Emina Milovanović; Şerife Burcu Bozkurt Altındağ; Marjan Matejić; Igor Milovanović

Inequalities for real number sequences with applications in spectral graph theory

Czechoslovak Mathematical Journal, Vol. 72 (2022), No. 3, 783–799

Persistent URL: `http://dml.cz/dmlcz/150617`

Czechoslovak Mathematical Journal, 72 (147) (2022), 783–799

INEQUALITIES FOR REAL NUMBER SEQUENCES WITH APPLICATIONS IN SPECTRAL GRAPH THEORY

Emina Milovanović, Niš, Şerife Burcu Bozkurt Altındağ, Konya, Marjan Matejić, Igor Milovanović, Niš

Received April 25, 2021. Published online March 23, 2022.

Abstract. Let $a = (a_1, a_2, \ldots, a_n)$ be a nonincreasing sequence of positive real numbers. Denote by $S = \{1, 2, \ldots, n\}$ the index set and by $J_k = \{I = \{r_1, r_2, \ldots, r_k\},\ 1 \leqslant r_1 < r_2 < \ldots < r_k \leqslant n\}$ the set of all subsets of S of cardinality k, $1 \leqslant k \leqslant n-1$. In addition, denote by $a_I = a_{r_1} + a_{r_2} + \ldots + a_{r_k}$, $1 \leqslant k \leqslant n-1$, $1 \leqslant r_1 < r_2 < \ldots < r_k \leqslant n$, the sum of k arbitrary elements of sequence a, where $a_{I_1} = a_1 + a_2 + \ldots + a_k$ and $a_{I_n} = a_{n-k+1} + a_{n-k+2} + \ldots + a_n$. We consider bounds of the quantities $RS_k(a) = a_{I_1}/a_{I_n}$, $LS_k(a) = a_{I_1} - a_{I_n}$ and $S_{k,\alpha}(a) = \sum\limits_{I \in J_k} a_I^{\alpha}$ in terms of $A = \sum\limits_{i=1}^{n} a_i$ and $B = \sum\limits_{i=1}^{n} a_i^2$. Then we use the obtained results to generalize some results regarding Laplacian and normalized Laplacian eigenvalues of graphs.

Keywords: inequality; real number sequence; Laplacian eigenvalue of graph; normalized Laplacian eigenvalue

MSC 2020: 15A18, 05C30

1. Introduction

Let $a = (a_1, a_2, \ldots, a_n)$, $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$, be a positive real number sequence. Designate by A and B the following sums:

$$A = \sum_{i=1}^{n} a_i \quad \text{and} \quad B = \sum_{i=1}^{n} a_i^2. \tag{1.1}$$

The research has been supported by the Serbian Ministry of Education, Science and Technological Development.

DOI: 10.21136/CMJ.2022.0155-21

The notation for A and B will be used through the entire paper. Further, denote by $S = \{1, 2, \ldots, n\}$ an index set and by

$$J_k = \{I = \{r_1, r_2, \ldots, r_k\} \colon 1 \leqslant r_1 < r_2 < \ldots < r_k \leqslant n\}$$

the set of all subsets of S of cardinality k, $1 \leqslant k \leqslant n-1$. In addition, denote by $a_I = a_{r_1} + a_{r_2} + \ldots + a_{r_k}$, $1 \leqslant k \leqslant n-1$, $1 \leqslant r_1 < r_2 < \ldots < r_k \leqslant n$, the sum of k arbitrary numbers of a, where $a_{I_1} = a_1 + a_2 + \ldots + a_k$ and $a_{I_n} = a_{n-k+1} + a_{n-k+2} + \ldots + a_n$. It is easy to verify that $a_{I_n} \leqslant a_I \leqslant a_{I_1}$ for any $I, I \in J_k$.

Let us define the quantities $RS_k(a)$, $LS_k(a)$ and $S_{k,\alpha}$ in the following way:

$$RS_k(a) = \frac{a_{I_1}}{a_{I_n}}, \quad LS_k(a) = a_{I_1} - a_{I_n}, \quad S_{k,\alpha}(a) = \sum_{I \in J_k} a_I^{\alpha},$$

where α is an arbitrary real number. In this paper we determine bounds of these quantities depending on A and B. As direct consequences of the obtained results we obtain a number of old/new inequalities for the Laplacian and normalized Laplacian eigenvalues of graphs.

2. Preliminaries

In this section we recall some discrete inequalities for real number sequences that will be used later in the paper.

Let $x = (x_i)$, $y = (y_i)$, $i = 1, 2, \ldots, n$, be two positive real number sequences with properties $0 < r_1 \leqslant x_i \leqslant R_1$ and $0 < r_2 \leqslant y_i \leqslant R_2$. In [25] the following inequality was proven:

$$\sum_{i=1}^{n} y_i^2 \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i y_i \right)^2 \leqslant \frac{n^2}{4} (R_1 R_2 - r_2 r_1)^2. \tag{2.1}$$

Let $p = (p_i)$, $i = 1, 2, \ldots, n$ be a sequence of nonnegative real numbers and $x = (x_i)$, $i = 1, 2, \ldots, n$, a sequence of positive real numbers. In [15] (see also [22]) it was proven that for any real r, $r \leqslant 0$ or $r \geqslant 1$,

$$\left(\sum_{i=1}^{n} p_i \right)^{r-1} \sum_{i=1}^{n} p_i x_i^r \geqslant \left(\sum_{i=1}^{n} p_i x_i \right)^r. \tag{2.2}$$

For $0 \leqslant r \leqslant 1$, the opposite inequality is valid. Equality holds if and only if either $r = 0$, or $r = 1$, or $p_1 = p_2 = \ldots = p_t = 0$ and $x_{t+1} = \ldots = x_n$ for some t, $1 \leqslant t \leqslant n-1$, or $x_1 = x_2 = \ldots = x_n$.

3. Inequalities for positive real number sequences

In the next theorem we establish a lower bound for $RS_k(a)$ in terms of A, B and parameters n and k, $1 \leqslant k \leqslant n-1$.

