) - B_{p^2(p-1)}}{p^2(p-1)} \right)^2 \right. \\
 &\quad \left. + \frac{1}{2}(-1)^{\langle x \rangle_p} E_{p-3}(-x) \right) \pmod{p^3}.
 \end{aligned}$$

For  $m = 3, 4, 6$  we see that  $(-1)^{\langle -1/m \rangle_p} = (-1)^{[p/m]}$ . Taking  $x = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$  in (6.3) and then applying (2.17)–(2.19) and (2.11)–(2.12) we deduce that

$$\begin{aligned}
 G_{p-1}(-1, 3) &= 27^{p-1} G_{p-1}\left(-\frac{1}{3}\right) \\
 &\equiv 27^{p-1} (-1)^{[p/3]} \left( 1 - 2p \left( -\frac{3}{2}q_p(3) + \frac{3}{4}pq_p(3)^2 \right) + 2p^2 \left( -\frac{3}{2}q_p(3) \right)^2 \right) \\
 &\quad + p^2 \left( -\frac{1}{3} \cdot \frac{2}{3} + 1 \right) \cdot 9U_{p-3} \\
 &\equiv (-1)^{[p/3]} 27^{p-1} (1 + pq_p(3))^3 + 7p^2 U_{p-3} \\
 &= (-1)^{[p/3]} 729^{p-1} + 7p^2 U_{p-3} \pmod{p^3},
 \end{aligned}$$

$$\begin{aligned}
 G_{p-1}(-1, 4) &= 64^{p-1} G_{p-1}\left(-\frac{1}{4}\right) \\
 &\equiv 64^{p-1} (-1)^{[p/4]} \left( 1 - 2p \left( -3q_p(2) + \frac{3}{2}pq_p(2)^2 \right) + 2p^2 (-3q_p(2))^2 \right) \\
 &\quad + p^2 \left( -\frac{1}{4} \cdot \frac{3}{4} + 1 \right) \cdot 16s_{p-3} \\
 &\equiv (-1)^{[p/4]} 64^{p-1} (1 + pq_p(2))^6 + 13p^2 s_{p-3} \\
 &= (-1)^{[p/4]} 4096^{p-1} + 13p^2 s_{p-3} \pmod{p^3},
 \end{aligned}$$

$$\begin{aligned}
 G_{p-1}(-1, 6) &= 432^{p-1} G_{p-1}(-1/6) \\
 &\equiv 432^{p-1} (-1)^{[p/6]} \left( 1 - 2p \left( -2q_p(2) - \frac{3}{2}q_p(3) + p \left( q_p(2)^2 + \frac{3}{4}q_p(3)^2 \right) \right) \right. \\
 &\quad \left. + 2p^2 \left( 2q_p(2) + \frac{3}{2}q_p(3) \right)^2 \right) + p^2 \left( -\frac{1}{6} \cdot \frac{5}{6} + 1 \right) \cdot 20E_{p-3} \\
 &\equiv (-1)^{(p-1)/2} 432^{p-1} (1 + pq_p(2))^4 (1 + pq_p(3))^3 + \frac{155}{9} p^2 E_{p-3} \\
 &= (-1)^{(p-1)/2} 186624^{p-1} + \frac{155}{9} p^2 E_{p-3} \pmod{p^3}.
 \end{aligned}$$

This proves the theorem. □

**Remark 6.1.** Taking  $x = -\frac{1}{2}$  in (6.3) and then applying (2.16) yields

$$(6.4) \quad G_{p-1} \equiv (-1)^{(p-1)/2} 256^{p-1} + 3p^2 E_{p-3} \pmod{p^3} \quad \text{for any prime } p > 3.$$



This was conjectured by the author earlier and later solved by Liu and Ni, see [7].

**Theorem 6.2.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} G_p &\equiv 12 + 64(-1)^{(p-1)/2}p^2E_{p-3} \pmod{p^3}, \\ G_p(-1, 3) &\equiv 21 + 243(-1)^{\lfloor p/3 \rfloor}p^2U_{p-3} \pmod{p^3}, \\ G_p(-1, 4) &\equiv 52 + 1024(-1)^{\lfloor p/4 \rfloor}p^2s_{p-3} \pmod{p^3}, \\ G_p(-1, 6) &\equiv 372 + 8640(-1)^{(p-1)/2}p^2E_{p-3} \pmod{p^3}. \end{aligned}$$

*Proof.* Suppose that  $x \in \mathbb{Z}_p$  and  $x \not\equiv 0 \pmod{p}$ . Since  $\binom{p-1}{k}(-1)^k \equiv 1 - pH_k \pmod{p^2}$ ,

$$\begin{aligned} G_p(x) &= \sum_{k=0}^p \binom{p}{k} (-1)^k \binom{x}{k} \binom{-1-x}{k} \\ &= 1 - \binom{x}{p} \binom{-1-x}{p} - \sum_{k=1}^{p-1} \frac{p}{k} \binom{p-1}{k-1} (-1)^{k-1} \binom{x}{k} \binom{-1-x}{k} \\ &\equiv 1 - \binom{x}{p} \binom{-1-x}{p} - p \sum_{k=1}^{p-1} \frac{\binom{x}{k} \binom{-1-x}{k}}{k} (1 - pH_{k-1}) \\ &= 1 - \binom{x}{p} \binom{-1-x}{p} - p \sum_{k=1}^{p-1} \frac{\binom{x}{k} \binom{-1-x}{k}}{k} \\ &\quad + p^2 \sum_{k=1}^{p-1} \binom{x}{k} \binom{-1-x}{k} \left( \frac{H_k}{k} - \frac{1}{k^2} \right) \pmod{p^3}. \end{aligned}$$