Theorem 3.1. *For any k, $1 \leqslant k \leqslant n-1$, we have that*

$$RS_k(a) = \frac{a_{I_1}}{a_{I_n}} \geqslant \frac{\left(\sqrt{n((n-k)B+(k-1)A^2)} + \sqrt{(n-k)(nB-A^2)}\right)^2}{k(n-1)A^2}. \tag{3.1}$$

When $2 \leqslant k \leqslant n-1$, equality holds if and only if $a_1 = a_2 = \ldots = a_n$. For $k=1$, equality holds if and only if $a_1 = a_2 = \ldots = a_n$, or $a_1 = a_2 = \ldots = a_p = a$ and $a_{p+1} = \ldots = a_n = p/(n-p)$, $1 \leqslant p \leqslant \frac{1}{2}n$.

Proof. For any k-tuple I, $I \in J_k$, the following is valid:

$$(a_I - a_{I_1})(a_I - a_{I_n}) \leqslant 0,$$

that is,

$$a_I^2 + a_{I_1}a_{I_n} \leqslant (a_{I_1} + a_{I_n})a_I.$$

After summation over all I, $I \in J_k$, of the above inequality, we obtain

$$\sum_{I\in J_k} a_I^2 + a_{I_1}a_{I_n}\sum_{I\in J_k} 1 \leqslant (a_{I_1}+a_{I_n})\sum_{I\in J_k} a_I. \tag{3.2}$$

From the arithmetic-geometric mean inequality for real numbers (see, e.g., [22]), and (3.2) we obtain

$$2\sqrt{a_{I_1}a_{I_n}\sum_{I\in J_k}1\sum_{I\in J_k}a_I^2} \leqslant (a_{I_1}+a_{I_n})\sum_{I\in J_k}a_I,$$

that is,

$$\left(\sqrt{\frac{a_{I_1}}{a_{I_n}}} + \sqrt{\frac{a_{I_n}}{a_{I_1}}}\right)^2 \geqslant \frac{4\sum_{I\in J_k}1\sum_{I\in J_k}a_I^2}{\left(\sum_{I\in J_k}a_I\right)^2}.$$

On the other hand, since

$$\left(\sqrt{\frac{a_{I_1}}{a_{I_n}}} - \sqrt{\frac{a_{I_n}}{a_{I_1}}}\right)^2 = \left(\sqrt{\frac{a_{I_1}}{a_{I_n}}} + \sqrt{\frac{a_{I_n}}{a_{I_1}}}\right)^2 - 4,$$

we have that

$$\left(\sqrt{\frac{a_{I_1}}{a_{I_n}}} - \sqrt{\frac{a_{I_n}}{a_{I_1}}}\right)^2 \geqslant \frac{4\left(\sum_{I\in J_k}1\sum_{I\in J_k}a_I^2 - \left(\sum_{I\in J_k}a_I\right)^2\right)}{\left(\sum_{I\in J_k}a_I\right)^2},$$

and therefore it holds that

$$\sqrt{\frac{a_{I_1}}{a_{I_n}}}+\sqrt{\frac{a_{I_n}}{a_{I_1}}}\geqslant\frac{2\sqrt{\sum_{I\in J_k}1\sum_{I\in J_k}a_I^2}}{\sum_{I\in J_k}a_I} \tag{3.3}$$

and

$$\sqrt{\frac{a_{I_1}}{a_{I_n}}}-\sqrt{\frac{a_{I_n}}{a_{I_1}}}\geqslant\frac{2\sqrt{\sum_{I\in J_k}1\sum_{I\in J_k}a_I^2-\left(\sum_{I\in J_k}a_I\right)^2}}{\sum_{I\in J_k}a_I}. \tag{3.4}$$

From (3.3) and (3.4) we obtain

$$\sqrt{\frac{a_{I_1}}{a_{I_n}}}\geqslant\frac{\sqrt{\sum_{I\in J_k}1\sum_{I\in J_k}a_I^2}+\sqrt{\sum_{I\in J_k}1\sum_{I\in J_k}a_I^2-\left(\sum_{I\in J_k}a_I\right)^2}}{\sum_{I\in J_k}a_I}. \tag{3.5}$$

On the other hand, the following is valid:

$$\sum_{I\in J_k}1=\binom{n}{k}, \tag{3.6}$$

$$\sum_{I\in J_k}a_I=\binom{n-1}{k-1}\sum_{i=1}^{n}a_i=\binom{n-1}{k-1}A, \tag{3.7}$$

$$\begin{aligned}\sum_{I\in J_k}a_I^2&=\binom{n-2}{k-1}\sum_{i=1}^{n}a_i^2+\binom{n-2}{k-2}\left(\sum_{i=1}^{n}a_i\right)^2=\binom{n-2}{k-1}B+\binom{n-2}{k-2}A^2\\&=\frac{\binom{n-1}{k-1}((n-k)B+(k-1)A^2)}{n-1}.\end{aligned} \tag{3.8}$$

Now from the above identities and inequality (3.5) we obtain (3.1).

Let $2\leqslant k\leqslant n-1$. Then equality in (3.2) holds if and only if $a_I\in\{a_{I_1},a_{I_n}\}$ for every $I\in J_k$. Equality in (3.3), and thus in (3.4), holds if and only if a_I is a constant for every $I\in J_k$. This implies that equality in (3.1) holds if and only if $a_1=a_2=\ldots=a_n$.

Let $k=1$. Suppose that for some p, $1\leqslant p\leqslant n$, $a=a_1=\ldots=a_p\geqslant a_{p+1}=\ldots=a_n=b$. Assume that in (3.3), equality is reached, that is,

$$\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}=\frac{2\sqrt{n(pa^2+(n-p)b^2)}}{pa+(n-p)b}.$$

In that case it holds that

$$(a-b)(pa-(n-p)b)=0,$$

which means that equality in (3.3) holds if and only if $a=b$ or $pa-(n-p)b=0$, which implies that equality in (3.1) holds if and only if $a_1=a_2=\ldots=a_n$ or $a=a_1=\ldots=a_p$ and $pa/(n-p)=a_{p+1}=\ldots=a_n$ for any p, $1\leqslant p\leqslant n/2$. □

In the next theorem we determine a lower bound for $LS_k(a)$ in terms of A, B and parameters n and k, $1 \leqslant k \leqslant n-1$.

Theorem 3.2. *For any k, $1 \leqslant k \leqslant n-1$, we have that*

$$LS_k(a) = a_{I_1} - a_{I_n} \geqslant \frac{2}{n}\sqrt{\frac{k(n-k)(nB - A^2)}{n-1}}. \tag{3.9}$$

For $2 \leqslant k \leqslant n-1$, equality holds if and only if $a_1 = a_2 = \ldots = a_n$. For $k = 1$, equality holds if and only if $a_1 = a_2 = \ldots = a_n$ or $a_1 = a_2 = \ldots = a_{n/2} \geqslant a_{n/2+1} = \ldots = a_n$, for even n.

Proof. For $n := \binom{n}{k}$, $1 \leqslant k \leqslant n-1$, $x_i := a_I$, $y_i := 1$, $R_1 = a_{I_1}$, $r_1 = a_{I_n}$, $r_2 = R_2 = 1$, with summation performed over all k-tuples I, $I \in J_k$, the inequality (2.1) becomes

$$\sum_{I\in J_k} 1 \sum_{I\in J_k} a_I^2 - \Big(\sum_{I\in J_k} a_I\Big)^2 \leqslant \frac{\binom{n}{k}^2 (a_{I_1} - a_{I_n})^2}{4},$$

that is,

$$LS_k(a) = a_{I_1} - a_{I_n} \geqslant \frac{2\sqrt{\sum_{I\in J_k} 1 \sum_{I\in J_k} a_I^2 - \big(\sum_{I\in J_k} a_I\big)^2}}{\binom{n}{k}}. \tag{3.10}$$

From the above and identities (3.6), (3.7) and (3.8) we obtain

$$LS_k(a) = a_{I_1} - a_{I_n} \geqslant \frac{2\sqrt{\binom{n}{k}\binom{n-1}{k-1}((n-k)B + (k-1)A^2) - (n-1)\binom{n-1}{k-1}^2 A^2}}{\sqrt{n-1}\binom{n}{k}},$$

from which (3.9) is obtained.

For $2 \leqslant k \leqslant n-1$, equality in (3.10) holds if and only if a_I is constant for every $I \in J_k$, which implies that equality in (3.9) holds if and only if $a_1 = a_2 = \ldots = a_n$.

Let $k = 1$. Suppose that there exists p, $1 \leqslant p \leqslant n-1$, such that

$$a = a_1 = a_2 = \ldots = a_p \geqslant a_{p+1} = \ldots = a_n = b.$$

Then, according to (3.10),

$$a - b = \frac{2\sqrt{p(n-p)}(a-b)}{n}.$$

If $a \neq b$, then it follows that $p = \frac{1}{2}n$, where n is even. Thus, for $k = 1$, equality in (3.9) holds if and only if $a_1 = a_2 = \ldots = a_n$ or $a_1 = a_2 = \ldots = a_{n/2} \geqslant a_{n/2+1} = \ldots = a_n$, for even n. □

In the following theorems we establish lower bounds for $S_{k,\alpha}(a)$.

Theorem 3.3. *For any real α, $\alpha \leqslant 0$ or $\alpha \geqslant 1$, we have that*

$$S_{k,\alpha}(a) = \sum_{I \in J_k} a_I^\alpha \geqslant \frac{k^{\alpha-1}\binom{n-1}{k-1}A^\alpha}{n^{\alpha-1}}. \tag{3.11}$$

For $0 \leqslant \alpha \leqslant 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$ or $\alpha = 1$ or $a_1 = a_2 = \ldots = a_n$.

Proof. For $n := \binom{n}{k}$, $r = \alpha$, $\alpha \leqslant 0$ or $\alpha \geqslant 1$, $p_i := 1$, $x_i := a_I$, with summation performed over all k-tuples I, $I \in J_k$, inequality (2.2) becomes

$$\Bigl(\sum_{I \in J_k} 1\Bigr)^{\alpha-1} \sum_{I \in J_k} a_I^\alpha \geqslant \Bigl(\sum_{I \in J_k} a_I\Bigr)^\alpha. \tag{3.12}$$

From the above and identities (3.6), (3.7) and (3.8) we obtain (3.11).

In a similar way, one can prove that for the case $0 \leqslant \alpha \leqslant 1$, the opposite inequality is valid in (3.11).

Bearing in mind the conditions for the equality cases in (2.2), we conclude that equality in (3.12) holds if and only if $\alpha = 0$ or $\alpha = 1$ or a_I is constant for every $I \in J_k$. This implies that equality in (3.11) holds if and only if either $\alpha = 0$ or $\alpha = 1$ or $a_1 = a_2 = \ldots = a_n$. □

The proof of the next theorem is fully analogous to that of Theorem 3.3, thus omitted.

Theorem 3.4. *For any real α, $\alpha \leqslant 1$ or $\alpha \geqslant 2$, we have that*

$$S_{k,\alpha}(a) = \sum_{I \in J_k} a_I^\alpha \geqslant \frac{\binom{n-1}{k-1}((n-k)B + (k-1)A^2)^{\alpha-1}}{(n-1)^{\alpha-1}A^{\alpha-2}}.$$

For $1 \leqslant \alpha \leqslant 2$, the opposite inequality is valid. Equality holds either if $\alpha = 1$ or $\alpha = 2$ or $a_1 = a_2 = \ldots = a_n$.

4. Applications

4.1. Inequalities for Laplacian eigenvalues of graphs. Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, $|E| = m$, be a simple connected graph with a sequence of vertex degrees $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n > 0$, $d_i = d(v_i)$. Denote by $\mathcal{A} = (a_{ij})_{n\times n}$ and $D = (d_{ij})_{n\times n}$ the adjacency and the diagonal degree matrix of G, respectively. Then $L = D - \mathcal{A}$ is the Laplacian matrix of G. The eigenvalues of matrix L, $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{n-1} > \mu_n = 0$ form the Laplacian spectrum of G, see [11].