Using the WZ method or the summation package Sigma in Mathematica, one can prove the identity

$$(6.5) \quad \sum_{k=1}^n \binom{n}{k} \binom{n+k}{k} \frac{(-1)^k}{k} H_k = 2 \sum_{k=1}^n \frac{(-1)^k}{k^2}.$$

Thus, appealing to (2.9),

$$\begin{aligned} \sum_{k=1}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{H_k}{k} &= \sum_{k=1}^{p-1} \binom{x}{k} \binom{x+k}{k} (-1)^k \frac{H_k}{k} \\ &\equiv \sum_{k=1}^{\langle x \rangle_p} \binom{\langle x \rangle_p}{k} \binom{\langle x \rangle_p + k}{k} (-1)^k \frac{H_k}{k} \\ &= 2 \sum_{k=1}^{\langle x \rangle_p} \frac{(-1)^k}{k^2} \equiv (-1)^{\langle x \rangle_p} E_{p-3}(-x) \pmod{p}. \end{aligned}$$

Now, combining the above with (2.14) and (2.15) gives

$$(6.6) \quad G_p(x) \equiv 1 - \binom{x}{p} \binom{-1-x}{p} + 2p \frac{B_{p^2(p-1)}(-x) - B_{p^2(p-1)}}{p^2(p-1)} \\ + 2p^2 \left( \frac{B_{p^2(p-1)}(-x) - B_{p^2(p-1)}}{p^2(p-1)} \right)^2 + p^2 (-1)^{\langle x \rangle_p} E_{p-3}(-x) \pmod{p^3}.$$

Taking  $x = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$  in (6.6) and then applying (2.17)–(2.19), (2.11)–(2.12) and (2.4) yields

$$\begin{aligned} G_p &= 16^p G_p\left(-\frac{1}{2}\right) \equiv 16^p \left( 1 - \binom{-\frac{1}{2}}{p} \right)^2 + 2p(-2q_p(2) + pq_p(2)^2) \\ &\quad + 2p^2(-2q_p(2))^2 + p^2(-1)^{(p-1)/2} E_{p-3}\left(\frac{1}{2}\right) \\ &\equiv 16^p \left( 1 - \binom{2p}{p} \right)^2 16^{-p} - 4pq_p(2) + 10p^2q_p(2)^2 + p^2(-1)^{(p-1)/2} \cdot 4E_{p-3} \\ &\equiv -4 + 16(1 + pq_p(2))^4(1 - 4pq_p(2) + 10p^2q_p(2)^2) \\ &\quad + 16^p \cdot 4(-1)^{(p-1)/2} p^2 E_{p-3} \\ &\equiv 12 + 64(-1)^{(p-1)/2} p^2 E_{p-3} \pmod{p^3}, \\ G_p(-1, 3) &= 27^p G_p\left(-\frac{1}{3}\right) \equiv 27^p \left( 1 - \binom{-\frac{1}{3}}{p} \binom{-\frac{2}{3}}{p} \right) + 2p \left( -\frac{3}{2}q_p(3) + \frac{3}{4}pq_p(3)^2 \right) \\ &\quad + 2p^2 \left( -\frac{3}{2}q_p(3) \right)^2 + p^2(-1)^{\lfloor p/3 \rfloor} E_{p-3}\left(\frac{1}{3}\right) \\ &\equiv 27^p \left( 1 - \binom{2p}{p} \binom{3p}{p} \right) 27^{-p} - 3pq_p(3) + 6p^2q_p(3)^2 + 9(-1)^{\lfloor p/3 \rfloor} p^2 U_{p-3} \\ &\equiv -\binom{2}{1} \binom{3}{1} + 27(1 + pq_p(3))^3(1 - 3pq_p(3) + 6p^2q_p(3)^2) \\ &\quad + 27^p \cdot 9(-1)^{\lfloor p/3 \rfloor} p^2 U_{p-3} \\ &\equiv 21 + 243(-1)^{\lfloor p/3 \rfloor} p^2 U_{p-3} \pmod{p^3}, \\ G_p(-1, 4) &= 64^p G_p\left(-\frac{1}{4}\right) \equiv 64^p \left( 1 - \binom{-\frac{1}{4}}{p} \binom{-\frac{3}{4}}{p} \right) + 2p \left( -3q_p(2) + \frac{3}{2}pq_p(2)^2 \right) \\ &\quad + 2p^2(-3q_p(2))^2 + p^2(-1)^{\lfloor p/4 \rfloor} E_{p-3}\left(\frac{1}{4}\right) \\ &\equiv -\binom{2p}{p} \binom{4p}{2p} + 64^p \cdot 16(-1)^{\lfloor p/4 \rfloor} p^2 s_{p-3} \\ &\quad + 64(1 + pq_p(2))^6(1 - 6pq_p(2) + 21p^2q_p(2)^2) \\ &\equiv -\binom{2}{1} \binom{4}{2} + 1024(-1)^{\lfloor p/4 \rfloor} p^2 s_{p-3} \\ &\quad + 64(1 + 6pq_p(2) + 15p^2q_p(2)^2)(1 - 6pq_p(2) + 21p^2q_p(2)^2) \\ &= 52 + 1024(-1)^{\lfloor p/4 \rfloor} p^2 s_{p-3} \pmod{p^3} \end{aligned}$$