For $n := n-1$, $a_i = \mu_i$, $i = 1,2,\ldots,n-1$, the sums A and B defined by (1.1) become, see [18],

$$A = \sum_{i=1}^{n-1} \mu_i = 2m \quad \text{and} \quad B = \sum_{i=1}^{n-1} \mu_i^2 = M_1(G) + 2m,$$

where $M_1(G) = \sum_{i=1}^{n} d_i^2$ is the first Zagreb index introduced in [12]. Now, we have the following corollaries of the theorems proved in the previous section.

Corollary 4.1. *Let G be a simple connected graph with $n \geqslant 3$ vertices and m edges. Then for any k, $1 \leqslant k \leqslant n-2$, we have that*

$$\frac{\mu_{I_1}}{\mu_{I_n}} \geqslant \frac{1}{4m^2}\left(\sqrt{\frac{(n-1)((n-k-1)(M_1(G)+2m)+4m^2(k-1))}{k(n-2)}} + \sqrt{\frac{(n-k-1)((n-1)(M_1(G)+2m)-4m^2)}{k(n-2)}}\right)^2. \tag{4.1}$$

Corollary 4.2. *Let G be a simple connected graph with $n \geqslant 3$ vertices and m edges. Then for any k, $1 \leqslant k \leqslant n-2$, we have that*

$$\mu_{I_1} - \mu_{I_n} \geqslant \frac{2}{n-1}\sqrt{\frac{k(n-k-1)((n-1)(M_1(G)+2m)-4m^2)}{n-2}}. \tag{4.2}$$

For $2 \leqslant k \leqslant n-2$, equality holds if and only if $G \cong K_n$. When $k=1$, then equality holds if and only if $G \cong K_n$ or $\mu_1 = \mu_2 = \ldots = \mu_{(n-1)/2} > \mu_{(n+1)/2} = \ldots = \mu_{n-1}$, where n is odd.

In [8] it was proven that

$$M_1(G) \geqslant \frac{4m^2}{n},$$

with equality if and only if G is a regular graph. Therefore, the next result is also valid.

Corollary 4.3. *Let G be a simple connected graph with $n \geqslant 3$ vertices and m edges. Then for any k, $1 \leqslant k \leqslant n-2$, we have that*

$$\mu_{I_1} - \mu_{I_n} \geqslant \frac{2}{n-1}\sqrt{\frac{2mk(n-k-1)(n(n-1)-2m)}{n(n-2)}} \tag{4.3}$$

and

$$\frac{\mu_{I_1}}{\mu_{I_n}} \geqslant \frac{1}{2m}\left(\sqrt{\frac{(n-1)((n-k-1)(2m+n)+2mn(k-1))}{kn(n-2)}} + \sqrt{\frac{(n-k-1)(n(n-1)-2m)}{kn(n-2)}}\right)^2.$$

Equalities hold if and only if $G \cong K_n$.

Inequalities (4.1), (4.2) and (4.3) were proven in [20].

Corollary 4.4. *Let* G *be a simple connected graph with* $n \geqslant 3$ *vertices and* m *edges. Then for any real* α, $\alpha \leqslant 0$ *or* $\alpha \geqslant 1$ *we have that*

$$S_{k,\alpha}(G) = \sum_{I \in J_k} \mu_I^\alpha \geqslant \frac{(2m)^\alpha k^{\alpha-1}\binom{n-2}{k-1}}{(n-1)^{\alpha-1}}.$$

For $0 \leqslant \alpha \leqslant 1$, *the opposite inequality is valid. Equality holds if and only if* $G \cong K_n$.

Corollary 4.5. *Let* G *be a simple connected graph with* $n \geqslant 3$ *vertices and* m *edges. Then*

$$\mu_1 - \mu_{n-1} \geqslant \frac{2}{n-1}\sqrt{(n-1)(M_1(G)+2m)-4m^2} \tag{4.4}$$

and

$$\mu_1 - \mu_{n-1} \geqslant \frac{2}{n-1}\sqrt{\frac{2m(n(n-1)-2m)}{n}}. \tag{4.5}$$

Equality in (4.4) *holds if and only if* $G \cong K_n$ *or* $\mu_1 = \ldots = \mu_{(n-1)/2} > \mu_{(n+1)/2} = \ldots = \mu_{n-1}$, *where* n *is odd. Equality in* (4.5) *holds if and only if* $G \cong K_n$.

Inequality (4.4) was proven in [9] and [31], and in [5] the equality case was determined. Inequality (4.5) was proven in [20].

Corollary 4.6. *Let* G *be a simple connected* r*-regular graph,* $2 \leqslant r \leqslant n-1$, *with* n *vertices. Then*

$$\mu_1 - \mu_{n-1} \geqslant \frac{2}{n-1}\sqrt{nr(n-r-1)}. \tag{4.6}$$

Equality holds if and only if $G \cong K_n$ *or* G *is a conference graph.*

Inequality (4.6) was proven in [31] (see also [10]), and the equality case was proven in [5].

Corollary 4.7. *Let G be a simple connected graph with $n \geqslant 3$ vertices and m edges. Then*

$$\sqrt{\frac{\mu_1}{\mu_{n-1}}} + \sqrt{\frac{\mu_{n-1}}{\mu_1}} \geqslant \frac{\sqrt{(n-1)(M_1(G)+2m)}}{m}, \tag{4.7}$$

$$\sqrt{\frac{\mu_1}{\mu_{n-1}}} - \sqrt{\frac{\mu_{n-1}}{\mu_1}} \geqslant \frac{\sqrt{(n-1)(M_1(G)+2m)-4m^2}}{m} \tag{4.8}$$

and

$$\frac{\mu_1}{\mu_{n-1}} \geqslant \frac{\left(\sqrt{(n-1)(M_1(G)+2m)} + \sqrt{(n-1)(M_1(G)+2m)-4m^2}\right)^2}{4m^2}. \tag{4.9}$$

Equalities hold if and only if $G \cong K_n$.

Inequality (4.7) was proven in [9] and [30], while (4.8) and (4.9) in [30].

Corollary 4.8. *Let G be a simple connected graph with $n \geqslant 3$ vertices and m edges. Then for any real α, $\alpha \leqslant 0$ or $\alpha \geqslant 1$ we have that*

$$S_\alpha(G) = \sum_{i=1}^{n-1} \mu_i^\alpha \geqslant \frac{(2m)^\alpha}{(n-1)^{\alpha-1}}. \tag{4.10}$$

When $0 \leqslant \alpha \leqslant 1$, the opposite inequality is valid.

The graph invariant $S_\alpha(G)$ was introduced in [32]. Inequality (4.10) for $0 \leqslant \alpha \leqslant 1$ was proven in [26].

Corollary 4.9. *Let G be a simple connected graph with $n \geqslant 3$ vertices and m edges. Then for any real α, $\alpha \leqslant 1$ or $\alpha \geqslant 2$ we have that*

$$S_\alpha(G) = \sum_{i=1}^{n-1} \mu_i^\alpha \geqslant \frac{(M_1(G)+2m)^{\alpha-1}}{(2m)^{\alpha-2}}. \tag{4.11}$$

When $1 \leqslant \alpha \leqslant 2$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 1$ or $\alpha = 2$ or $G \cong K_n$.

Inequality (4.11) was proven in [6].

4.2. The Nordhaus-Gaddum type inequality for Laplacian eigenvalues of graphs. A Nordhaus-Gaddum type inequality, or NG-inequality for simplicity, gives a relationship between any parameter of a graph and its complement.

Denote by $\mu_i(G)$, $i = 1, 2, \ldots, n-1$, the Laplacian eigenvalues of graph G. Let $\overline{G}$ be a complement of G and $\mu_i(\overline{G})$ Laplacian eigenvalues of $\overline{G}$. The following identity is valid, see, e.g., [11], [30],

$$\mu_i(G) = n - \mu_{n-i}(\overline{G}), \quad i = 1, 2, \ldots, n-1. \tag{4.12}$$

Lemma 4.1. *For any k, $1 \leqslant k \leqslant n-2$, we have that*

$$\mu_{I_1}(G) + \mu_{I_1}(\overline{G}) = kn + LS_k(G) \tag{4.13}$$

and

$$\mu_{I_n}(G) + \mu_{I_n}(\overline{G}) = kn - LS_k(G). \tag{4.14}$$

P r o o f. Since

$$\mu_{I_1}(\overline{G}) = \mu_1(\overline{G}) + \mu_2(\overline{G}) + \ldots + \mu_k(\overline{G}),$$

from (4.12) it follows that

$$\mu_{I_1}(\overline{G}) = kn - \mu_{I_n}(G).$$

Therefore

$$\mu_{I_1}(\overline{G}) + \mu_{I_1}(G) = kn + \mu_{I_1}(G) - \mu_{I_n}(G),$$

from which (4.13) is obtained.

Similarly, since

$$\mu_{I_1}(G) = \mu_1(G) + \mu_2(G) + \ldots + \mu_k(G),$$

from (4.12) it follows that

$$\mu_{I_1}(G) = kn - \mu_{I_n}(\overline{G}),$$

and therefore

$$\mu_{I_1}(G) - \mu_{I_n}(G) = kn - \mu_{I_n}(\overline{G}) - \mu_{I_n}(G),$$

from which (4.14) is obtained. □

For $k = 1$, inequalities (4.13) and (4.14) become

$$\mu_1(G) + \mu_1(\overline{G}) = n + \mu_1(G) - \mu_{n-1}(G)$$

and

$$\mu_{n-1}(G) + \mu_{n-1}(\overline{G}) = n - \mu_1(G) + \mu_{n-1}(G),$$

which were proven in [31].

From identities (4.13) and (4.14) and inequalities (4.2) and (4.3), the next inequality of Nordhaus-Gaddum type is obtained, see [24].

Corollary 4.10. *Let G be a simple connected graph with $n \geqslant 3$ vertices and m edges. Then for any k, $1 \leqslant k \leqslant n-2$, we have that*

$$\mu_{I_1}(G) + \mu_{I_1}(\overline{G}) \geqslant kn + \frac{2}{n-1}\sqrt{\frac{k(n-k-1)((n-1)(M_1(G)+2m)-4m^2)}{n-2}}$$

and

$$\mu_{I_n}(G) + \mu_{I_n}(\overline{G}) \leqslant kn - \frac{2}{n-1}\sqrt{\frac{k(n-k-1)((n-1)(M_1(G)+2m)-4m^2)}{n-2}}.$$

When $2 \leqslant k \leqslant n-1$, equalities hold if and only if $G \cong K_n$. When $k=1$, equalities hold if and only if $G \cong K_n$ or $\mu_1 = \mu_2 = \ldots = \mu_{(n-1)/2} \geqslant \mu_{(n+1)/2} = \ldots = \mu_{n-1}$, for odd n.

Corollary 4.11. *Let G be a simple connected graph with $n \geqslant 3$ vertices and m edges. Then for any k, $1 \leqslant k \leqslant n-2$, we have that*

$$\mu_{I_1}(G) + \mu_{I_1}(\overline{G}) \geqslant kn + \frac{2}{n-1}\sqrt{\frac{2mk(n-k-1)(n(n-1)-2m)}{n(n-2)}}$$

and

$$\mu_{I_n}(G) + \mu_{I_n}(\overline{G}) \leqslant kn - \frac{2}{n-1}\sqrt{\frac{2mk(n-k-1)(n(n-1)-2m)}{n(n-2)}}.$$

Equalities hold if and only if $G \cong K_n$.

4.3. Inequalities for the normalized Laplacian eigenvalues of graphs. Denote by $\mathcal{L} = D^{-1/2}LD^{-1/2}$ the normalized Laplacian matrix of G, and by $\varrho_1 \geqslant \varrho_2 \geqslant \ldots \geqslant \varrho_{n-1} > \varrho_n = 0$ its eigenvalues, see [7]. The following identities are valid for the normalized Laplacian eigenvalues of G, see [33]:

$$A = \sum_{i=1}^{n-1} \varrho_i = n \quad \text{and} \quad B = \sum_{i=1}^{n-1} \varrho_i^2 = n + 2R_{-1}(G),$$

where

$$R_{-1}(G) = \sum_{i \sim j} \frac{1}{d_i d_j}$$

is a graph invariant known as general Randić index, see [4].