and

$$\begin{aligned}
G_p(-1, 6) &= 432^p G_p\left(-\frac{1}{6}\right) \equiv 432^p \left(1 - \binom{-\frac{1}{6}}{p} \binom{-\frac{5}{6}}{p}\right) \\
&\quad + 2p \left(-2q_p(2) - \frac{3}{2}q_p(3) + p \left(q_p(2)^2 + \frac{3}{4}q_p(3)^2\right)\right) \\
&\quad + 2p^2 \left(-2q_p(2) - \frac{3}{2}q_p(3)\right)^2 + p^2 (-1)^{[p/6]} E_{p-3} \left(\frac{1}{6}\right) \\
&\equiv 432^p \left(1 - \binom{3p}{p} \binom{6p}{3p}\right) 432^{-p} - p(4q_p(2) + 3q_p(3)) \\
&\quad + p^2(10q_p(2)^2 + 6q_p(3)^2 + 12q_p(2)q_p(3)) \\
&\quad + 432^p \cdot 20(-1)^{(p-1)/2} p^2 E_{p-3} \\
&\equiv -\binom{3}{1} \binom{6}{3} + 432(1 + pq_p(2))^4 (1 + pq_p(3))^3 (1 - p(4q_p(2) + 3q_p(3))) \\
&\quad + p^2(10q_p(2)^2 + 6q_p(3)^2 + 12q_p(2)q_p(3)) + 432 \cdot 20(-1)^{(p-1)/2} p^2 E_{p-3} \\
&\equiv 372 + 8640(-1)^{(p-1)/2} p^2 E_{p-3} \pmod{p^3}.
\end{aligned}$$

This completes the proof.  $\square$

**Theorem 6.3.** *Let  $p > 3$  be a prime,  $x \in \mathbb{Z}_p$  and  $x \not\equiv 0 \pmod{p}$ . Then*

$$\begin{aligned}
&\sum_{k=0}^{p-1} (2k(k+1) + x(x+1) + 1)(-1)^k G_k(x)^2 \\
&\equiv p^2 (-1)^{\langle x \rangle_p} \left(1 - \binom{x}{p} \binom{-1-x}{p}\right) \\
&\quad + 2p^3 (-1)^{\langle x \rangle_p} \binom{x}{p} \binom{-1-x}{p} H_{\langle x \rangle_p} \pmod{p^4}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\sum_{k=0}^{p-1} (8k^2 + 8k + 3) \frac{G_k^2}{(-256)^k} \equiv 3(-1)^{(p-1)/2} p^2 + 25p^4 E_{p-3} \pmod{p^5}, \\
&\sum_{k=0}^{p-1} (18k^2 + 18k + 7) \frac{G_k(-1, 3)^2}{(-729)^k} \equiv 7(-1)^{[p/3]} p^2 + 130p^4 U_{p-3} \pmod{p^5}, \\
&\sum_{k=0}^{p-1} (32k^2 + 32k + 13) \frac{G_k(-1, 4)^2}{(-4096)^k} \equiv 13(-1)^{[p/4]} p^2 + 425p^4 s_{p-3} \pmod{p^5}, \\
&\sum_{k=0}^{p-1} (72k^2 + 72k + 31) \frac{G_k(-1, 6)^2}{(-186624)^k} \equiv 31(-1)^{(p-1)/2} p^2 + \frac{11285}{9} p^4 E_{p-3} \pmod{p^5}.
\end{aligned}$$

**Proof.** By (3.1) and [23], Theorem 3.3 (with  $r = 2$ ,  $u_k = G_k(x)$ ,  $b(k) = 2k(k+1) + x^2 + x + 1$  and  $c = 1$ ), for  $n \in \mathbb{Z}^+$ ,

$$(6.7) \quad \sum_{k=0}^{n-1} (2k(k+1) + x^2 + x + 1)(-1)^{n-1-k} G_k(x)^2 = n^2 G_n(x) G_{n-1}(x).$$

Hence, appealing to (6.3), (6.6) and (2.14),

$$\begin{aligned} & \sum_{k=0}^{p-1} (2k(k+1) + x^2 + x + 1)(-1)^k G_k(x)^2 \\ &= p^2 G_{p-1}(x) G_p(x) \\ &\equiv p^2 (-1)^{\langle x \rangle_p} (1 - 2p H_{\langle x \rangle_p}) \left( 1 - \binom{x}{p} \binom{-1-x}{p} + 2p H_{\langle x \rangle_p} \right) \\ &\equiv p^2 (-1)^{\langle x \rangle_p} \left( 1 - \binom{x}{p} \binom{-1-x}{p} \right) \\ &\quad + 2p^3 (-1)^{\langle x \rangle_p} \binom{x}{p} \binom{-1-x}{p} H_{\langle x \rangle_p} \pmod{p^4}. \end{aligned}$$

Taking  $n = p$ ,  $x = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$  in (6.7) and then applying Theorems 6.1–6.2 and (6.4) yields the remaining results.  $\square$

## 7. CONJECTURES ON CONGRUENCES INVOLVING $G_n$ , $G_n(-1, 3)$ , $G_n(-1, 4)$ AND $G_n(-1, 6)$

Calculations by Maple suggest the following challenging conjectures.

**Conjecture 7.1.** *Let  $p$  be a prime,  $p > 3$ . Then*

$$\begin{aligned} G_{2p} &\equiv G_2 + 3072(-1)^{(p-1)/2} p^2 E_{p-3} \pmod{p^3}, \\ G_{2p-1} &\equiv (-1)^{(p-1)/2} 16^4 (p-1) G_1 + 164p^2 E_{p-3} \pmod{p^3}, \\ G_{3p} &\equiv G_3 + 94464(-1)^{(p-1)/2} p^2 E_{p-3} \pmod{p^3}, \\ G_{2p}(-1, 3) &\equiv G_2(-1, 3) + 20412(-1)^{[p/3]} p^2 U_{p-3} \pmod{p^3}, \\ G_{2p-1}(-1, 3) &\equiv (-1)^{[p/3]} 27^4 (p-1) G_1(-1, 3) + 660p^2 U_{p-3} \pmod{p^3}, \\ G_{2p-1}(-1, 6) &\equiv (-1)^{(p-1)/2} 432^4 (p-1) G_1(-1, 6) + \frac{82580}{3} p^2 E_{p-3} \pmod{p^3}. \end{aligned}$$

**Conjecture 7.2.** Let  $p$  be a prime,  $p > 3$ .

(i) If  $p \equiv 1 \pmod{4}$  and so  $p = x^2 + y^2$  with  $4 \mid x - 1$ , then

$$\sum_{n=0}^{p-1} \frac{G_n}{(-16)^n} \equiv (-1)^{(p-1)/4} \sum_{n=0}^{p-1} \frac{G_n}{8^n} \equiv (-1)^{(p-1)/4} \sum_{n=0}^{p-1} \frac{G_n}{32^n} \equiv 2x - \frac{p}{2x} \pmod{p^2},$$

$$\sum_{n=0}^{p-1} \frac{nG_n}{(-16)^n} \equiv \frac{(-1)^{(p-1)/4}}{3} \sum_{n=0}^{p-1} \frac{nG_n}{8^n} \equiv -(-1)^{(p-1)/4} \sum_{n=0}^{p-1} \frac{nG_n}{32^n} \equiv -x + \frac{p}{2x} \pmod{p^2}.$$

(ii) If  $p \equiv 3 \pmod{4}$ , then

$$\sum_{n=0}^{p-1} \frac{G_n}{(-16)^n} \equiv \frac{2}{3} (-1)^{(p+1)/4} \sum_{n=0}^{p-1} \frac{G_n}{8^n} \equiv 2(-1)^{(p-3)/4} \sum_{n=0}^{p-1} \frac{G_n}{32^n} \equiv \frac{p}{\binom{(p-3)/2}{(p-3)/4}} \pmod{p^2}.$$

**Conjecture 7.3.** Let  $p$  be an odd prime. Then

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n} G_n}{128^n} \equiv \begin{cases} 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ \frac{1}{3} p^2 \left( \left[ \frac{1}{4} p \right] \right)^{-2} \pmod{p^3} & \text{if } p \equiv 5 \pmod{8}, \\ -\frac{3}{2} p^2 \left( \left[ \frac{1}{8} p \right] \right)^{-2} \pmod{p^3} & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

**Conjecture 7.4.** Let  $p$  be an odd prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} G_k}{64^k (2k-1)} \equiv 2(-1)^{(p-1)/2} p^2 \pmod{p^3}.$$

**Conjecture 7.5.** Let  $p > 3$  be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} G_k}{72^k (2k-1)} \equiv \begin{cases} -\frac{4}{9} x^2 + \frac{8}{27} p \pmod{p^2} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod{4}, \\ -\frac{2}{9} R_1(p) - \frac{2}{27} p \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} G_k}{576^k (2k-1)} \equiv \begin{cases} -\frac{8}{3} x^2 + \frac{32}{27} p \pmod{p^2} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod{4}, \\ -\frac{2}{9} R_1(p) + \frac{4}{27} p \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

where  $R_1(p) = (2p + 2 - 2^{p-1}) \binom{(p-1)/2}{(p-3)/4}^2$ .

**Conjecture 7.6.** Let  $p$  be an odd prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} G_k}{128^k (2k-1)} \equiv \begin{cases} -\frac{3}{8}(4x^2 - 2p) \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ -\frac{1}{8}R_2(p) \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}, \end{cases}$$

where

$$R_2(p) = (5 - 4(-1)^{(p-1)/2}) \times \left(1 + (4 + 2(-1)^{(p-1)/2})p - 4(2^{p-1} - 1) - \frac{p}{2}H_{[p/8]}\right) \binom{\frac{1}{2}(p-1)}{[\frac{1}{8}p]}^2.$$

**Conjecture 7.7.** Let  $p > 3$  be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} G_k}{48^k (2k-1)} \equiv \begin{cases} \frac{4}{9}x^2 \pmod{p^2} & \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ -\frac{8}{9}R_3(p) - \frac{2}{9}p \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} G_k}{(-192)^k (2k-1)} \equiv \begin{cases} -\frac{32}{9}x^2 + \frac{4}{3}p \pmod{p^2} & \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ -\frac{8}{9}R_3(p) + \frac{4}{9}p \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

where

$$R_3(p) = \left(1 + 2p + \frac{4}{3}(2^{p-1} - 1) - \frac{3}{2}(3^{p-1} - 1)\right) \binom{\frac{1}{2}(p-1)}{[\frac{1}{6}p]}^2.$$