Let $a_i = \varrho_i$, $i = 1, 2, \ldots, n$. From (3.7) and (3.8) we have that

$$\sum_{I \in J_k} \varrho_I = n\binom{n-2}{k-1} \quad \text{and} \quad \sum_{I \in J_k} \varrho_I^2 = \frac{\binom{n-2}{k-1}}{n-2}((n-k-1)(n+2R_{-1}(G)) + n^2(k-1)).$$

In this section we give some corollaries of the results presented in Section 3.

Corollary 4.12. *Let G be a simple connected graph with $n \geqslant 3$ vertices. Then for any k, $1 \leqslant k \leqslant n-2$, we have that*

$$\sqrt{\frac{\varrho_{I_1}}{\varrho_{I_n}}}+\sqrt{\frac{\varrho_{I_n}}{\varrho_{I_1}}} \geqslant \frac{2}{n}\sqrt{\frac{(n-1)((n-k-1)(n+2R_{-1}(G))+n^2(k-1))}{k(n-2)}}. \tag{4.15}$$

Equality holds if and only if $G \cong K_n$.

In [28] it was proven that

$$\frac{n}{\Delta} \leqslant 2R_{-1}(G) \leqslant \frac{n}{\delta},$$

so we have the following consequence of Corollary 4.12.

Corollary 4.13. *Let G be a simple connected graph with $n \geqslant 3$ vertices. Then for every k, $1 \leqslant k \leqslant n-2$, we have that*

$$\sqrt{\frac{\varrho_{I_1}}{\varrho_{I_n}}}+\sqrt{\frac{\varrho_{I_n}}{\varrho_{I_1}}} \geqslant 2\sqrt{\frac{(n-1)((n-k-1)(1+\Delta)+n\Delta(k-1))}{n(n-2)k\Delta}}.$$

Equality holds if and only if $G \cong K_n$.

For $k=1$ we have the next corollary of Corollary 4.12.

Corollary 4.14. *Let G be a simple connected graph with $n \geqslant 3$ vertices. Then*

$$\sqrt{\frac{\varrho_1}{\varrho_{n-1}}}+\sqrt{\frac{\varrho_{n-1}}{\varrho_1}} \geqslant \frac{2}{n}\sqrt{(n-1)(n+2R_{-1}(G))}. \tag{4.16}$$

Equality holds if and only if $G \cong K_n$.

Inequality (4.16) was proven in [3], see also [13].

Corollary 4.15. *Let G be a simple connected graph with $n \geqslant 2$ vertices. Then*

$$\sqrt{\frac{\varrho_1}{\varrho_{n-1}}}+\sqrt{\frac{\varrho_{n-1}}{\varrho_1}} \geqslant 2\sqrt{\frac{(n-1)(1+\Delta)}{n\Delta}}.$$

Equality holds if and only if $G \cong K_n$.

Corollary 4.16. *Let G be a simple connected graph with $n \geqslant 3$ vertices. Then for every k, $1 \leqslant k \leqslant n-2$, we have that*

$$\sqrt{\frac{\varrho_{I_1}}{\varrho_{I_n}}}-\sqrt{\frac{\varrho_{I_n}}{\varrho_{I_1}}} \geqslant \frac{2}{n}\sqrt{\frac{(n-k-1)((n-1)(n+2R_{-1}(G))-n^2)}{(n-2)k}}.$$

Equality holds if and only if $G \cong K_n$.

Corollary 4.17. *Let G be a simple connected graph with $n \geqslant 3$ vertices. Then for any k, $1 \leqslant k \leqslant n-2$, we have that*

$$\frac{\varrho_{I_1}}{\varrho_{I_n}} \geqslant \Bigg(\frac{\sqrt{(n-1)((n-k-1)(n+2R_{-1}(G))+n^2(k-1))}}{n\sqrt{(n-2)k}} + \frac{\sqrt{(n-k-1)((n-1)(n+2R_{-1}(G))-n^2)}}{n\sqrt{(n-2)k}} \Bigg)^2.$$

Equality holds if and only if $G \cong K_n$.

Corollary 4.18. *Let G be a simple connected graph with $n \geqslant 3$ vertices. Then for any k, $1 \leqslant k \leqslant n-2$, we have that*

$$\frac{\varrho_{I_1}}{\varrho_{I_n}} \geqslant \Bigg(\frac{\sqrt{(n-1)((n-k-1)(1+\Delta)+n(k-1)\Delta)}}{\sqrt{n(n-2)\Delta k}} + \frac{\sqrt{(n-k-1)((n-1)(1+\Delta)-n\Delta)}}{\sqrt{n(n-2)\Delta k}} \Bigg)^2.$$

Equality holds if and only if $G \cong K_n$.

Corollary 4.19. *Let G be a simple connected graph with $n \geqslant 2$ vertices. Then*

$$\frac{\varrho_1}{\varrho_{n-1}} \geqslant \left(\frac{\sqrt{(n-1)(n+2R_{-1}(G))} + \sqrt{(n-1)(n+2R_{-1}(G))-n^2}}{n} \right)^2.$$

Equality holds if and only if $G \cong K_n$.

Corollary 4.20. *Let G be a simple connected graph with $n \geqslant 2$ vertices. Then*

$$\frac{\varrho_1}{\varrho_{n-1}} \geqslant \left(\sqrt{\frac{(n-1)(1+\Delta)}{\Delta n}} + \sqrt{\frac{n-1-\Delta}{\Delta n}} \right)^2.$$

Equality holds if and only if $G \cong K_n$.

Remark 4.1. If G is a d-regular graph, $2 \leqslant d \leqslant n-1$, then

$$\frac{\varrho_1}{\varrho_{n-1}} \geqslant \frac{1}{nd}(\sqrt{(n-1)(1+d)} + \sqrt{n-1-d})^2,$$

with equality if and only if $G \cong K_n$. The above inequality was proven in [10].

Corollary 4.21. *Let G be a simple connected graph with $n \geqslant 3$ vertices. Then for any k, $1 \leqslant k \leqslant n-2$, we have that*

$$\varrho_{I_1} - \varrho_{I_n} \geqslant \frac{2}{n-1}\sqrt{\frac{k(n-k-1)(2(n-1)R_{-1}(G)-n)}{n-2}}.$$

Equality holds if and only if $G \cong K_n$.

Corollary 4.22. *Let G be a simple connected graph with $n \geqslant 3$ vertices. Then for any k, $1 \leqslant k \leqslant n-2$, we have that*

$$\varrho_{I_1} - \varrho_{I_n} \geqslant \frac{2}{n-1}\sqrt{\frac{kn(n-k-1)(n-1-\Delta)}{(n-2)\Delta}}.$$

Equality holds if and only if $G \cong K_n$.

Corollary 4.23. *Let G be a simple connected graph with $n \geqslant 2$ vertices. Then*

$$\varrho_1 - \varrho_{n-1} \geqslant \frac{2}{n-1}\sqrt{2(n-1)R_{-1}(G) - n}. \tag{4.17}$$

Equality holds if and only if $G \cong K_n$.

Inequality (4.17) was proven in [3], see also [1], [13].

Corollary 4.24. *Let G be a simple connected graph with $n \geqslant 3$ vertices. Then*

$$\varrho_1 - \varrho_{n-1} \geqslant \frac{2}{n-1}\sqrt{\frac{n(n-1-\Delta)}{\Delta}}.$$

Equality holds if and only if $G \cong K_n$.

The above inequality was given in [1].

For $a_i = \varrho_i$, $i = 1, 2, \ldots, n$, based on Theorem 3.4 we obtain the following result.

Corollary 4.25. *Let G be a simple connected graph with $n \geqslant 3$ vertices. Then for any real α such that $\alpha \leqslant 1$ or $\alpha \geqslant 2$ we have that*

$$S_{\alpha,k}(G) = \sum_{I \in J_k} \varrho_I^\alpha \geqslant \frac{\binom{n-2}{k-1}((n-k-1)(n+2R_{-1}(G)) + n^2(k-1))^{\alpha-1}}{n^{\alpha-2}(n-2)^{\alpha-1}}.$$

When $1 \leqslant \alpha \leqslant 2$, the sense of the inequality reverses. Equality holds if and only if either $\alpha = 1$ or $\alpha = 2$ or $G \cong K_n$.

For $k = 1$, from Corollary 4.25 we obtain the following result.

Corollary 4.26. *Let G be a simple connected graph with n vertices. Then for any real α, $\alpha \leqslant 1$ or $\alpha \geqslant 2$ we have that*

$$S_\alpha(G) = \sum_{i=1}^{n-1} \varrho_i^\alpha \geqslant \frac{(n+2R_{-1}(G))^{\alpha-1}}{n^{\alpha-2}}.$$

For $1 \leqslant \alpha \leqslant 2$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 1$ or $\alpha = 2$ or $G \cong K_n$.

The above inequality was proven in [17].

Corollary 4.27. *Let G be a simple connected graph with n vertices. Then*

$$\sum_{i=1}^{n-1} \sqrt{\varrho_i} \geqslant \sqrt{\frac{n^3}{n+2R_{-1}(G)}} \tag{4.18}$$

and

$$\sum_{i=1}^{n-1} \frac{1}{\varrho_i} \geqslant \frac{n^3}{(n+2R_{-1}(G))^2}. \tag{4.19}$$

In both cases, equality is valid if and only if $G \cong K_n$.

Inequality (4.18) was proven in [29], whereas (4.19) in [17].

Remark 4.2. The invariant

$$LIE(G) = \sum_{i=1}^{n-1} \sqrt{\varrho_i}$$

was conceived in [29] (see also [23]) and named the Laplacian incidence energy, whereas

$$K(G) = \sum_{i=1}^{n-1} \frac{1}{\varrho_i}$$

is Kemeny's constant, see [16].

For $a_i = \varrho_i$, $i = 1, 2, \ldots, n$, from Theorem 3.3 we obtain the following result.

Corollary 4.28. *Let G be a simple connected graph with n vertices. Then for any real α, $\alpha \leqslant 0$ or $\alpha \geqslant 1$ we have that*

$$S_{\alpha,k}(G) \geqslant \frac{n^\alpha k^{\alpha-1}\binom{n-2}{k-1}}{(n-1)^{\alpha-1}}.$$

When $0 \leqslant \alpha \leqslant 1$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$ or $\alpha = 1$ or $G \cong K_n$.

Corollary 4.29. *Let G be a simple connected graph with n vertices. Then for any real α, $\alpha \leqslant 0$ or $\alpha \geqslant 1$ we have that*

$$S_\alpha(G) \geqslant \frac{n^\alpha}{(n-1)^{\alpha-1}}.$$

When $0 \leqslant \alpha \leqslant 1$, the sense of the inequality reverses. Equality holds if and only if either $\alpha = 0$ or $\alpha = 1$ or $G \cong K_n$.

Corollary 4.30. *Let G be a simple connected graph with n vertices. Then*

$$LIE(G) = \sum_{i=1}^{n-1} \sqrt{\varrho_i} \leqslant \sqrt{n(n-1)} \tag{4.20}$$

and

$$K(G) = \sum_{i=1}^{n-1} \frac{1}{\varrho_i} \geqslant \frac{(n-1)^2}{n}. \tag{4.21}$$

In both cases, equality holds if and only if $G \cong K_n$.

Inequality (4.20) was proven in [29], whereas (4.21) in [27], see also [2], [14], [19], [21].

References

[1] *E. Andrade, M. A. A. de Freitas, M. Robbiano, J. Rodríguez*: New lower bounds for the Randić spread. Linear Algebra Appl. *544* (2018), 254–272. zbl MR doi

[2] *M. Bianchi, A. Cornaro, J. L. Palacios, A. Torriero*: Bounds for the Kirchhoff index via majorization techniques. J. Math. Chem. *51* (2013), 569–587. zbl MR doi

[3] *S. K. Butler*: Eigenvalues and Structures of Graphs: Ph.D. Thesis. University of California, San Diego, 2008. MR

[4] *M. Cavers, S. Fallat, S. Kirkland*: On the normalized Laplacian energy and general Randić index R_{-1} of graphs. Linear Algebra Appl. *433* (2010), 172–190. zbl MR doi

[5] *X. Chen, K. C. Das*: Some results on the Laplacian spread of a graph. Linear Algebra Appl. *505* (2016), 245–260. zbl MR doi