**Conjecture 7.8.** Let  $p$  be a prime,  $p \neq 2, 3, 7$ . Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} G_k}{63^k (2k-1)} \equiv \begin{cases} \frac{4}{7}y^2 + \frac{26}{1323}p \pmod{p^2} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod{7}, \\ \frac{32}{63} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{k+1} + \frac{688}{1323}p \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} G_k}{(-4032)^k (2k-1)} \equiv \begin{cases} \frac{1408}{63}y^2 - \frac{2368}{1323}p \pmod{p^2} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod{7}, \\ \frac{32}{63} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{k+1} + \frac{1024}{1323}p \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

**Conjecture 7.9.** Let  $p$  be an odd prime. Then

$$\sum_{n=0}^{p-1} (2n+1) \frac{G_n^2}{(-256)^n} \equiv \begin{cases} p(-1)^{(p-1)/4} \left(2x - \frac{p}{2x}\right) \pmod{p^3} \\ \quad \text{if } p = x^2 + y^2 \equiv 1 \pmod{4} \text{ and } 4 \mid x-1, \\ 0 \pmod{p^3} \quad \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

**Conjecture 7.10.** Let  $p > 3$  be a prime. Then

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{G_n(-1, 3)}{(-27)^n} \\ & \equiv \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3} \text{ and } 3 \mid x-1, \\ -\frac{3}{2}p \left(\frac{\frac{1}{2}(p-1)}{\frac{1}{6}(p-5)}\right)^{-1} \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \end{cases} \\ & \sum_{n=0}^{p-1} \frac{G_n(-1, 3)}{3^n} \\ & \equiv \begin{cases} -L + \frac{p}{L} \pmod{p^2} & \text{if } 3 \mid p-1, 4p = L^2 + 27M^2 \text{ and } 3 \mid L-1, \\ -\frac{4}{3}p \left(\frac{p-2}{3}\right)!^3 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \end{cases} \\ & \sum_{n=0}^{p-1} \frac{G_n(-1, 3)}{243^n} \\ & \equiv \begin{cases} -L + \frac{p}{L} \pmod{p^2} & \text{if } 3 \mid p-1, 4p = L^2 + 27M^2 \text{ and } 3 \mid L-1, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3} \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{nG_n(-1, 3)}{(-27)^n} \equiv -x + \frac{p}{2x} \pmod{p^2} \\ & \quad \text{for } p = x^2 + 3y^2 \equiv 1 \pmod{3} \text{ with } 3 \mid x-1, \\ & \sum_{n=0}^{p-1} \frac{nG_n(-1, 3)}{3^n} \equiv L - \frac{2p}{L} \pmod{p^2} \\ & \quad \text{for } 3 \mid p-1 \text{ with } 4p = L^2 + 27M^2 \text{ and } 3 \mid L-1, \\ & \sum_{n=0}^{p-1} \frac{nG_n(-1, 3)}{243^n} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 1 \pmod{3}. \end{aligned}$$

**Conjecture 7.11.** Let  $p$  be a prime,  $p > 3$ . Then

$$\begin{aligned} \left(\frac{-3}{p}\right) \sum_{n=0}^{p-1} G_n(-1, 4) &\equiv \frac{21\left(\frac{p}{7}\right) - 19}{2} \left(\frac{6}{p}\right) \sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{4096^n} \\ &\equiv \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod{7} \text{ with } \left(\frac{x}{7}\right) = 1, \\ \frac{10p}{\binom{[3p/7]}{[p/7]}} \pmod{p^2} & \text{if } p \equiv 3 \pmod{7}, \\ -\frac{5p}{2\binom{[3p/7]}{[p/7]}} \pmod{p^2} & \text{if } p \equiv 5 \pmod{7}, \\ \frac{5p}{\binom{[3p/7]}{[p/7]}} \pmod{p^2} & \text{if } p \equiv 6 \pmod{7}, \end{cases} \\ \left(\frac{6}{p}\right) \sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{(-8)^n} &\equiv (3 - 2(-1)^{(p-1)/2}) \left(\frac{3}{p}\right) \sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{(-512)^n} \\ &\equiv \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod{4} \text{ with } 4 \mid x - 1, \\ \frac{5p}{2} \left(\frac{1}{4}(p-3)\right)^{-1} \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases} \\ \sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{16^n} &\equiv \left(4 - 3\left(\frac{-3}{p}\right)\right) \left(\frac{-2}{p}\right) \sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{256^n} \\ &\equiv \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3} \text{ with } 3 \mid x - 1, \\ -\frac{7}{2}p \left(\frac{1}{6}(p-1)\right)^{-1} \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \end{cases} \\ \sum_{n=0}^{p-1} \frac{G_n(-1, 4)}{(-64)^n} &\equiv \begin{cases} (-1)^{[p/8] + (p-1)/2} \left(2x - \frac{p}{2x}\right) \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8} \text{ with } 4 \mid x - 1, \\ -\frac{4}{2 - (-1)^{(p-1)/2}} p \left(\frac{1}{8}p\right)^{-1} \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

**Conjecture 7.12.** Let  $p > 3$  be a prime.

(i) If  $p \equiv 1, 2, 4 \pmod{7}$  and so  $p = x^2 + 7y^2$  with  $x \equiv 1, 2, 4 \pmod{7}$ , then

$$\left(\frac{-3}{p}\right) \sum_{n=0}^{p-1} nG_n(-1, 4) \equiv 20 \left(\frac{6}{p}\right) \sum_{n=0}^{p-1} \frac{nG_n(-1, 4)}{4096^n} \equiv -\frac{40}{21} \left(x - \frac{p}{2x}\right) \pmod{p^2}.$$



(ii) If  $p \equiv 1 \pmod{4}$  and so  $p = x^2 + 4y^2$  with  $x \equiv 1 \pmod{4}$ , then

$$\left(\frac{6}{p}\right) \sum_{n=0}^{p-1} \frac{nG_n(-1, 4)}{(-8)^n} \equiv 5 \left(\frac{3}{p}\right) \sum_{n=0}^{p-1} \frac{nG_n(-1, 4)}{(-512)^n} \equiv -\frac{5}{3} \left(x - \frac{p}{2x}\right) \pmod{p^2}.$$

(iii) If  $p \equiv 1 \pmod{3}$  and so  $p = x^2 + 3y^2$  with  $x \equiv 1 \pmod{3}$ , then

$$\sum_{n=0}^{p-1} \frac{nG_n(-1, 4)}{16^n} \equiv -7 \left(\frac{-2}{p}\right) \sum_{n=0}^{p-1} \frac{nG_n(-1, 4)}{256^n} \equiv -\frac{7}{3} \left(x - \frac{p}{2x}\right) \pmod{p^2}.$$

(iv) If  $p \equiv 1, 3 \pmod{8}$  and so  $p = x^2 + 2y^2$  with  $x \equiv 1 \pmod{4}$ , then

$$(-1)^{[p/8] + (p-1)/2} \sum_{n=0}^{p-1} \frac{nG_n(-1, 4)}{(-64)^n} \equiv -x + \frac{p}{2x} \pmod{p^2}.$$

**Conjecture 7.13.** Let  $p > 3$  be a prime,  $m \in \{-3267, -1350, -108, 44, 100, 135, 300, 1836, 8748, 110700, 27000108\}$  and  $p \nmid m(108 - m)$ . Then

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{G_k(-1, 3)}{m^k} \equiv \left(\frac{m(m-108)}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(108-m)^k} \pmod{p^2}.$$

**Conjecture 7.14.** Let  $p > 3$  be a prime. Then

$$\begin{aligned} \sum_{k=0}^{p-1} (15k+2) \frac{\binom{2k}{k} G_k(-1, 3)}{(-3267)^k} &\equiv 2 \left(\frac{-15}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 11, \\ \sum_{k=0}^{p-1} (81k+13) \frac{\binom{2k}{k} G_k(-1, 3)}{(-1350)^k} &\equiv 13 \left(\frac{p}{3}\right) p \pmod{p^2} \quad \text{for } p \neq 5, \\ \sum_{k=0}^{p-1} (3k+1) \frac{\binom{2k}{k} G_k(-1, 3)}{(-108)^k} &\equiv \left(\frac{-6}{p}\right) p \pmod{p^2}, \\ \sum_{k=0}^{p-1} (32k+21) \frac{\binom{2k}{k} G_k(-1, 3)}{44^k} &\equiv 21 \left(\frac{p}{11}\right) p \pmod{p^2} \quad \text{for } p \neq 11, \\ \sum_{k=0}^{p-1} (k+3) \frac{\binom{2k}{k} G_k(-1, 3)}{100^k} &\equiv 3 \left(\frac{-2}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 5, \\ \sum_{k=0}^{p-1} (3k-2) \frac{\binom{2k}{k} G_k(-1, 3)}{135^k} &\equiv -2 \left(\frac{-15}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 5, \\ \sum_{k=0}^{p-1} (32k+1) \frac{\binom{2k}{k} G_k(-1, 3)}{300^k} &\equiv \left(\frac{p}{3}\right) p \pmod{p^2} \quad \text{for } p \neq 5, \end{aligned}$$

$$\sum_{k=0}^{p-1} (96k + 11) \frac{\binom{2k}{k} G_k(-1, 3)}{1836^k} \equiv 11 \left( \frac{-51}{p} \right) p \pmod{p^2} \quad \text{for } p \neq 17,$$

$$\sum_{k=0}^{p-1} (160k + 17) \frac{\binom{2k}{k} G_k(-1, 3)}{8748^k} \equiv 17 \left( \frac{p}{3} \right) p \pmod{p^2},$$

$$\sum_{k=0}^{p-1} (6144k + 527) \frac{\binom{2k}{k} G_k(-1, 3)}{110700^k} \equiv 527 \left( \frac{-123}{p} \right) p \pmod{p^2} \quad \text{for } p \neq 5, 41,$$

$$\sum_{k=0}^{p-1} (1500000k + 87659) \frac{\binom{2k}{k} G_k(-1, 3)}{27000108^k} \equiv 87659 \left( \frac{-267}{p} \right) p \pmod{p^2} \quad \text{for } p \neq 53, 89.$$