[6] *X. Chen, J. Qian*: Bounding the sum of powers of the Laplacian eigenvalues of graphs. Appl. Math., Ser. B (Engl. Ed.) *26* (2011), 142–150. zbl MR doi

[7] *F. R. K. Chung*: Spectral Graph Theory. Regional Conference Series in Mathematics 92. AMS, Providence, 1997. zbl MR doi

[8] *C. S. Edwards*: The largest vertex degree sum for a triangle in a graph. Bull. Lond. Math. Soc. *9* (1977), 203–208. zbl MR doi

[9] *G. H. Fath-Tabar, A. R. Ashrafi*: Some remarks on Laplacian eigenvalues and Laplacian energy of graphs. Math. Commun. *15* (2010), 443–451. zbl MR

[10] *F. Goldberg*: Bounding the gap between extremal Laplacian eigenvalues of graphs. Linear Algebra Appl. *416* (2006), 68–74. zbl MR doi

[11] *R. Grone, R. Merris*: The Laplacian spectrum of graph. II. SIAM J. Discrete Math. *7* (1994), 221–229. zbl MR doi

[12] *I. Gutman, N. Trinajstić*: Graph theory and molecular orbitals. Total φ-electron energy of alternant hydrocarbons. Chem. Phys. Lett. *17* (1972), 535–538. doi

[13] *M. Hakimi-Nezhaad, A. R. Ashrafi*: A note on normalized Laplacian energy of graphs. J. Contemp. Math. Anal., Armen. Acad. Sci. *49* (2014), 207–211. zbl MR doi

[14] *J. Huang, S. Li*: On the normalized Laplacian spectrum, degree-Kirchhoff index and spanning trees of graphs. Bul. Aust. Math. Soc. *91* (2015), 353–367. zbl MR doi

[15] *J. L. W. V. Jensen*: Sur les functions convexes et les inéqualités entre les valeurs moyennes. Acta Math. *30* (1906), 175–193. (In French.) zbl MR doi

[16] *J. G. Kemeny, J. L. Snell*: Finite Markov Chains. The University Series in Undergraduate Mathematics. Van Nostrand, Princeton, 1960. zbl MR
[17] *J. Li, J.-M. Guo, W. C. Shiu, Ş. B. B. Altındağ, D. Bozkurt*: Bounding the sum of powers of normalized Laplacian eigenvalues of a graph. Appl. Math. Comput. *324* (2018), 82–92. zbl MR doi
[18] *R. Merris*: Laplacian matrices of graphs: A survey. Linear Algebra Appl. *197–198* (1994), 143–176. zbl MR doi
[19] *I. Ž. Milovanović, E. I. Milovanović, E. Glogić*: Lower bounds of the Kirchhoff and degree Kirchhoff indices. Sci. Publ. State Univ. Novi Pazar, Ser. A, Appl. Math. Inf. Mech. *7* (2015), 25–31. doi
[20] *I. Ž. Milovanović, E. I. Milovanović, E. Glogić*: On Laplacian eigenvalues of connected graphs. Czech. Math. J. *65* (2015), 529–535. zbl MR doi
[21] *I. Ž. Milovanović, E. I. Milovanović*: Bounds for the Kirchhoff and degree Kirchhoff indices. Bounds in Chemical Graph Theory: Mainstreams. Mathematical Chemistry Monographs 20. University of Kragujevac, Kragujevac, 2017, pp. 93–119.
[22] *D. S. Mitrinović, J. E. Pečarić, A. M. Fink*: Classical and New Inequalities in Analysis. Mathematics and Its Applications. East European Series 61. Kluwer Academic Publishers, Dorchrecht, 1993. zbl MR doi
[23] *V. Nikiforov*: The energy of graphs and matrices. J. Math. Anal. Appl. *326* (2007), 1472–1475. zbl MR doi
[24] *E. A. Nordhaus, J. W. Gaddum*: On complementary graphs. Am. Math. Mon. *63* (1956), 175–177. zbl MR doi
[25] *N. Ozeki*: On the estimation of the inequalities by the maximum, or minimum values. J. College Arts Sci. Chiba Univ. *5* (1968), 199–203. (In Japanese.) MR
[26] *J. L. Palacios*: Some inequalities for Laplacian descriptors via majorization. MATCH Commun. Math. Comput. Chem. *77* (2017), 189–194. zbl MR
[27] *J. L. Palacios, J. M. Renom*: Broder and Karlin's formula for hitting times and the Kirchhoff index. Int. J. Quantum Chem. *111* (2011), 35–39. doi
[28] *L. Shi*: Bounds on Randić indices. Discr. Math. *309* (2009), 5238–5241. zbl MR doi
[29] *L. Shi, H. Wang*: The Laplacian incidence energy of graphs. Linear Algebra Appl. *439* (2013), 4056–4062. zbl MR doi
[30] *Z. You, B. Liu*: On the Laplacian spectral ratio of connected graphs. Appl. Math. Lett. *25* (2012), 1245–1250. zbl MR doi
[31] *Z. You, B. Liu*: The Laplacian spread of graphs. Czech. Math. J. *62* (2012), 155–168. zbl MR doi
[32] *B. Zhou*: On sum of powers of the Laplacian eigenvalues of graphs. Linear Algebra Appl. *429* (2008), 2239–2246. zbl MR doi
[33] *P. Zumstein*: Comparison of Spectral Methods Through the Adjacency Matrix and the Laplacian of a Graph: Diploma Thesis. ETH, Zürich, 2005.

Authors' addresses: Emina Milovanović, University of Niš, Faculty of Electronic Engineering, Aleksandra Medvedeva 14, Niš 18106, Serbia, e-mail: ema@elfak.ni.ac.rs; Şerife Burcu Bozkurt Altındağ, Yenikent Kardelen Konutlari, Selçuklu, 42070 Konya, Turkey, e-mail: srf_burcu_bozkurt@hotmail.com; Marjan Matejić, Igor Milovanović, University of Niš, Faculty of Electronic Engineering, Aleksandra Medvedeva 14, Niš 18106, Serbia, e-mail: marjan.matejic@elfak.ni.ac.rs, igor@elfak.ni.ac.rs.