**Conjecture 7.15.** Let  $p > 3$  be a prime,

$$m \in \{ -24591257600, -2508800, -614400, -20480, -2048, -392, \\ 175, 400, 1280, 4225, 12544, 83200, 6635776, 199148800 \}$$

and  $p \nmid m(256 - m)$ . Then

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{G_k(-1, 4)}{m^k} \equiv \left( \frac{m(m - 256)}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(256 - m)^k} \pmod{p^2}.$$

**Conjecture 7.16.** Let  $p > 3$  be a prime. Then

$$\sum_{k=0}^{p-1} (81k + 23) \frac{\binom{2k}{k} G_k(-1, 4)}{(-392)^k} \equiv 23(-1)^{(p-1)/2} p \pmod{p^2} \quad \text{for } p \neq 7,$$

$$\sum_{k=0}^{p-1} (6k + 1) \frac{\binom{2k}{k} G_k(-1, 4)}{(-2048)^k} \equiv \left( \frac{-6}{p} \right) p \pmod{p^2},$$

$$\sum_{k=0}^{p-1} (162k + 17) \frac{\binom{2k}{k} G_k(-1, 4)}{(-20480)^k} \equiv 17 \left( \frac{-10}{p} \right) p \pmod{p^2} \quad \text{for } p \neq 5,$$

$$\sum_{k=0}^{p-1} (4802k + 361) \frac{\binom{2k}{k} G_k(-1, 4)}{(-614400)^k} \equiv 361 \left( \frac{-2}{p} \right) p \pmod{p^2} \quad \text{for } p \neq 5,$$

$$\sum_{k=0}^{p-1} (162k + 11) \frac{\binom{2k}{k} G_k(-1, 4)}{(-2508800)^k} \equiv 11 \left( \frac{-22}{p} \right) p \pmod{p^2} \quad \text{for } p \neq 5, 7,$$

$$\sum_{k=0}^{p-1} (192119202k + 8029841) \frac{\binom{2k}{k} G_k(-1, 4)}{(-24591257600)^k} \\ \equiv 8029841 \left( \frac{-58}{p} \right) p \pmod{p^2} \quad \text{for } p \neq 5, 7, 13, 29$$

and

$$\sum_{k=0}^{p-1} (81k + 88) \frac{\binom{2k}{k} G_k(-1, 4)}{175^k} \equiv 88 \left(\frac{-7}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 5, 7,$$

$$\sum_{k=0}^{p-1} (3k - 1) \frac{\binom{2k}{k} G_k(-1, 4)}{400^k} \equiv -\left(\frac{-3}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 5,$$

$$\sum_{k=0}^{p-1} (16k + 1) \frac{\binom{2k}{k} G_k(-1, 4)}{1280^k} \equiv \left(\frac{-5}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 5,$$

$$\sum_{k=0}^{p-1} (81k + 8) \frac{\binom{2k}{k} G_k(-1, 4)}{4225^k} \equiv 8 \left(\frac{-7}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 5, 13,$$

$$\sum_{k=0}^{p-1} (192k + 19) \frac{\binom{2k}{k} G_k(-1, 4)}{12544^k} \equiv 19 \left(\frac{-1}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 7,$$

$$\sum_{k=0}^{p-1} (1296k + 113) \frac{\binom{2k}{k} G_k(-1, 4)}{83200^k} \equiv 113 \left(\frac{-13}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 13,$$

$$\sum_{k=0}^{p-1} (103680k + 6599) \frac{\binom{2k}{k} G_k(-1, 4)}{6635776^k} \equiv 6599 \left(\frac{-1}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 7, 23,$$

$$\sum_{k=0}^{p-1} (3111696k + 162833) \frac{\binom{2k}{k} G_k(-1, 4)}{199148800^k} \equiv 162833 \left(\frac{-37}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 5, 29, 37.$$

**Conjecture 7.17.** Let  $p > 3$  be a prime,

$$m \in \{ -16579647, -285768, -52272, -6272, 5103, 34496, 886464, 12289728, \\ 884737728, 147197953728, 262537412640769728 \}$$

and  $p \nmid m(1728 - m)$ . Then

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{G_k(-1, 6)}{m^k} \equiv \left(\frac{m(m - 1728)}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(1728 - m)^k} \pmod{p^2}.$$

**Conjecture 7.18.** Let  $p > 3$  be a prime. Then

$$\sum_{k=0}^{p-1} (125k + 24) \frac{\binom{2k}{k} G_k(-1, 6)}{(-6272)^k} \equiv 24 \left(\frac{-2}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 7,$$

$$\sum_{k=0}^{p-1} (125k + 13) \frac{\binom{2k}{k} G_k(-1, 6)}{(-52272)^k} \equiv 13 \left(\frac{-3}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 11,$$

$$\sum_{k=0}^{p-1} (1331k + 109) \frac{\binom{2k}{k} G_k(-1, 6)}{(-285768)^k} \equiv 109 \left(\frac{-1}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 7,$$

$$\sum_{k=0}^{p-1} (614125k + 36968) \frac{\binom{2k}{k} G_k(-1, 6)}{(-16579647)^k} \equiv 36968 \left(\frac{-7}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 7, 19$$

and

$$\sum_{k=0}^{p-1} (125k - 8) \frac{\binom{2k}{k} G_k(-1, 6)}{5103^k} \equiv -8 \left(\frac{-7}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 7,$$

$$\sum_{k=0}^{p-1} (512k + 39) \frac{\binom{2k}{k} G_k(-1, 6)}{34496^k} \equiv 39 \left(\frac{-11}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 7, 11,$$

$$\sum_{k=0}^{p-1} (512k + 37) \frac{\binom{2k}{k} G_k(-1, 6)}{886464^k} \equiv 37 \left(\frac{-19}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 19,$$

$$\sum_{k=0}^{p-1} (64000k + 3917) \frac{\binom{2k}{k} G_k(-1, 6)}{12289728^k} \equiv 3917 \left(\frac{-3}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 11, 23,$$

$$\sum_{k=0}^{p-1} (512000k + 24853) \frac{\binom{2k}{k} G_k(-1, 6)}{884737728^k} \equiv 24853 \left(\frac{-43}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 7, 43,$$

$$\sum_{k=0}^{p-1} (440^3 k + 3312613) \frac{\binom{2k}{k} G_k(-1, 6)}{147197953728^k}$$

$$\equiv 3312613 \left(\frac{-67}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 7, 31, 67,$$

$$\sum_{k=0}^{p-1} (53360^3 k + 3787946075413) \frac{\binom{2k}{k} G_k(-1, 6)}{262537412640769728^k}$$

$$\equiv 3787946075413 \left(\frac{-163}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 7, 11, 19, 127, 163.$$

**Remark 7.1.** Let  $p$  be a prime,  $p > 3$ . For the values of  $m$  in Conjecture 7.13, Conjecture 7.15 and Conjecture 7.17, the congruences for

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(108 - m)^k}, \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(256 - m)^k} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(1728 - m)^k}$$

modulo  $p^2$  were conjectured by Sun in [25], [27] and the author in [13], and partially solved by the author in [15]–[17].

**Conjecture 7.19.** Let  $p$  be an odd prime and  $m, r \in \mathbb{Z}^+$ . Then

$$\begin{aligned} G_{mp^r} &\equiv G_{mp^{r-1}} \pmod{p^{2r}}, \\ G_{mp^r}(-1, 3) &\equiv G_{mp^{r-1}}(-1, 3) \pmod{p^{2r}}, \\ G_{mp^r}(-1, 4) &\equiv G_{mp^{r-1}}(-1, 4) \pmod{p^{2r}}, \\ G_{mp^r}(-1, 6) &\equiv G_{mp^{r-1}}(-1, 6) \pmod{p^{2r}}. \end{aligned}$$

**Conjecture 7.20.** Suppose that  $p$  is an odd prime,  $m \in \{1, 3, 5, \dots\}$  and  $r \in \{2, 3, 4, \dots\}$ . Then

$$\begin{aligned} G_{(mp^r-1)/2} &\equiv p^2 G_{(mp^{r-2}-1)/2} \pmod{p^{2r-1}} \quad \text{for } p \equiv 3 \pmod{4}, \\ G_{(mp^r-1)/2}(-1, 3) &\equiv p^2 G_{(mp^{r-2}-1)/2}(-1, 3) \pmod{p^{2r-1}} \quad \text{for } p \equiv 2 \pmod{3}, \\ G_{(mp^r-1)/2}(-1, 4) &\equiv p^2 G_{(mp^{r-2}-1)/2}(-1, 4) \pmod{p^{2r-1}} \quad \text{for } p \equiv 5, 7 \pmod{8}, \\ G_{(mp^r-1)/2}(-1, 6) &\equiv p^2 G_{(mp^{r-2}-1)/2}(-1, 6) \pmod{p^{2r-1}} \quad \text{for } p \equiv 3 \pmod{4}. \end{aligned}$$

**Conjecture 7.21.** Suppose that  $p$  is an odd prime,  $m \in \{1, 3, 5, \dots\}$  and  $r \in \{2, 3, 4, \dots\}$ . Then

$$\begin{aligned} G_{(mp^r-1)/2} &\equiv (4x^2 - 2p)G_{(mp^{r-1}-1)/2} - p^2 G_{(mp^{r-2}-1)/2} \pmod{p^r} \\ &\quad \text{for } p = x^2 + 4y^2 \equiv 1 \pmod{4}, \\ G_{(mp^r-1)/2}(-1, 3) &\equiv (-1)^{(p-1)/2} (4x^2 - 2p)G_{(mp^{r-1}-1)/2}(-1, 3) \\ &\quad - p^2 G_{(mp^{r-2}-1)/2}(-1, 3) \pmod{p^r} \\ &\quad \text{for } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ G_{(mp^r-1)/2}(-1, 4) &\equiv (4x^2 - 2p)G_{(mp^{r-1}-1)/2}(-1, 4) \\ &\quad - p^2 G_{(mp^{r-2}-1)/2}(-1, 4) \pmod{p^r} \\ &\quad \text{for } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ G_{(mp^r-1)/2}(-1, 6) &\equiv (4x^2 - 2p)G_{(mp^{r-1}-1)/2}(-1, 6) \\ &\quad - p^2 G_{(mp^{r-2}-1)/2}(-1, 6) \pmod{p^r} \\ &\quad \text{for } p = x^2 + 4y^2 \equiv 1 \pmod{4}. \end{aligned}$$

**Conjecture 7.22.** Suppose  $n \in \mathbb{Z}^+$ . If  $x \in (-1, 0)$ , then  $G_n(x)^2 < G_{n+1}(x) \times G_{n-1}(x)$ . If  $x \notin [-1, 0]$ , then  $G_n(x)^2 > G_{n+1}(x)G_{n-1}(x)$ .

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